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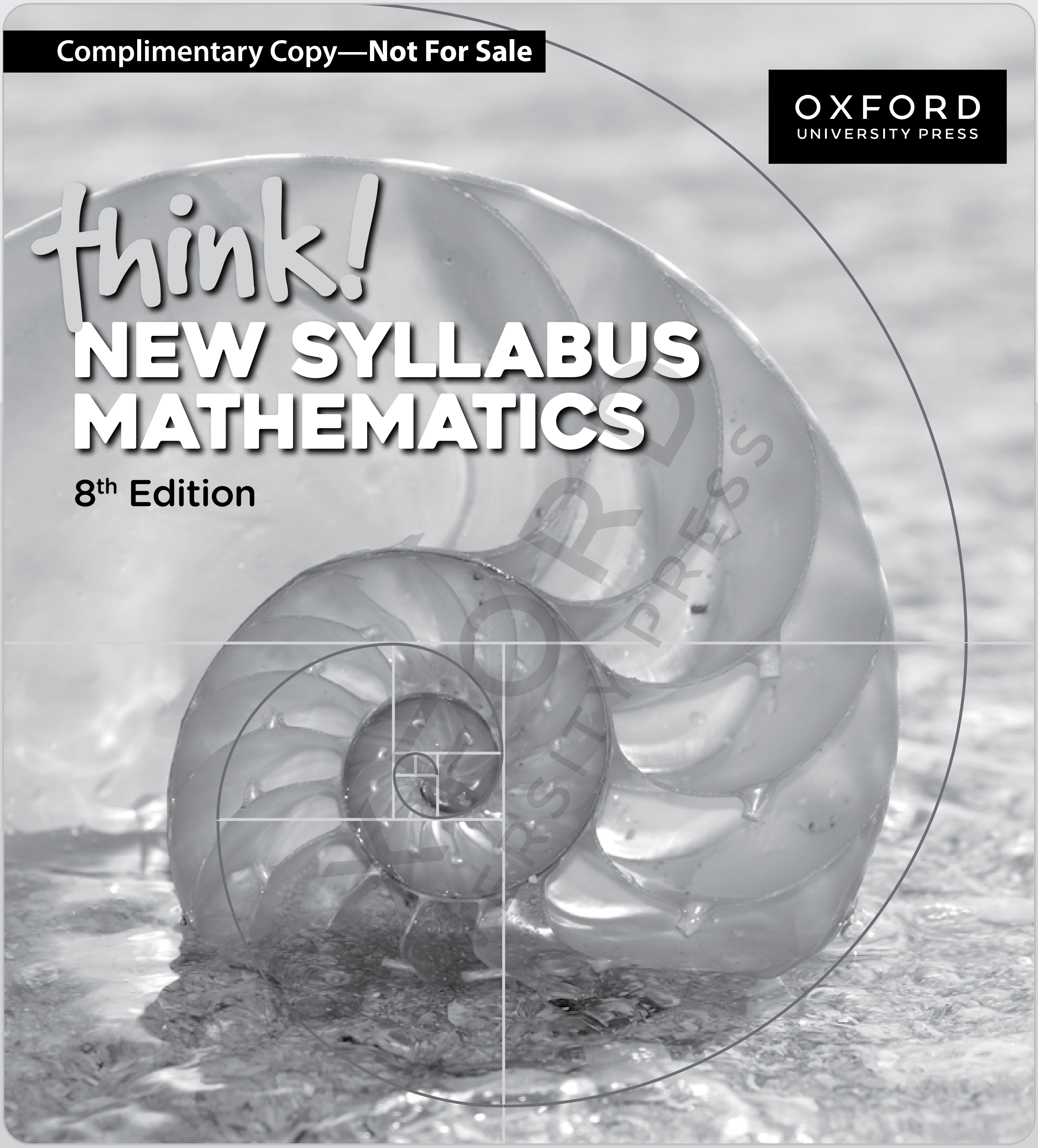


think! NEW SYLLABUS MATHEMATICS

8th Edition

Teacher's Resource Book 4

TV

CONTENTS

Syllabus Matching Grid

Cambridge O Level Mathematics (Syllabus D) 4024. Syllabus for examination in 2025, 2026 and 2027.

Theme or Topic	Subject Content	Reference
1. Number	Identify and use: • Natural numbers • Integers (positive, negative and zero) • Prime numbers • Square numbers • Cube numbers • Common factors and common multiples • Rational and irrational numbers (e.g. π, $\sqrt{2}$) • Reciprocals.	Book 1: Chapter 1 Chapter 2 Chapter 4
2. Sets	• Use set language, set notation and Venn diagrams to describe sets and represent relationships between sets	Book 2: Chapter 12 Book 4: Chapter 1
2. Powers and roots	Calculate with the following: • Squares • Square roots • Cubes and cube roots of numbers • Cube roots • other powers and roots of numbers	Book 1: Chapter 1 Chapter 2 Chapter 3 Chapter 4
4. Fractions, decimals and percentages	• Use the language and notation of the following in appropriate contexts: (a) proper fractions (b) improper fractions (c) mixed numbers (d) decimals (e) percentages. • Recognise equivalence and convert between these forms.	Book 1: Chapter 2 Chapter 3 Chapter 8
5. Ordering	• Order quantities by magnitude and demonstrate familiarity with the symbols $=, \neq, <, >, \leqslant, \geqslant$.	Book 1: Chapter 2 Chapter 3 Chapter 4 Book 2: Chapter 3
6. The four operations	• Use the four operations for calculations with integers, fractions and decimals, including correct ordering of operations and use of brackets.	Book 1: Chapter 2 Chapter 3 Chapter 4
7. Indices I	• Understand and use indices (positive, zero, negative and fractional). • Understand and use the rules of indices.	Book 3: Chapter 4
8. Standard form	• Use the standard form $A \times 10^n$, where n is a positive or negative integer, and $1 \leqslant A < 10$. • Convert numbers into and out of standard form. • Calculate with values in standard form.	Book 3: Chapter 4

9.	Estimation	• Round values to a specified degree of accuracy. • Make estimates for calculations involving numbers, quantities and measurements. • Round answers to a reasonable degree of accuracy in the context of a given problem.	Book 1: Chapter 5
10.	Limits of accuracy	• Give upper and lower bounds for data rounded to a specified accuracy. • Find upper and lower bounds of the results of calculations which have used data rounded to a specified accuracy.	Book 1: Chapter 5
11.	Ratio and proportion	Understand and use ratio and proportion to: • give ratios in their simplest form • divide a quantity in a given ratio • use proportional reasoning and ratios in context.	Book 1: Chapter 9
12.	Rates	• Use common measures of rate. • Apply other measures of rate. • Solve problems involving average speed.	Book 1: Chapter 9
13.	Percentages	• Calculate a given percentage of a quantity. • Express one quantity as a percentage of another. • Calculate percentage increase or decrease. • Calculate with simple and compound interest. • Calculate using reverse percentages.	Book 1: Chapter 8
14.	Using a calculator	• Use a calculator efficiently. • Enter values appropriately on a calculator. • Interpret the calculator display appropriately.	Book 1: Chapter 1 Chapter 4 Chapter 12
15.	Time	• Calculate with time: seconds (s), minutes (min), hours (h), days, weeks, months, years, including the relationship between units. • Calculate times in terms of the 24-hour and 12-hour clock. • Read clocks and timetables.	Book 1: Chapter 9
16.	Money	• Calculate with money. • Convert from one currency to another.	Book 1: Chapter 9 Book 2: Chapter 6
17.	Exponential growth and decay	• Use exponential growth and decay.	Book 3: Chapter 4
18.	Surds	• Understand and use surds, including simplifying expressions. • Rationalise the denominator.	Book 3: Chapter 4
19.	Introduction to algebra	• Know that letters can be used to represent generalised numbers. • Substitute numbers into expressions and formulas.	Book 1: Chapter 6
20.	Algebraic manipulation	• Simplify expressions by collecting like terms. • Expand products of algebraic expressions. • Factorise by extracting common factors. • Factorise expressions of the form: (a) $ax + bx + kay + kby$ (b) $a^2x^2 - b^2y^2$ (c) $a^2 + 2ab + b^2$ (d) $ax^2 + bx + c$ (e) $ax^3 + bx^2 + cx$ • Complete the square for expressions in the form $ax^2 + bx + c$.	Book 1: Chapter 6 Book 2: Chapter 4

21. Algebraic fractions	• Manipulate algebraic fractions. • Factorise and simplify rational expressions.	Book 3: Chapter 1
22. Indices II	• Understand and use indices (positive, zero, negative and fractional). • Understand and use the rules of indices.	Book 3: Chapter 4
23. Equations	• Construct expressions, equations and formulas. • Solve linear equations in one unknown. • Solve fractional equations with numerical and linear algebraic denominators. • Solve simultaneous linear equations in two unknowns. • Solve quadratic equations by factorisation, completing the square and by use of the quadratic formula. • Change the subject of formulas.	Book 1: Chapter 7 Book 2: Chapter 2 Book 3: Chapter 1 Chapter 2 Chapter 3
24. Inequalities	• Represent and interpret inequalities, including on a number line. • Construct, solve and interpret linear inequalities. • Represent and interpret linear inequalities in two variables graphically. • List inequalities that define a given region.	Book 2: Chapter 3
25. Sequences	• Continue a given number sequence or pattern. • Recognise patterns in sequences, including the term-to-term rule, and relationships between different sequences. • Find and use the *n*th term of sequences.	Book 2: Chapter 5
26. Proportion	• Express direct and inverse proportion in algebraic terms and use this form of expression to find unknown quantities.	Book 2: Chapter 7
27. Graphs in practical situations	• Use and interpret graphs in practical situations including travel graphs and conversion graphs. • Draw graphs from given data. • Apply the idea of rate of change to simple kinematics involving distance-time and speed-time graphs, acceleration and deceleration. • Calculate distance travelled as area under a speed-time graph.	Book 2: Chapter 1 Book 3: Chapter 6
28. Graphs of functions	• Construct tables of values, and draw, recognise and interpret graphs for functions of the following forms: (a) ax^n (includes sums of no more than three of these) (b) $ab^x + c$ where $n = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2, 3$; a and c are rational numbers; and b is a positive integer. • Solve associated equations graphically, including finding and interpreting roots by graphical methods. • Draw and interpret graphs representing exponential growth and decay problems. • Estimate gradients of curves by drawing tangents.	Book 3: Chapter 6
29. Sketching curves	• Recognise, sketch and interpret graphs of the following functions: (a) linear (b) quadratic (c) cubic (d) reciprocal (e) exponential.	Book 2: Chapter 1 Book 3: Chapter 2 Chapter 3 Chapter 6
30. Functions	• Understand functions, domain and range, and use function notation. • Understand and find inverse functions $f^{-1}(x)$. • Form composite functions as defined by $gf(x) = g(f(x))$.	Book 4: Chapter 5

31. Coordinates	• Use and interpret Cartesian coordinates in two dimensions.	Book 2: Chapter 1
32. Drawing linear graphs	• Draw straight-line graphs for linear equations.	Book 2: Chapter 1
33. Gradient of linear graphs	• Find the gradient of a straight line. • Calculate the gradient of a straight line from the coordinates of two points on it.	Book 2: Chapter 1 Book 3: Chapter 5
34. Length and midpoint	• Calculate the length of a line segment. • Find the coordinates of the midpoint of a line segment.	Book 3: Chapter 5
35. Equations of linear graphs	• Interpret and obtain the equation of a straight-line graph.	Book 2: Chapter 1 Book 3: Chapter 5
36. Parallel lines	• Find the gradient and equation of a straight line parallel to a given line.	Book 3: Chapter 5
37. Perpendicular lines	• Find the gradient and equation of a straight line perpendicular to a given line.	Book 3: Chapter 5
38. Geometrical terms	• Use and interpret the following geometrical terms: (a) point (b) vertex (c) line (d) plane (e) parallel (f) perpendicular (g) perpendicular bisector (h) bearing (i) right angle (j) acute, obtuse and reflex angles (k) interior and exterior angles (l) similar (m) congruent (n) scale factor. • Use and interpret the vocabulary of: (a) triangles (b) special quadrilaterals (c) polygons (d) nets (e) solids. • Use and interpret the vocabulary of a circle.	Book 1: Chapter 10 Chapter 11 Book 2: Chapter 8 Chapter 11
39. Geometrical constructions	• Measure and draw lines and angles. • Construct a triangle, given the lengths of all sides, using a ruler and pair of compasses only. • Draw, use and interpret nets.	Book 1: Chapter 11

40. Scale drawings	• Draw and interpret scale drawings. • Use and interpret three-figure bearings.	Book 2: Chapter 8 Book 4: Chapter 7
41. Similarity	• Calculate lengths of similar shapes. • Use the relationships between lengths and areas of similar shapes and lengths, surface areas and volumes of similar solids. • Solve problems and give simple explanations involving similarity.	Book 2: Chapter 8 Book 4: Chapter 11
42. Symmetry	• Recognise line symmetry and order of rotational symmetry in two dimensions. • Recognise symmetry properties of prisms, cylinders, pyramids and cones.	Book 1: Chapter 11 Book 2: Chapter 11 Book 3: Chapter 7
43. Angles	• Calculate unknown angles and give simple explanations using the following geometrical properties: (a) sum of angles at a point = 360° (b) sum of angles at a point on a straight line = 180° (c) vertically opposite angles are equal (d) angle sum of a triangle = 180° and angle sum of a quadrilateral = 360°. • Calculate unknown angles and give geometric explanations for angles formed within parallel lines: (a) corresponding angles are equal (b) alternate angles are equal (c) co-interior (supplementary) angles sum to 180°. • Know and use angle properties of regular and irregular polygons.	Book 1: Chapter 10 Chapter 11
44. Circle theorems I	• Calculate unknown angles and give explanations using the following geometrical properties of circles: • angle in a semicircle = 90° • angle between tangent and radius = 90° • angle at the centre is twice the angle at the circumference • angles in the same segment are equal • opposite angles of a cyclic quadrilateral sum to 180° (supplementary) • alternate segment theorem.	Book 4: Chapter 9
45. Circle theorems II	• Use the following symmetry properties of circles: • equal chords are equidistant from the centre • the perpendicular bisector of a chord passes through the centre • tangents from an external point are equal in length.	Book 4: Chapter 9
46. Units of measure	• Use metric units of mass, length, area, volume and capacity in practical situations and convert quantities into larger or smaller units.	Book 1: Chapter 3 Chapter 12 Book 2: Chapter 11
47. Area and perimeter	• Carry out calculations involving the perimeter and area of a rectangle, triangle, parallelogram and trapezium.	Book 1: Chapter 12

48. Circles, arcs and sectors	• Carry out calculations involving the circumference and area of a circle. • Carry out calculations involving arc length and sector area as fractions of the circumference and area of a circle.	Book 1: Chapter 12 Book 4: Chapter 8
49. Surface area and volume	Carry out calculations and solve problems involving the surface area and volume of a: • cuboid • prism • cylinder • sphere • pyramid • cone.	Book 2: Chapter 11 Book 3: Chapter 7
50. Compound shapes and parts of shapes	• Carry out calculations and solve problems involving perimeters and areas of: (a) compound shapes (b) parts of shapes. • Carry out calculations and solve problems involving surface areas and volumes of: (a) compound solids (b) parts of solids.	Book 1: Chapter 12 Book 2: Chapter 11 Book 3: Chapter 7
51. Pythagoras' theorem	• Know and use Pythagoras' theorem.	Book 2: Chapter 9
52. Right-angled triangles	• Know and use the sine, cosine and tangent ratios for acute angles in calculations involving sides and angles of a right-angled triangle. • Solve problems in two dimensions using Pythagoras' theorem and trigonometry. • Know that the perpendicular distance from a point to a line is the shortest distance to the line. • Carry out calculations involving angles of elevation and depression.	Book 2: Chapter 9 Chapter 10 Book 4: Chapter 6 Chapter 7
53. Non-right-angled triangles	• Use the sine and cosine rules in calculations involving lengths and angles for any triangle. • Use the formula area of triangle = $\frac{1}{2}ab \sin C$	Book 4: Chapter 6
54. Pythagoras' theorem and trigonometry in 3D	• Carry out calculations and solve problems in three dimensions using Pythagoras' theorem and trigonometry, including calculating the angle between a line and a plane.	Book 4: Chapter 7
55. Transformations	Recognise, describe and draw the following transformations: • Reflection of a shape in a straight line. • Rotation of a shape about a centre through multiples of 90°. • Enlargement of a shape from a centre by a scale factor. • Translation of a shape by a vector $\begin{pmatrix} x \\ y \end{pmatrix}$.	Book 4: Chapter 10
56. Vectors in two dimensions	• Describe a translation using a vector represented by $\begin{pmatrix} x \\ y \end{pmatrix}$, $\overrightarrow{AB}$ or $\mathbf{a}$ • Add and subtract vectors. • Multiply a vector by a scalar.	Book 4: Chapter 7
57. Magnitude of a vector	• Calculate the magnitude of a vectorvector $\begin{pmatrix} x \\ y \end{pmatrix}$ as $\sqrt{x^2 + y^2}$	Book 4: Chapter 4

58. Vector geometry	• Represent vectors by directed line segments. • Use position vectors. • Use the sum and difference of two or more vectors to express given vectors in terms of two coplanar vectors. • Use vectors to reason and to solve geometric problems.	Book 4: Chapter 4
59. Introduction to probability	• Understand and use the probability scale from 0 to 1. • Understand and use probability notation. • Calculate the probability of a single event. • Understand that the probability of an event not occurring = 1 – the probability of the event occurring.	Book 2: Chapter 12
60. Relative and expected frequencies	• Understand relative frequency as an estimate of probability. • Calculate expected frequencies.	Book 2: Chapter 12
61. Probability of combined events	Calculate the probability of combined events using, where appropriate: • sample space diagrams • Venn diagrams • tree diagrams.	Book 4: Chapter 2
62. Classifying statistical data	• Classify and tabulate statistical data.	Book 1: Chapter 13
63. Interpreting statistical data	• Read, interpret and draw inferences from tables and statistical diagrams. • Compare sets of data using tables, graphs and statistical measures. • Appreciate restrictions on drawing conclusions from given data.	Book 1: Chapter 13
64. Averages and measures of spread	• Calculate the mean, median, mode and range for individual data and distinguish between the purposes for which these are used. • Calculate an estimate of the mean for grouped discrete or grouped continuous data. • Identify the modal class from a grouped frequency distribution.	Book 3: Chapter 8
65. Statistical charts and diagrams	• Draw and interpret: (a) bar charts (b) pie charts (c) pictograms (d) simple frequency distributions.	Book 1: Chapter 13
66. Scatter diagrams	• Draw and interpret scatter diagrams. • Understand what is meant by positive, negative and zero correlation. • Draw by eye, interpret and use a straight line of best fit.	Book 4: Chapter 3
67. Cumulative frequency diagrams	• Draw and interpret cumulative frequency tables and diagrams. • Estimate and interpret the median, percentiles, quartiles and interquartile range from cumulative frequency diagrams.	Book 4: Chapter 3
68. Histograms	• Draw and interpret histograms. • Calculate with frequency density.	Book 2: Chapter 13

Secondary 4 Mathematics Scheme of Work

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
1	**1 Further Sets**	**1.1** Intersection and union of two sets (pp. 2 – 7)	• Use Venn diagrams to represent the intersection and union of two sets	Understand and use set language, notation and Venn diagrams to describe sets and represent relationships between sets			Practise Now 2 Q 2(iv) (p. 5) Ex 1A Q 8(iv), 9(ii), 11(iv) (p. 6)
1		**1.2** Applications of sets in real-world contexts (pp. 7 – 21)	• Use Venn diagrams to represent the intersection and union of two or three sets • Solve problems involving sets and Venn diagrams		Thinking Time (p. 11) Investigation – Exploring related sets (p. 14) Performance Task (p. 16)		
1		**Miscellaneous**					
2	**2 Probability of Combined Events**	**2.1** Probability of single event (pp. 24 – 28)	• Find probability of single events using sample space diagrams and tree diagrams	Understand and use probability notation Calculate the probability of a single event	Class Discussion – Range of values of the probability of an event (p. 25)		
2		**2.2** Probability of combined events (pp. 28 – 41)	• Find probability of combined events using sample space diagrams and tree diagrams	Calculate the probability of combined events using, where appropriate: • sample space diagrams • Venn diagrams • tree diagrams	Thinking Time (p. 29) Thinking Time (p. 34)		Ex 2A Q 9(iv) (p. 38)
3		**2.3** Addition Law of Probability and mutually exclusive events (pp. 42 – 47)	• Use Addition Law of Probability to solve problems involving mutually exclusive events		Investigation – Mutually exclusive and non-mutually exclusive events (p. 42)		Worked Example 8(iii) (p. 44) Practise Now 8(iii) (p. 45) Ex 2B Q 8(iii), 9(i), (iii), 11(ii) (pp. 46 – 47)
3		**2.4** Multiplication Law of Probability and independent events (pp. 47 – 59)	• Use Multiplication Law of Probability to solve problems involving independent events		Class Discussion – Choosing an appropriate diagram to represent sample space (p. 47) Thinking Time (p. 52) Investigation – Dependent events (p. 53)		Worked Example 11(ii) (p. 54) Practise Now 11 Q 1(ii), 2(ii) (pp. 54 – 55) Ex 2C Q 6(ii), 7(ii), 8(ii), 18(iv), 19(ii) (pp. 56, 59)
3		**Miscellaneous**					

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
4	**3 Statistical Data Analysis**	**3.1** Cumulative frequency table and curve (pp. 62 – 75)	• Solve problems involving cumulative frequency tables and cumulative frequency curves	Draw and interpret cumulative frequency tables and diagrams	Class Discussion – Constructing table of cumulative frequencies (pp. 62 – 63)		Practise Now 1A(iii) (p. 64) Worked Example 1(iii) (p. 65) Ex 3A Q 1(iii), 2(iii), 3(iii), 9(ii), (v)(c), 10(i)(d), (ii)(c), 11(ii), 14(ii), 15 (pp. 67 – 68, 71 – 73, 75)
5		**3.2** Median, quartiles, percentiles, range and interquartile range (pp. 76 – 94)	• Estimate the median, quartiles and percentiles from cumulative frequency curves • Calculate quartiles of a set of discrete data • Calculate range and interquartile range of a set of data	Estimate and interpret the median, percentiles, quartiles and interquartile range from cumulative frequency diagrams	Thinking Time (p. 80)		Worked Example 5(iii), (iv) (p. 87 – 88) Practise Now 5(iii), (iv) (p. 88) Ex 3B Q 6(iii), 7(iii), 8(ii), 10(iii), 11(v), 12(d), 13(iv), 15(iv) (pp. 90 – 94)
6		**3.3** Further comparison of data (pp. 95 – 100)	• Compare two sets of data using the median and the interquartile range, or the mean and the range	Compare sets of data using tables, graphs and statistical measures			Worked Example 7(ii) (pp. 96 – 97) Practise Now 7 (p. 97) Ex 3C Q 3(iii), 5(ii), (iii), 6(ii), (iii), 7(i), (ii) (pp. 98 – 100)
6		**3.4** Scatter diagrams (pp. 100 – 109)	• Solve problems involving scatter diagrams	Draw and interpret scatter diagrams Understand what is meant by positive, negative and zero correlation Draw by eye, interpret and use a straight line of best fit	Thinking Time (p. 102) Class Discussion – Scatter diagram with no correlation (p. 106) Thinking Time (p. 106) Journal Writing (p. 106)		Worked Example 8(v) (pp. 103 – 104) Practise Now 8(v) (p. 104) Worked Example 9(v) (pp. 104 – 105) Practise Now 9(v) (p. 106) Ex 3D Q 4(v), 5(v) (pp. 108 – 109)
6		**Miscellaneous**					

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
7	**4** **Vectors**	**4.1** Vectors in two dimensions (pp. 112 – 123)	• Use vector notations • Represent vectors as directed line segments • Represent vectors in column vector form	Calculate the magnitude of a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as $\sqrt{x^2+y^2}$ Represent vectors by directed line segments	Class Discussion – Equal vectors (pp. 114 – 115) Thinking Time (p. 116)		Worked Example 2(ii)(b) (pp. 117 – 118) Practise Now 2(ii)(b) (p. 119) Worked Example 3(iv) (pp. 119 – 120) Practise Now 3(iv) (p. 120) Ex 4A Q 7(ii)(b), 8(ii), 9(iv), 13(i)(a), (iii), (iv) (pp. 122 – 123)
7		**4.2** Addition of vectors (pp. 123 – 130)	• Add vectors	Add vectors	Class Discussion – The zero vector (p. 128)		
8		**4.3** Vector subtraction (pp. 130 – 141)	• Subtract vectors	Subtract vectors	Thinking Time (p. 132)		Ex 4B Q 16(ii), (iii) (p. 141)
9		**4.4** Scalar multiples of a vector (pp. 141 – 145)	• Multiply a vector by a scalar	Multiply a vector by a scalar	Thinking Time (p. 142) Class Discussion – Graphical representation of vectors (p. 145)		
9		**4.5** Expression of a vector in terms of two other vectors (pp. 145 – 147)	• Express a vector in terms of two non-zero and non-parallel coplanar vectors	Use the sum and difference of two or more vectors to express given vectors in terms of two coplanar vectors	Class Discussion – Expressing a vector in terms of two other vectors (p. 147)		
9		**4.6** Position vectors (pp. 147 – 152)	• Express a vector in terms of position vectors • Express translation by a vector	Use position vectors Describe a translation using a vector represented by $\begin{pmatrix} x \\ y \end{pmatrix}$, $\overrightarrow{AB}$ or $\mathbf{a}$			Ex 4C Q 13, 15(iii) (p. 152)

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
10		**4.7** Applications of vectors (pp. 152 – 163)	• Solve geometric problems involving the use of vectors	Use vectors to reason and to solve geometric problems	Class Discussion – Real-life applications of the resultant vector of two vectors or of the difference between two vectors (p. 153)		Worked Example 18 (p. 159) Practise Now 18 (p. 159) Ex 4D Q 18 (p. 163)
10		**Miscellaneous**					
11	**5 Relations and Functions**	**5.1** Relations (pp. 168 – 169)	• Identify a relation • Illustrate a relation using a mapping diagram	Understand functions, domain and range, and use function notation			
11		**5.2** Functions (pp. 169 – 176)	• Identify a function • Verify if a given relation is a function		Thinking Time (p. 170)		Worked Example 1 (p. 170) Practise Now 1 (p. 171) Ex 5A Q 1 (p. 175)
12		**5.3** Inverse functions (pp. 177 – 181)	• Determine the inverse of a given function	Understand and find inverse functions $f^{-1}(x)$			
12		**5.4** Composite functions (pp. 182 – 185)	• Find the composite of a function	Form composite functions as defined by $gf(x) = g(f(x))$			
12		**Miscellaneous**					
13	**6 Further Trigonometry**	**6.1** Sine and cosine of obtuse angles (pp. 188 – 196)	• Determine the trigonometric values of obtuse angles		Investigation – Relationship between trigonometric ratios of acute and obtuse angles (p. 190)		
13		**6.2** Area of triangle (pp. 196 – 201)	• Find the area of a triangle given two sides and an included angle	Use the formula area of triangle $= \frac{1}{2}ab \sin C$			

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
14		**6.3** Sine Rule (pp. 201 – 210)	• Find unknown sides and angles of a triangle, given two sides and one angle, two angles and one side, or three sides	Use the sine rule in calculations involving lengths and angles for any triangle	Investigation – Sine Rule (pp. 202 – 203) Investigation – Ambiguous case (p. 206)	Investigation – Sine Rule (pp. 202 – 203) Investigation – Ambiguous case (p. 206)	Ex 6C Q 10(i) (p. 209)
14		**6.4** Cosine Rule (pp. 210 – 217)		Use the cosine rule in calculations involving lengths and angles for any triangle	Investigation – Cosine Rule (p. 211) Thinking Time (p. 212) Thinking Time (p. 215)	Investigation – Cosine Rule (p. 211)	Ex 6D Q 15(i) (p. 217)
14		**Miscellaneous**					
15	**7 Applications of Trigonometry**	**7.1** Angles of elevation and depression (pp. 220 – 226)	• Solve problems in two and three dimensions involving angles of elevation and depression	Carry out calculations involving angles of elevation and depression	Performance Task (pp. 220 – 221)		
15		**7.2** Bearings (pp. 226 – 234)	• Solve problems in two and three dimensions involving bearings	Carry out calculations and solve problems in three dimensions using Pythagoras' theorem and trigonometry, including calculating the angle between a line and a plane			
16		**7.3** Three-dimensional problems (pp. 234 – 247)	• Solve problems in three dimensions • Find angle between a line and a plane		Investigation – Visualising right angles in 3D figures (pp. 235 – 236) Thinking Time (p. 242)	Investigation – Visualising right angles in 3D figures (pp. 235 – 236)	Thinking Time (p. 242)
16		**Miscellaneous**					
17	**8 Arc Length and Sector Area**	**8.1** Length of arc (pp. 250 – 257)	• Identify the chord, arc, segment and sector of a circle • Find the length of an arc	Use and interpret the vocabulary of a circle Carry out calculations involving arc length as fractions of the circumference of a circle	Class Discussion – Understanding the parts of a circle (pp. 250 – 251) Investigation – Discovering how to calculate arc length (pp. 251 – 252)		

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
17		**8.2** Area of sector (pp. 257 – 265)	• Find the area of a sector and the area of a segment of a circle when given an angle in degrees	Carry out calculations involving sector area as fractions of the area of a circle	Investigation – Discovering how to calculate sector area (pp. 257 – 259)	Investigation – Discovering how to calculate sector area (pp. 257 – 259)	
17		**Miscellaneous**					
18	**9 Geometrical Properties of Circles**	**9.1** Symmetric properties of circles (pp. 268 – 285)	• Apply symmetric properties of circles to solve problems	Calculate unknown angles and give explanations using the following geometrical properties of circles: • angle between tangent and radius = 90° Use the following symmetry properties of circles: • equal chords are equidistant from the centre • the perpendicular bisector of a chord passes through the centre • tangents from an external point are equal in length	Investigation – Discovering circle symmetric property 1 (pp. 268 – 270) Thinking Time (p. 271) Investigation – Discovering circle symmetric property 2 (pp. 274 – 275) Investigation – Discovering circle symmetric property 3 (pp. 276 – 277) Investigation – Discovering circle symmetric property 4 (pp. 278 – 279)	Investigation – Discovering circle symmetric property 1 (pp. 268 – 270) Investigation – Discovering circle symmetric property 2 (pp. 274 – 275) Investigation – Discovering circle symmetric property 3 (pp. 276 – 277) Investigation – Discovering circle symmetric property 4 (pp. 278 – 279)	

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
19		**9.2** Angle properties of circles (pp. 286 – 298)	• Apply angle properties of circles to solve problems	Calculate unknown angles and give explanations using the following geometrical properties of circles: • angle in a semicircle = 90° • angle at the centre is twice the angle at the circumference • angles in the same segment are equal • opposite angles of a cyclic quadrilateral sum to 180° (supplementary)	Class Discussion – Identifying angles at centre and at circumference (p. 287) Investigation – Discovering circle angle property 1 (pp. 287 – 288) Class Discussion – Finding unknown angle using circle angle property 1 (p. 288) Investigation – Discovering circle angle property 2 (pp. 289 – 290) Class Discussion – Angles in same segment or in opposite segments (p. 292) Investigation – Discovering circle angle property 3 (pp. 292 – 293) Investigation – Discovering circle angle property 4 (p. 294)	Investigation – Discovering circle angle property 1 (pp. 287 – 288) Investigation – Discovering circle angle property 2 (pp. 289 – 290) Investigation – Discovering circle angle property 3 (pp. 292 – 293) Investigation – Discovering circle angle property 4 (p. 294)	
19		**9.3** Alternate Segment Theorem (pp. 299 – 306)	• Apply Alternate Segment Theorem to solve problems	Calculate unknown angles and give explanations using the following geometrical properties of circles: • alternate segment theorem	Investigation – Alternate Segment Theorem (p. 299) Thinking Time (p. 300)	Investigation – Alternate Segment Theorem (p. 299)	Ex 9B Q 23(a), 24, 25(i), 29(i) (pp. 305 – 306)
19		**Miscellaneous**					

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
20	**10 Geometrical Transformation**	**10.1** Reflection (pp. 310 – 314)	• Reflect an object and find the line of reflection by construction	Recognise, describe and draw the reflection of a shape in a straight line	Thinking Time (p. 312)		
20		**10.2** Rotation (pp. 315 – 319)	• Rotate an object and find the centre of rotation by construction	Recognise, describe and draw the rotation of a shape about a centre through multiples of 90°	Thinking Time (p. 316)		
20		**10.3** Translation (pp. 319 – 323)	• Translate an object • Express translation by a vector	Recognise, describe and draw the translation of a shape by a vector $\begin{pmatrix} x \\ y \end{pmatrix}$	Thinking Time (p. 319) Thinking Time (p. 321) Journal Writing (p. 321)		
21		**10.4** Enlargement (pp. 323 – 332)	• Enlarge a figure, and find the centre and scale factor of enlargement	Recognise, describe and draw the enlargement of a shape from a centre by a scale factor	Class Discussion – Enlargement in our surroundings (p. 328)		
21		**10.5** Combined transformation (pp. 332 – 337)	• Find image figure of an object under a combination of transformation		Thinking Time (p. 333)		
21		**Miscellaneous**					
22	**11 Area and Volume of Similar Figures and Solids**	**11.1** Area of similar figures (pp. 340 – 348)	• Compare ratios between the lengths and areas of similar figures	Use the relationship between lengths and areas of similar shapes and lengths, surface areas and volumes of similar solids	Investigation – Areas of similar figures (pp. 340 – 341)		
22		**11.2** Volume of similar solids (pp. 348 – 353)	• Compare ratios between the lengths and volumes of similar solids		Investigation – Volume and mass of similar solids (pp. 348 – 349)		
23		**11.3** Solving problems involving similar solids (pp. 354 – 358)	• Solve real-world problems using the relationship between similar figures and solids	Solve problems and give simple explanations involving similarity	Thinking Time (p. 355)		Ex 11B Q 17(b), (c) (p. 358)
23		**Miscellaneous**					

Chapter 1 Further Sets

TEACHING NOTES

Suggested Approach

Students have learnt about the basic set notation, Venn diagrams, universal sets, complements of sets and subsets in Book 2. As such, teachers should do a revision or quick recap of these prior to this chapter.

In addition, teachers should not take an abstract approach when introducing the union and intersection of sets. Instead, teachers should try to apply the set language to describe things in daily life to arouse students' interest to learn this topic.

Section 1.1: Intersection and union of two sets

Teachers may wish to use the Chapter Opener to introduce the intersection of two sets and guide students to think how they can represent the intersection of the two sets on the Venn diagram since all the elements in a set are distinct. Students should be able to conclude that the intersection of two sets refers to the set of elements which are common to both sets, which is represented by the overlapping region of the two sets on the Venn diagram.

Teachers could use Practise Now 1, Question 2, to reinforce the meaning of subset and Practise Now 1, Question 3, to introduce disjoint sets.

Similarly, teachers are recommended to use a Venn diagram to help students visualise the union of two sets, i.e. all the elements which are in either set.

Section 1.2: Applications of sets in real-world contexts

In this section, students will consolidate all that they have learnt about sets to solve mathematical and real-life problems involving sets.

Teachers may use a step-by-step approach as shown in Worked Example 5 to identify and shade the required region. It should be reinforced that for a union, students should shade all the regions with at least one tick; for an intersection, students should shade all the regions with exactly two ticks; for a complement, students should shade all the regions without any tick.

Introductory Problem

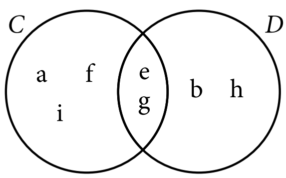
The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 7).

1.1 Intersection and union of two sets

Practise Now 1

1. (i) $\boldsymbol{C \cap D}$ = {**e, g**}

 (ii)

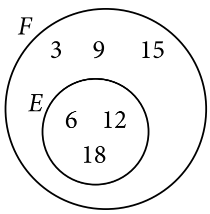


2. (i) $\boldsymbol{E}$ = {**6, 12, 18**}

 $\boldsymbol{F}$ = {**3, 6, 9, 12, 15, 18**}

 (ii) $\boldsymbol{E \cap F}$ = {**6, 12, 18**}

 (iii)



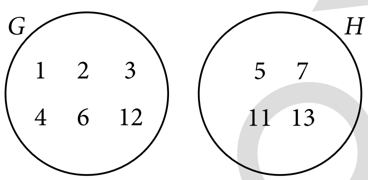
 (iv) **Yes**, $E \cap F = E$ because all the elements of E are also in F.

3. (i) $\boldsymbol{G}$ = {**1, 2, 3, 4, 6, 12**}

 $\boldsymbol{H}$ = {**5, 7, 11, 13**}

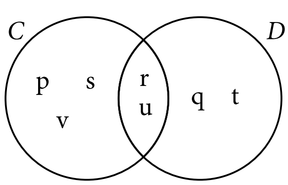
 (ii) $\boldsymbol{G \cap H = \varnothing}$ because G and H do not share any common elements.

 (iii)



Practise Now 2

1. (i)

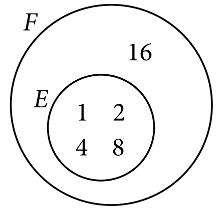


 (ii) $\boldsymbol{C \cup D}$ = {**p, q, r, s, t, u, v**}

2. (i) $\boldsymbol{E}$ = {**1, 2, 4, 8**}

 $\boldsymbol{F}$ = {**1, 2, 4, 8, 16**}

 (ii)



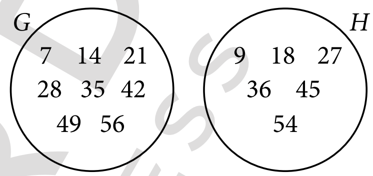
 (iii) $\boldsymbol{E \cup F}$ = {**1, 2, 4, 8, 16**}

 (iv) **Yes**, $E \cup F = F$ because all the elements of E are also in F.

3. (i) $\boldsymbol{G}$ = {**7, 14, 21, 28, 35, 42, 49, 56**}

 $\boldsymbol{H}$ = {**9, 18, 27, 36, 45, 54**}

 (ii)

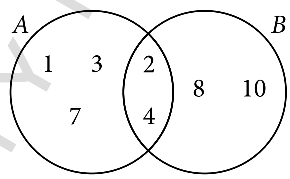


 (iii) $\boldsymbol{G \cup H}$ = {**7, 9, 14, 18, 21, 27, 28, 35, 36, 42, 45, 49, 54, 56**}

Exercise 1A

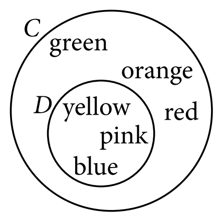
1. (i) $\boldsymbol{A \cap B}$ = {**2, 4**}

 (ii)



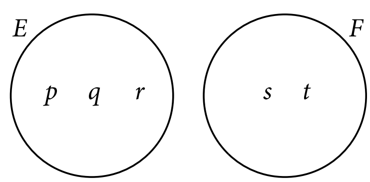
2. (i) $\boldsymbol{C \cap D}$ = {**yellow, pink, blue**}

 (ii)

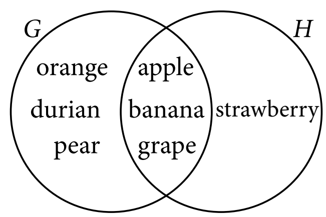


3. (i) $\boldsymbol{E \cap F = \varnothing}$

 (ii)

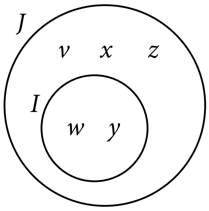


4. (i)



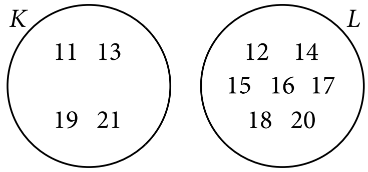
 (ii) $\boldsymbol{G \cup H}$ = {**apple, orange, banana, grape, durian, pear, strawberry**}

5. (i)



(ii) $\boldsymbol{I \cup J = \{v, w, x, y, z\}}$

6. (i)



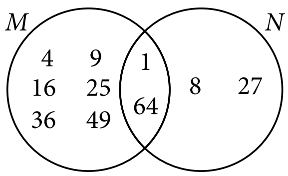
(ii) $\boldsymbol{K \cup L = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}}$

7. (i) $\boldsymbol{M = \{1, 4, 9, 16, 25, 36, 49, 64\}}$

$\boldsymbol{N = \{1, 8, 27, 64\}}$

(ii) $\boldsymbol{M \cap N = \{1, 64\}}$

(iii)

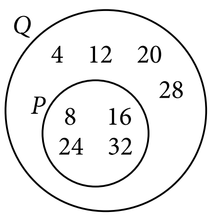


8. (i) $\boldsymbol{P = \{8, 16, 24, 32\}}$

$\boldsymbol{Q = \{4, 8, 12, 16, 20, 24, 28, 32\}}$

(ii) $\boldsymbol{P \cap Q = \{8, 16, 24, 32\}}$

(iii)



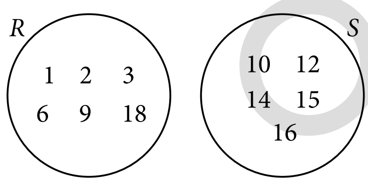
(iv) Yes, $P \cap Q = P$ because all the elements of P are also in Q.

9. (i) $\boldsymbol{R = \{1, 2, 3, 6, 9, 18\}}$

$\boldsymbol{S = \{10, 12, 14, 15, 16\}}$

(ii) $\boldsymbol{R \cap S = \varnothing}$ because R and S do not share any common elements.

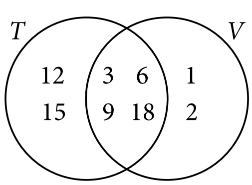
(iii)



10. (i) $\boldsymbol{T = \{3, 6, 9, 12, 15, 18\}}$

$\boldsymbol{V = \{1, 2, 3, 6, 9, 18\}}$

(ii)

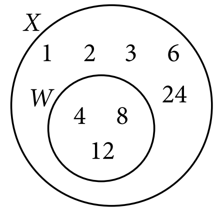


(iii) $\boldsymbol{T \cup V = \{1, 2, 3, 6, 9, 12, 15, 18\}}$

11. (i) $\boldsymbol{W = \{4, 8, 12\}}$

$\boldsymbol{X = \{1, 2, 3, 4, 6, 8, 12, 24\}}$

(ii)



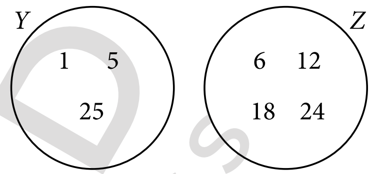
(iii) $\boldsymbol{W \cup X = \{1, 2, 3, 4, 6, 8, 12, 24\}}$

(iv) Yes, $W \cup X = X$ because all the elements of W are also in X.

12. (i) $\boldsymbol{Y = \{1, 5, 25\}}$

$\boldsymbol{Z = \{6, 12, 18, 24\}}$

(ii)



(iii) $\boldsymbol{Y \cup Z = \{1, 5, 6, 12, 18, 24, 25\}}$

13. $A \cap B = \{(x, y) : (x, y)$ are the coordinates of the points of intersection between the curve $y = x^2 - 3x + 2$ and the line $y = 0$ such that x and y are integers$\}$

14. $y = x^2 + x + 2$ — (1)

$y = 3x + 5$ — (2)

(1) = (2):

$x^2 + x + 2 = 3x + 5$

$x^2 - 2x - 3 = 0$

$(x + 1)(x - 3) = 0$

$x + 1 = 0$ or $x - 3 = 0$

$x = -1$ $\quad$ $x = 3$

Substitute into (2):

$y = 3(-1) + 5$ or $y = 3(3) + 5$

$= 2$ $\quad$ $= 14$

$\therefore$ $\boldsymbol{C \cap D = \{(-1, 2), (3, 14)\}}$

15. $E \cup F = \{y : y$ is a real number such that $y \geqslant -4\}$

1.2 Applications of sets in real-world contexts

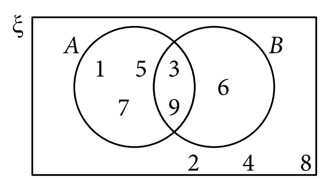
Practise Now 3

1. (i) $\boldsymbol{\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}}$

$\boldsymbol{A = \{1, 3, 5, 7, 9\}}$

$\boldsymbol{B = \{3, 6, 9\}}$

(ii)



(iii) (a) $(A \cup B)' = \{2, 4, 8\}$

(b) Method 1:

$A = \{1, 3, 5, 7, 9\}$

$B' = \{1, 2, 4, 5, 7, 8\}$

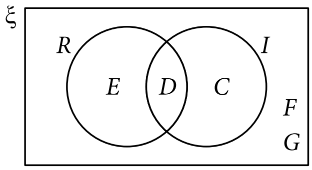
$A \cap B' = \{1, 5, 7\}$

Method 2:

From the Venn diagram,

$A \cap B' = \{1, 5, 7\}$

2.



Practise Now 4

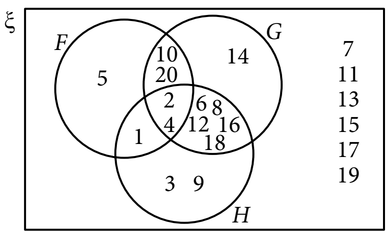
(i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

$F = \{1, 2, 4, 5, 10, 20\}$

$G = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

$H = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18\}$

(ii)

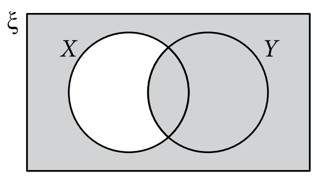


(iii) (a) $F \cap (G \cup H)' = \{5\}$

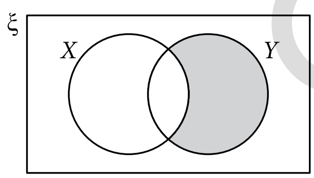
(b) $F \cup (G \cup H)' = \{1, 2, 4, 5, 7, 10, 11, 13, 15, 17, 19, 20\}$

Practise Now 5

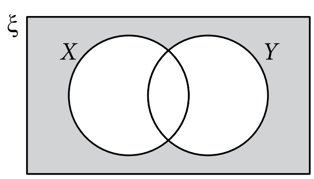
(i) $X' \cup Y$



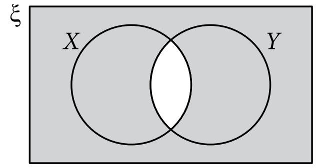
(ii) $X' \cap Y$



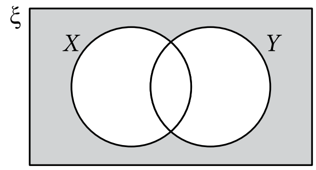
(iii) $(X \cup Y)'$



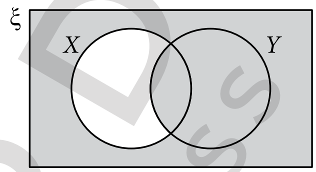
(iv) $X' \cup Y'$



(v) $X' \cap Y'$



(vi) $(X \cap Y')'$



Thinking Time (Page 11)

1. $(X \cup Y)' = X' \cap Y'$

2. $(X \cap Y)' = X' \cup Y'$

Practise Now 6

(a) $X' \cap Y$

(b) $(X' \cap Y) \cup (X \cap Y')$

(c) Y'

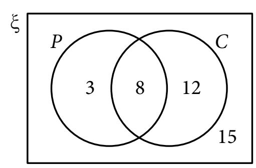
(d) $X \cup Y'$

Practise Now 7

1. Let P and C denote the set of students who study Physics and Chemistry respectively.

Number of students who study only Physics $= 11 - 8$
$= 3$

Number of students who study only Chemistry $= 20 - 8$
$= 12$

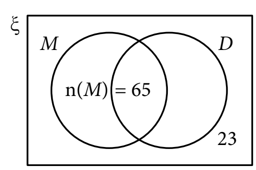


∴ number of students who study neither Physics nor Chemistry

$= 38 - 3 - 8 - 12$

$= \mathbf{15}$

2. Let M and D denote the set of participants who like movies and dramas respectively.



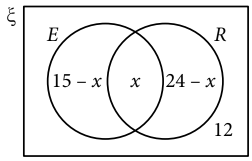
∴ number of participants who like dramas but not movies

$= 117 - 65 - 23$

$= \mathbf{29}$

Introductory Problem Revisited

Let E and R denote the set of students who listen to electronic music and rock music respectively. Let the number of students who listen to both electronic music and rock music be x.



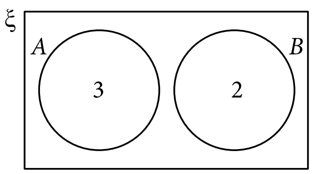
$$\begin{aligned}(15 - x) + x + (24 - x) + 12 &= 40\\ 51 - x &= 40\\ x &= 51 - 40\\ &= 11\end{aligned}$$

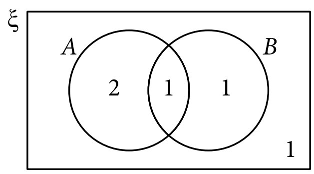
$\therefore$ number of students who listen to both genres of music is **11**.

Investigation (Exploring related sets)

Part 1

1.





2. **(i)** $n(A \cap B)$ is the largest when $\boldsymbol{B \subset A}$.
The largest value of $n(A \cap B)$ is **2**.

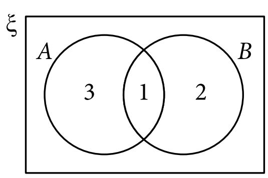
(ii) $n(A \cup B)$ is the smallest when $\boldsymbol{B \subset A}$.
The smallest value of $n(A \cup B)$ is **3**.

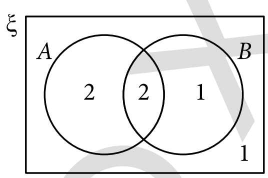
(iii) $n(A \cap B)$ is the smallest when $A \cap B = \boldsymbol{\varnothing}$.
The smallest value of $n(A \cap B)$ is **0**.

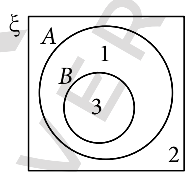
(iv) $n(A \cup B)$ is the largest when $A \cap B = \boldsymbol{\varnothing}$.
The largest value of $n(A \cup B)$ is **5**.

Part 2

3.







4. **(i)** $n(A \cap B)$ is the largest when $\boldsymbol{B \subset A}$.
The largest value of $n(A \cap B)$ is **3**.

(ii) $n(A \cup B)$ is the smallest when $\boldsymbol{B \subset A}$.
The smallest value of $n(A \cup B)$ is **4**.

(iii) $n(A \cap B)$ is the smallest when $A \cup B = \boldsymbol{\xi}$.
The smallest value of $n(A \cap B)$ is **1**.

(iv) $n(A \cup B)$ is the largest when $A \cup B = \boldsymbol{\xi}$.
The largest value of $n(A \cup B)$ is **6**.

Practise Now 8

1. **(i)** $n(A \cup B)'$ is largest when $n(A \cup B)$ is smallest.
This will occur when $B \subseteq A$.
$\therefore$ largest possible value of $n(A \cup B)' = 23 - 15$
$= \mathbf{8}$

(ii) $n(A \cup B)'$ is smallest when $n(A \cup B)$ is largest.
This will occur when $A \cup B = \xi$, i.e. $n(A \cup B) = n(\xi) = 23$.
$\therefore$ smallest possible value of $n(A \cup B)' = \mathbf{0}$

2. Let C and A denote the set of students who enjoy reading comic books and adventure story books respectively.

(i) $n(C \cup A)$ is largest when $C \cap A = \varnothing$, i.e. C and A are disjoint sets.
Largest possible value of $n(C \cup A) = 8 + 11$
$= 19$
$\therefore$ largest possible number of students who enjoy reading comic books or adventure story books is **19**.

(ii) $n(C \cup A)'$ is smallest when $n(C \cup A)$ is largest, i.e. $C \cap A = \varnothing$.
Smallest possible value of $n(C \cup A)' = 36 - 19$
$= 17$
$\therefore$ smallest possible number of students who do not enjoy reading comic books or adventure story books is **17**.

(iii) $n(C \cap A)$ is largest when $C \subset A$.
Largest possible value of $n(C \cap A) = 8$
$\therefore$ largest possible number of students who enjoy reading comic books and adventure story books is **8**.

(iv) $n(C' \cap A)$ is smallest when $n(C \cap A)$ is largest, i.e. when $C \subset A$.
Smallest possible value of $n(C' \cap A) = 11 - 8$
$= 3$
$\therefore$ smallest possible number of students who enjoy reading adventure story books but not comic books is **3**.

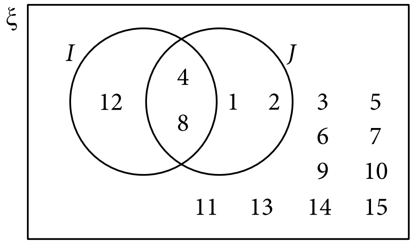
Exercise 1B

1. **(i)** ξ = **{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}**

I = **{4, 8, 12}**

J = **{1, 2, 4, 8}**

(ii)



(iii) **(a)** $(I \cup J)'$ = **{3, 5, 6, 7, 9, 10, 11, 13, 14, 15}**

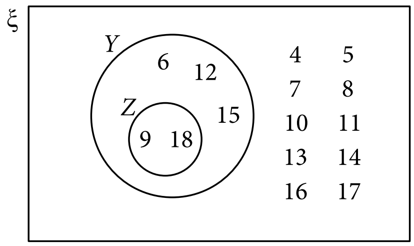
(b) $I \cap J'$ = **{12}**

2. **(i)** ξ = **{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}**

Y = **{6, 9, 12, 15, 18}**

Z = **{9, 18}**

(ii)



(iii) **(a)** $(Y \cup Z)'$ = **{4, 5, 7, 8, 10, 11, 13, 14, 16, 17}**

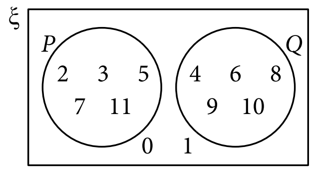
(b) $Y \cap Z'$ = **{6, 12, 15}**

3. **(i)** ξ = **{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}**

P = **{2, 3, 5, 7, 11}**

Q = **{4, 6, 8, 9, 10}**

(ii)

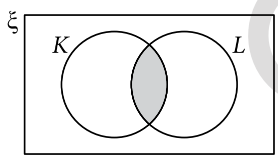


(iii) **(a)** $P \cup Q$ = **{2, 3, 4, 5, 6, 7, 8, 9, 10, 11}**

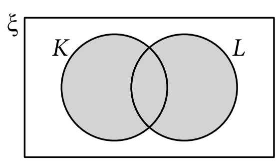
(b) $(P \cup Q)'$ = **{0, 1}**

(c) $P' \cap Q$ = **{4, 6, 8, 9, 10}**

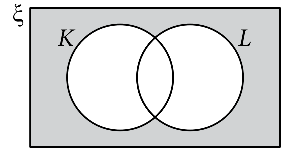
4. **(i)** $K \cap L$



(ii) $K \cup L$



(iii) $(K \cup L)'$



5. **(a)** A

(b) B

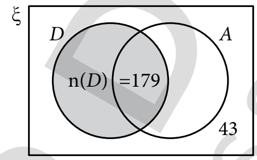
(c) $A \cap B$

(d) $A \cup B$

(e) A'

(f) B'

6. Let D and A denote the set of participants who like durian and apple respectively.

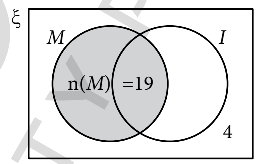


$n(D' \cap A) = 253 - 179 - 43$

$= 31$

$\therefore$ number of participants who like apple but not durian is **31**.

7. Let M and I denote the set of students who have travelled to Malaysia and Indonesia respectively.

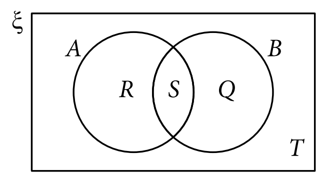


$n(M' \cap I) = 38 - 19 - 4$

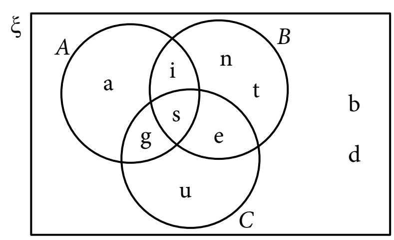
$= 15$

$\therefore$ number of participants who have travelled to Indonesia but not Malaysia is **15**.

8.



9. **(i)**



(ii) **(a)** $(A \cup B) \cap C$ = **{e, g, s}**

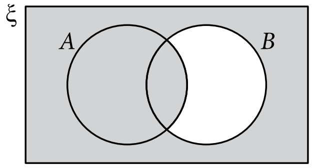
(b) $(A \cap B) \cup C$ = **{e, g, i, s, u}**

10. **(a)** **(i)** $(A \cup B) \cup C$ = **{1, 2, 3, 4, 5, 6, 7, 8}**

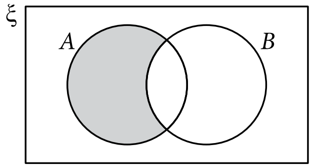
(ii) $(A \cup C) \cap B$ = **{2, 4}**

(b) $D \cap (E \cup F)$ = **{a, b, c}**

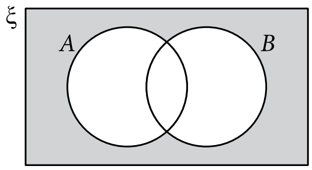
11. **(i)** $A \cup B'$



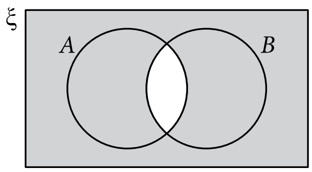
(ii) $A \cap B'$



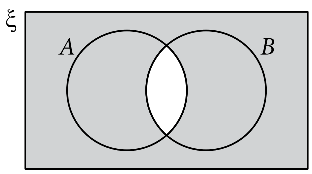
(iii) $(A \cup B)'$



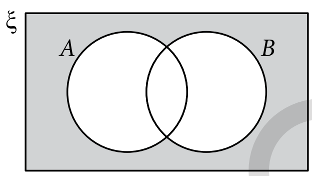
(iv) $(A \cap B)'$



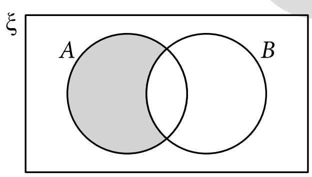
(v) $A' \cup B'$



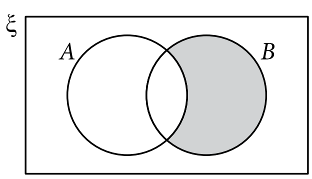
(vi) $A' \cap B'$



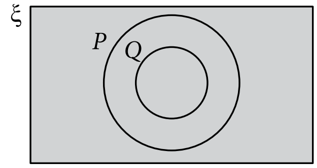
(vii) $(A' \cup B)'$



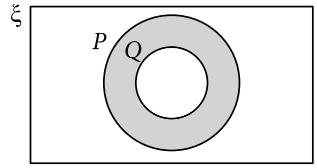
(viii) $(A \cup B')'$



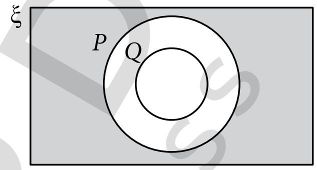
12. **(i)** $P \cup Q'$



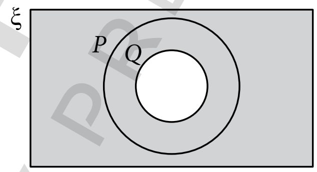
(ii) $P \cap Q'$



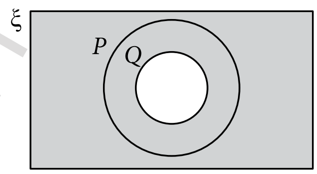
(iii) $(P \cup Q)'$



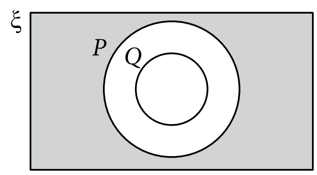
(iv) $(P \cap Q)'$



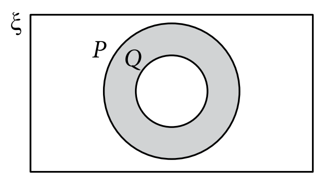
(v) $P' \cup Q'$



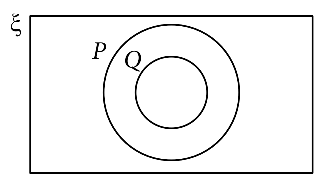
(vi) $P' \cap Q'$



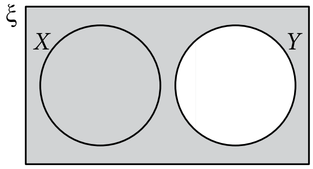
(vii) $(P' \cup Q)'$



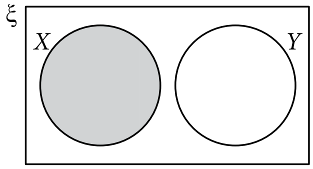
(viii) $(P \cup Q')'$



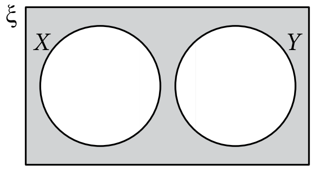
13. **(i)** $X \cup Y'$



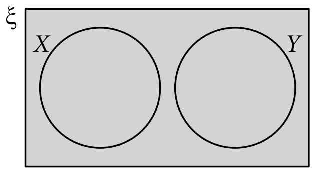
(ii) $X \cap Y'$



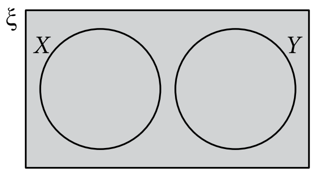
(iii) $(X \cup Y)'$



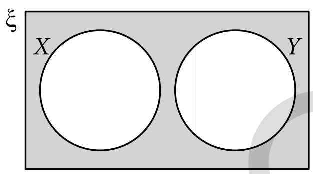
(iv) $(X \cap Y)'$



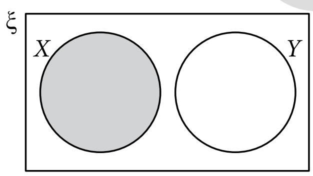
(v) $X' \cup Y'$



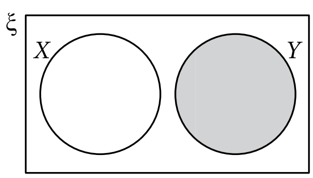
(vi) $X' \cap Y'$



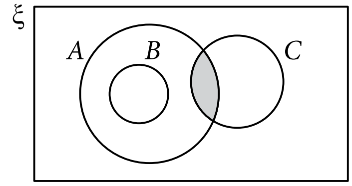
(vii) $(X' \cup Y)'$



(viii) $(X \cup Y')'$

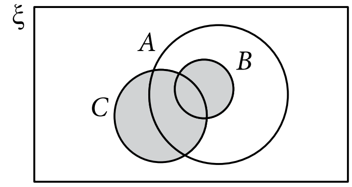


14. **(a)**



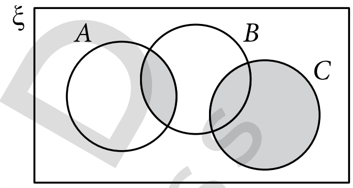
$C \cap B' \cap A$

(b)



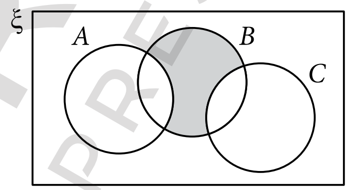
$C \cup (A \cap B)$

(c)



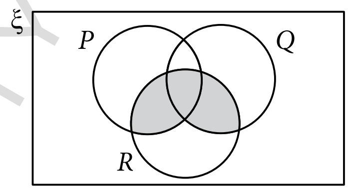
$(A \cap B) \cup C$

(d)



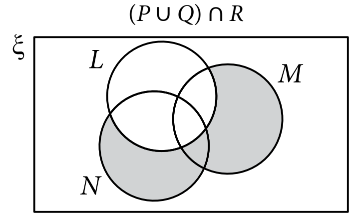
$A' \cap C' \cap B$

(e)



$(P \cup Q) \cap R$

(f)



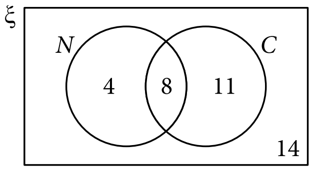
$L' \cap (M \cup N)$

15. **(a)** $X \cap Y'$
(b) $X' \cap Y$
(c) $(X \cup Y)'$
(d) $X' \cup Y'$
(e) $X \cup Y'$
(f) $X' \cup Y$
(g) $X \cap Y'$
(h) $X' \cup Y$
(i) $Y \cup Z$
(j) $X' \cap (Y \cup Z)$

16. Let N and C denote the set of students who like novels and comic books respectively.

Number of students who like only novels $= 12 - 8$
$= 4$

Number of students who like only comic books $= 19 - 8$
$= 11$



$\therefore$ number of students who like neither novels nor comic books
$= 37 - 4 - 8 - 11$
$=$ **14**

17. Number of people who own only laptops $= 23 - 9$
$= 14$

Number of people who own only desktop computers $= 17 - 9$
$= 8$

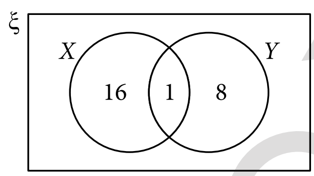
$\therefore$ number of people who own either laptops or desktop computers but not both $= 14 + 8$
$=$ **22**

18. **(i)** $n(A \cup B)'$ is largest when $n(A \cup B)$ is smallest.
This will occur when $A \subseteq B$.
$\therefore$ largest possible value of $n(A \cup B)' = 17 - 12$
$=$ **5**

(ii) $n(A \cup B)'$ is smallest when $n(A \cup B)$ is largest.
This will occur when $A \cup B = \xi$, i.e. $n(A \cup B) = n(\xi) = 17$.
$\therefore$ smallest possible value of $n(A \cup B)' =$ **0**

19. **(i)** $n(X \cap Y)'$ is smallest when $n(X \cap Y)$ is largest.
This will occur when $Y \subset X$.
$\therefore$ smallest possible value of $n(X \cap Y)' = 25 - 9$
$=$ **16**

(ii) $n(X \cap Y)'$ is largest when $n(X \cap Y)$ is smallest.
This will occur when $X \cup Y = \xi$, i.e. $n(X \cup Y) = n(\xi) = 25$.



$\therefore$ largest possible value of $n(X \cap Y)' = 16 + 8$
$=$ **24**

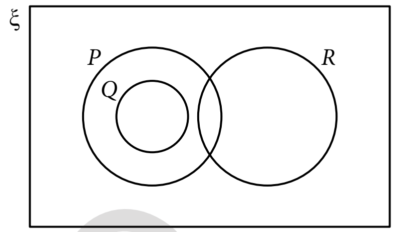
20. **(i)** $n(A \cup B)'$ is largest when $n(A \cup B)$ is smallest.
This will occur when $B \subseteq A$.
$\therefore$ largest possible value of $n(A \cup B)' = 25 - 15$
$=$ **10**

(ii) $n(A \cup B)'$ is smallest when $n(A \cup B)$ is largest.
This will occur when $A \cap B = \varnothing$, i.e. A and B are disjoint sets.
$\therefore$ smallest possible value of $n(A \cup B)' = 25 - 15 - 9$
$=$ **1**

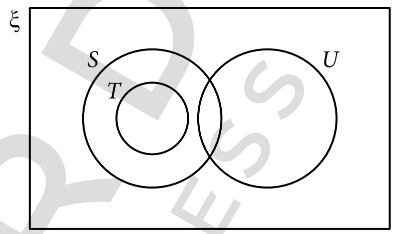
21. **(i)** $n(X \cap Y)'$ is smallest when $n(X \cap Y)$ is largest.
This will occur when $X \subset Y$.
$\therefore$ smallest possible value of $n(X \cap Y)' = 11 - 4$
$=$ **7**

(ii) $n(X \cap Y)'$ is largest when $n(X \cap Y)$ is smallest.
This will occur when $X \cap Y = \varnothing$, i.e. X and Y are disjoint sets.
$\therefore$ largest possible value of $n(X \cap Y)' = 11 - 0$
$=$ **11**

22.



23.

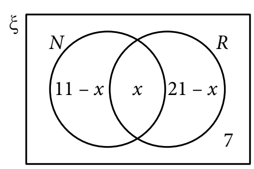


24. **(i)** $A \cap \xi = \boldsymbol{A}$
(ii) $A \cup \xi = \boldsymbol{\xi}$
(iii) $A \cap \varnothing = \boldsymbol{\varnothing}$
(iv) $A \cup \varnothing = \boldsymbol{A}$

25. **(i)** $A \cap B = \boldsymbol{A}$
(ii) $A \cup B = \boldsymbol{B}$
(iii) Not possible to simplify further.
(iv) Not possible to simplify further.
(v) $(B \cup C) \cap A = \boldsymbol{A}$
(vi) $(B \cap C) \cap A = \boldsymbol{\varnothing}$
(vii) Not possible to simplify further.
(viii) $(A \cap C) \cup B = \boldsymbol{B}$

26. **(i)** $\boldsymbol{(P \cap Q)' \cup (P' \cap Q)}$
(ii) $\boldsymbol{(P \cap Q)' \cup (P' \cap Q) \cup (P \cup Q)'}$

27. Let N and R denote the set of students who like noodles and rice respectively.
Let the number of students who like both noodles and rice be x.



$(11 - x) + x + (21 - x) + 7 = 35$
$39 - x = 35$
$x = 39 - 35$
$= 4$

$\therefore$ number of students who like both noodles and rice is **4**.

28. Let the number of adults who commute to work by both bus and train be x.

Number of adults who commute to work by only bus = $12 - x$

Number of adults who commute to work by only train = $25 - x$

Total number of adults surveyed = 40

$$(12 - x) + x + (25 - x) + 8 = 40$$
$$45 - x = 40$$
$$x = 45 - 40$$
$$= 5$$

Number of adults who commute to work by only bus = $12 - 5$
$= 7$

Number of adults who commute to work by only train = $25 - 5$
$= 20$

$\therefore$ number of adults who commute to work either by bus or by train but not both = $7 + 20$
= **27**

29. Let D, S and W denote the set of people whose hobbies are dancing, sewing and swimming respectively.

(i) $n(D \cap S \cap W') = 16 - x$

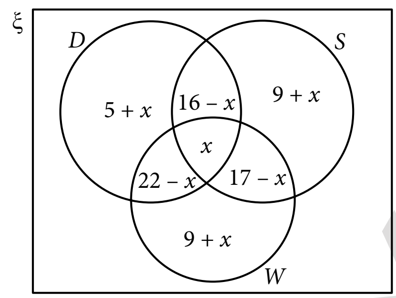
$n(S \cap W \cap D') = 17 - x$

$n(D \cap W \cap S') = 22 - x$

$n(D \cap (S \cup W)') = 43 - (16 - x) - x - (22 - x)$
$= 5 + x$

$n(S \cap (D \cup W)') = 42 - (16 - x) - x - (17 - x)$
$= 9 + x$

$n(W \cap (D \cup S)') = 48 - (22 - x) - x - (17 - x)$
$= 9 + x$



(ii) Since there are 90 people,

$$5 + x + 16 - x + 9 + x + x + 22 - x + 17 - x + 9 + x = 90$$
$$x + 78 = 90 \text{ (shown)}$$

(iii) $x + 78 = 90$

$x = \mathbf{12}$

30. Let H, K and L denote the set of people diagnosed with heart disease, kidney disease and lung disease respectively.

(i) $n(H \cap K \cap L') = 11 - x$

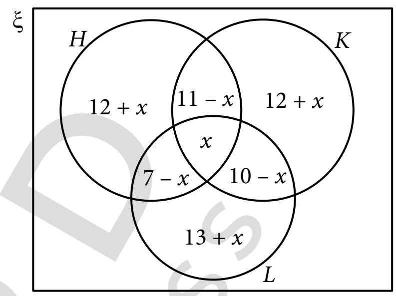
$n(K \cap L \cap H') = 10 - x$

$n(H \cap L \cap K') = 7 - x$

$n(H \cap (K \cup L)') = 30 - (11 - x) - x - (7 - x)$
$= 12 + x$

$n(K \cap (H \cup L)') = 33 - (11 - x) - x - (10 - x)$
$= 12 + x$

$n(L \cap (H \cup K)') = 30 - (7 - x) - x - (10 - x)$
$= 13 + x$



(ii) Since there are 68 men,

$$12 + x + 11 - x + 12 + x + x + 7 - x + 10 - x + 13 + x = 68$$
$$x + 65 = 68$$
$$x = \mathbf{3}$$

(iii) Number of men diagnosed with only lung disease = $13 + x$
$= 13 + 3$
= **16**

31. Let G, M and S denote the set of students who study Geography, Mathematics and Science respectively.

Let x be the number of students who study all 3 subjects.

$n(G \cap M \cap S') = 15 - x$

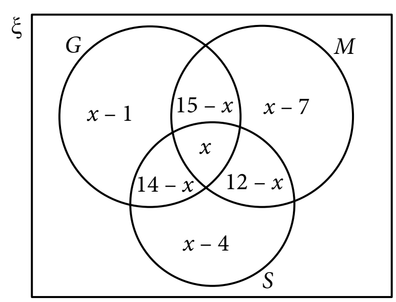
$n(M \cap S \cap G') = 12 - x$

$n(G \cap S \cap M') = 14 - x$

$n(G \cap (M \cup S)') = 28 - (15 - x) - x - (14 - x)$
$= x - 1$

$n(M \cap (G \cup S)') = 20 - (15 - x) - x - (12 - x)$
$= x - 7$

$n(S \cap (G \cup M)') = 22 - (14 - x) - x - (12 - x)$
$= x - 4$



Since there are 40 students,

$$x - 1 + 15 - x + x - 7 + x + 14 - x + 12 - x + x - 4 = 40$$
$$x + 29 = 40$$
$$x = \mathbf{11}$$

32. Let C and S denote the set of students who enjoy cycling and swimming respectively.

(i) $n(C \cup S)$ is largest when $C \cap S = \emptyset$, i.e. C and A are disjoint sets.

Largest possible value of $n(C \cup S) = 15 + 9$
$= 24$

$\therefore$ largest possible number of students who enjoy cycling or swimming is **24**.

(ii) $n(C \cup S)'$ is smallest when $n(C \cup S)$ is largest, i.e. $C \cap S = \emptyset$.

Smallest possible value of $n(C \cup S)' = 38 - 24$
$= 14$

$\therefore$ smallest possible number of students who do not enjoy cycling or swimming is **14**.

(iii) $n(C \cap S)$ is largest when $C \subset S$.

Largest possible value of $n(C \cap S) = 9$

$\therefore$ largest possible number of students who enjoy cycling and swimming is **9**.

(iv) $n(C' \cap S)$ is smallest when $n(C \cap S)$ is largest, i.e. when $C \subset S$.

Smallest possible value of $n(C' \cap S) = 9 - 9$
$= 0$

$\therefore$ smallest possible number of students who enjoy swimming but not cycling is **0**.

33. Let B and C denote the set of children who enjoy playing board games and card games respectively.

(i) $n(B \cap C)$ is smallest when $B \cup C = \xi$.

Smallest possible value of $n(B \cap C) = (13 + 28) - 35$
$= 6$

$\therefore$ smallest possible number of students who enjoy playing board games and card games is **6**.

(ii) $n(B \cup C)'$ is smallest when $B \cup C = \xi$.

Smallest possible value of $n(B \cup C)' = 35 - 35$
$= 0$

$\therefore$ smallest possible number of students who do not enjoy playing board games or card games is **0**.

(iii) $n(B \cup C)$ is smallest when $B \subset C$.

Smallest possible value of $n(B \cup C) = 28$

$\therefore$ smallest possible number of students who enjoy playing board games or card games is **28**.

(iv) $n(B' \cap C)$ is smallest when $n(B \cap C)$ is largest, i.e. when $B \subset C$.

Smallest possible value of $n(B' \cap C) = 28 - 13$
$= 15$

$\therefore$ smallest possible number of students who enjoy playing card games but not board games is **15**.

Chapter 2 Probability of Combined Events

TEACHING NOTES

Suggested Approach

As the pupils are already familiar with the concept of probability that they learnt in Book 2, it would be easier to approach this topic. However, a quick revision of what students have learnt in Book 2 is suggested.

Following the revision, teachers can get students to think about how games in sports such as tennis, football or hockey are started. Teachers can prompt students to notice that usually, a coin or something else with two sides is used, and a player from each team will choose either face, determining who has the first advantage based on the outcome of the toss. Why is the coin the norm in most cases? Why not use a die or any other objects?

Teachers can then get the whole class to throw a coin 20 times each and record the number of occurrences of heads and tails. Students can then tally the number of heads and tails to draw a distinction between the theoretical and actual probabilities occurring in an event. Teachers can urge students to think about whether the outcome of one toss affects the outcome of the next toss. Teachers may then discuss some cases where probability is useful in making real-life predictions and demonstrate why learning about the probability of combined events, and not just single events, is important in real-world contexts.

Section 2.1: Probability of single event

As the pupils are already familiar with set notations which they have already learnt in Book 2, teachers can introduce the concept of sample space and events using set notation.

Section 2.2: Probability of combined events

In this section, we introduce the sample space diagram and tree diagram when two events are taking place. Sample space diagrams and tree diagrams are useful for solving problems involving two events. Teachers may also highlight to students that tree diagrams, unlike sample space diagrams, can represent the sample space for more than two events. As some students may have difficulties understanding the use of a tree diagram, teachers are recommended to go through examples involving tree diagrams with students (see Thinking Time on page 34 of the textbook).

Section 2.3: Addition Law of Probability and mutually exclusive events

Before students learn about adding probabilities, they must learn about mutually exclusive events. However, some students may not be able to distinguish between mutually exclusive events and those that are not. As such, teachers are recommended to cite or go through examples to help students to do so (see Investigation: Mutually exclusive and non-mutually exclusive events). Teachers should lead students to conclude that mutually exclusive events cannot occur at the same time and P(A or B), which can also be written as $P(A \cup B)$, is equal to $P(A) + P(B)$.

Section 2.4: Multiplication Law of Probability and independent events

Discuss the concept of independent events and dependent events using simple everyday life examples such as the following:

(i) Throwing a coin followed by tossing a die. Will the first event affect the result of the second event?

(ii) Tossing a white die followed by tossing a red die. Will the first event affect the result of the second event?

(iii) A bag has 5 red marbles and 7 blue marbles. All the marbles are identical except for their colour. A marble is selected, its colour is noted and it is put back into the bag. A second marble is then picked and its colour noted. Will the first event affect the result of the second event?

(iv) Using the same bag of marbles in **(iii)**, a marble is selected, its colour is noted and it is put aside. A second marble is then picked and its colour is noted. Will the first event affect the result of the second event?

Teachers can give examples to help students understand dependent events (see Investigation: Dependent Events). After students have learnt about mutually exclusive events and independent events, it is common for students to mix up between both types of events. Thus, it is important for teachers to check if students are able to make a clear distinction between both types of events, and also be able to identify the type of event in various problems.

In addition, teachers can use some of the questions in Practise Now 10 and 11 to show how problems involving independent and dependent events can be solved, and teachers can get students to work with tree diagrams as well as possibility diagrams.

Introductory Problem
The solutions to this problem can be found in Exercise 2A Question 20(ii)(d).

2.1 Probability of single event

Class Discussion (Range of values of the probability of an event)
If an event is an impossible event, the probability of it occurring is 0. If it is a certain event, the probability of it occurring is 1. Since these two types of events define the limits of the range of values of the probability of an event, then all other events which are neither impossible nor certain will take on values between 0 and 1.
Hence, P(E) must lie between 0 and 1 inclusive.

Practise Now 1
(i) Let S represent the sample space. Then S = **{C, L, E_1, V, E_2, R}**.

(ii) (a) P(an 'E' is chosen) $= \frac{2}{6}$

$= \mathbf{\frac{1}{3}}$

(b) P(a 'C' is chosen) $= \mathbf{\frac{1}{6}}$

(c) P(a 'C' or an 'R' is chosen) $= \frac{1}{6} + \frac{1}{6}$

$= \frac{2}{6}$

$= \mathbf{\frac{1}{3}}$

(d) P(a 'T' is chosen) = **0**

(iii) (a) Let E be the event that the letter chosen is a vowel.
Then $E = \{E_1, E_2\}$.

$\therefore$ P(E) $= \frac{n(E)}{n(S)}$

$= \frac{2}{6}$

$= \mathbf{\frac{1}{3}}$

(b) E' is the event that the letter chosen is a consonant.

Method 1:

E' = {C, L, V, R}

P(E') $= \frac{n(E')}{n(S)}$

$= \frac{4}{6}$

$= \mathbf{\frac{2}{3}}$

Method 2:

P(E') = 1 – P(E)

$= 1 - \frac{1}{3}$

$= \mathbf{\frac{2}{3}}$

Practise Now 2
(i) Let R be the event that the point lies in the red sector.

P(R) $= \frac{\text{area of red sector}}{\text{area of circle}}$

$= \frac{\text{angle of red sector}}{\text{angle of circle}}$

$= \frac{90°}{360°}$

$= \mathbf{\frac{1}{4}}$

(ii) Method 1:

Sum of angles of all sectors that are not red = 360° – 90°
= 270°

P(R') $= \frac{\text{sum of areas of all sectors that are not red}}{\text{area of circle}}$

$= \frac{\text{sum of angles of all sectors that are not red}}{\text{angle of circle}}$

$= \frac{270°}{360°}$

$= \mathbf{\frac{3}{4}}$

Method 2:

P(R') = 1 – P(R)

$= 1 - \frac{1}{4}$

$= \mathbf{\frac{3}{4}}$

(iii) Angle of orange sector = 360° – 90° – 135° – 45° – 30°
= 60°

Let O be the event that the point lies in the orange sector.

P(O) $= \frac{\text{area of orange sector}}{\text{area of circle}}$

$= \frac{\text{angle of orange sector}}{\text{angle of circle}}$

$= \frac{60°}{360°}$

$= \mathbf{\frac{1}{6}}$

(iv) Let E be the event that the point lies in either the red or the orange sector.

Method 1:

Sum of angles of red and orange sectors = 90° + 60°
= 150°

P(E) $= \frac{\text{sum of areas of red and orange sectors}}{\text{area of circle}}$

$= \frac{\text{sum of angles of red and orange sectors}}{\text{angle of circle}}$

$= \frac{150°}{360°}$

$= \mathbf{\frac{5}{12}}$

Method 2:

P(E) = P(R) + P(O)

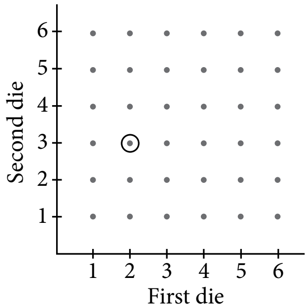
$= \frac{1}{4} + \frac{1}{6}$

$= \mathbf{\frac{5}{12}}$

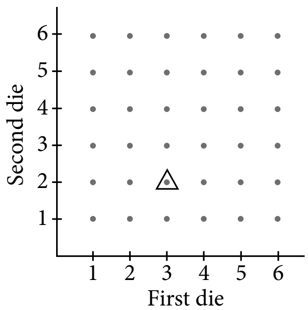
2.2 Probability of combined events

Thinking Time (Page 29)

(i)



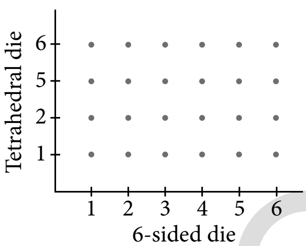
(ii)



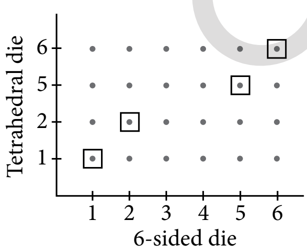
No, they represent different outcomes.
(2, 3) represents the outcome where the first die shows a '2' and the second die shows a '3'.
(3, 2) represents the outcome where the first die shows a '3' and the second die shows a '2'.

Practise Now 3

1. **(i)**



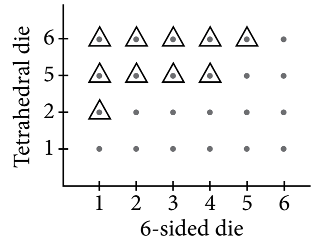
(ii) **(a)**



P(both dice show the same number) $= \frac{4}{24}$

$= \mathbf{\frac{1}{6}}$

(b)

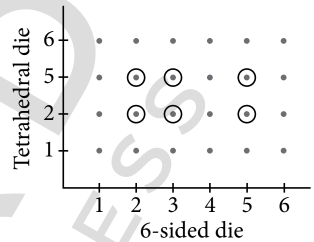


P(number shown on the tetrahedral die is greater than the number shown on the 6-sided die)

$= \frac{10}{24}$

$= \mathbf{\frac{5}{12}}$

(c)

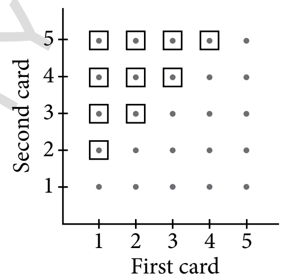


P(numbers shown on both dice are prime numbers)

$= \frac{6}{24}$

$= \mathbf{\frac{1}{4}}$

2. **(i)**

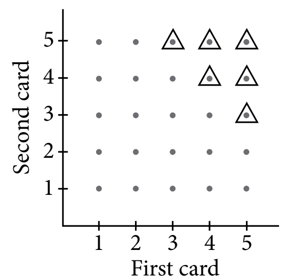


P(number shown on the second card is greater than the number shown on the first card)

$= \frac{10}{25}$

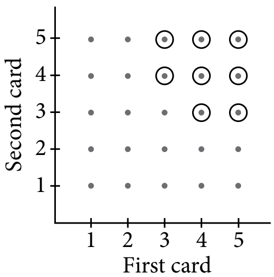
$= \mathbf{\frac{2}{5}}$

(ii)



P(sum of the two numbers shown is greater than 7) $= \mathbf{\frac{6}{25}}$

(iii)



P(product of the two numbers shown is greater than 10)

$= \mathbf{\frac{8}{25}}$

Practise Now 4

1. **(i)**

Tetrahedral die (columns); 6-sided die (rows)

+	**1**	**2**	**5**	**6**
1	**2**	3	**6**	**7**
2	**3**	**4**	**7**	**8**
3	**4**	**5**	8	**9**
4	**5**	**6**	**9**	**10**
5	**6**	**7**	**10**	**11**
6	7	**8**	**11**	**12**

Tetrahedral die (columns); 6-sided die (rows)

+	**1**	**2**	**5**	**6**
1	**1**	**2**	**5**	**6**
2	**2**	4	**10**	**12**
3	**3**	**6**	**15**	18
4	**4**	**8**	**20**	**24**
5	**5**	**10**	**25**	**30**
6	**6**	**12**	30	**36**

(ii) **(a)** P(sum is even) $= \frac{12}{24}$

$= \mathbf{\frac{1}{2}}$

(b) P(sum is divisible by 3) $= \frac{8}{24}$

$= \mathbf{\frac{1}{3}}$

(c) P(sum is a perfect square) $= \frac{4}{24}$

$= \mathbf{\frac{1}{6}}$

(d) P(sum is less than 2) = **0**

(iii) **(a)** P(product is odd) $= \frac{6}{24}$

$= \mathbf{\frac{1}{4}}$

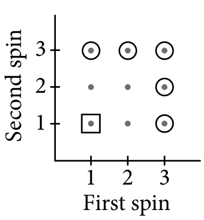
(b) P(product is larger than 12) $= \frac{8}{24}$

$= \mathbf{\frac{1}{3}}$

(c) P(product is a prime number) $= \mathbf{\frac{5}{24}}$

(d) P(product is less than 37) = **1**

2. **(i)**



(a) P(each score is a '1') $= \mathbf{\frac{1}{9}}$

(b) P(at least one score is a '3') $= \mathbf{\frac{5}{9}}$

(ii) **(a)**

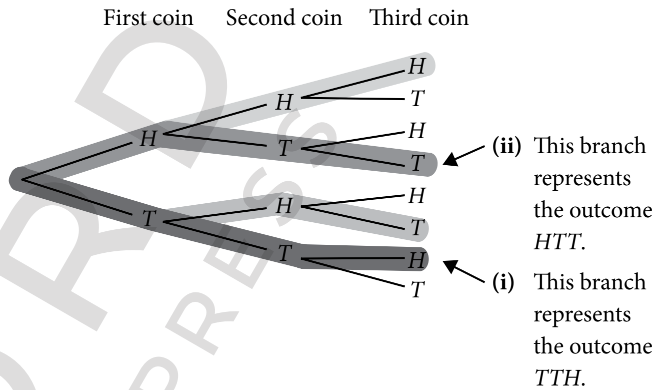
	1	**2**	**3**
1	1	**2**	**3**
2	**2**	**2**	**3**
3	**3**	3	**3**

(b) P(final score is even) $= \frac{3}{9}$

$= \mathbf{\frac{1}{3}}$

(c) P(final score is a prime number) $= \mathbf{\frac{8}{9}}$

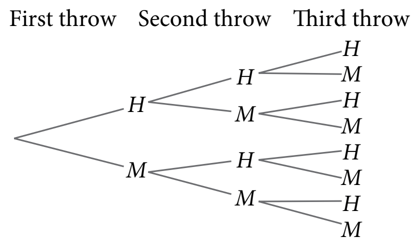
Thinking Time (Page 34)



The branches *TTH* and *HTT* do not represent the same outcome. *TTH* represents the outcome of obtaining a tail in the first and second tosses and a head in the third; *HTT* represents the outcome of obtaining a head in the first toss and a tail in the second and third tosses.

Practise Now 5

1. For each throw, Imran's possible outcomes are hitting the bullseye (*H*) or missng the bullseye (*M*).



(i) P(he misses the bullseye once) $= \mathbf{\frac{3}{8}}$

(ii) P(he hits the bullseye at least once)

= 1 – P(he misses the bullseye all three throws)

$= 1 - \frac{1}{8}$

$= \mathbf{\frac{7}{8}}$

2. **(i)**

Box A	Box B	Sum	Product
1	1	2	1
	2	3	2
2	1	3	2
	2	4	4
3	1	4	3
	2	5	6
4	1	5	4
	2	6	8

(ii) **(a)** P(at least one '1' is obtained) = $\mathbf{\frac{5}{8}}$

(b) P(sum of two numbers is 3) = $\frac{2}{8}$

$= \mathbf{\frac{1}{4}}$

(c) P(product is at least 4) = $\frac{4}{8}$

$= \mathbf{\frac{1}{2}}$

(d) P(sum is equal to the product) = $\mathbf{\frac{1}{8}}$

Practise Now 6

(a) P(selected element is an 'a') = $\mathbf{\frac{1}{9}}$

(b) P(selected element is a 'g' or a 'j') = $\mathbf{\frac{2}{9}}$

(c) Let V be the event that the selected element is a vowel.
Then V = {a, e, i}.

$\therefore P(V) = \frac{3}{9}$

$= \mathbf{\frac{1}{3}}$

(d) $P(C \cap D) = \mathbf{\frac{2}{9}}$

Exercise 2A

1. Let S represent the sample space.

(a) S = **{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}**

(b) S = **{head, tail}**

(c) $S = \{\boldsymbol{B_1, B_2, B_3, B_4, W_1, W_2}\}$

(d) $S = \{\mathbf{S_1, T_1, U, D, E, N, T_2, S_2}\}$

2. **(i)** Let S represent the sample space.
Then $S = \{\boldsymbol{G_1, G_2, Y_1, Y_2, Y_3}\}$.

(ii) **(a)** Let E be the event that the marble drawn is yellow.
Then $E = \{Y_1, Y_2, Y_3\}$.

$\therefore P(E) = \frac{n(E)}{n(S)}$

$= \mathbf{\frac{3}{5}}$

(b) E' is the event that the marble drawn is not yellow.

$P(E') = 1 - P(E)$

$= 1 - \frac{3}{5}$

$= \mathbf{\frac{2}{5}}$

(iii) **(a)** P(black) = **0**

(b) P(green or yellow) = **1**

3. **(i)** Let S represent the sample space.
Then $S = \{\mathbf{P, R, O, B_1, A, B_2, I_1, L, I_2, T, Y}\}$.

(ii) **(a)** P(a 'B') = $\mathbf{\frac{2}{11}}$

(b) P(a 'T') = $\mathbf{\frac{1}{11}}$

(c) P(an 'I' or an 'O') = $\mathbf{\frac{3}{11}}$

(d) P(an 'E') = **0**

(iii) **(a)** Let E be the event that the letter chosen is a vowel.
Then $E = \{O, A, I_1, I_2\}$.

$\therefore P(E) = \frac{n(E)}{n(S)}$

$= \mathbf{\frac{4}{11}}$

(b) E' is the event that the letter chosen is a consonant.

$P(E') = 1 - P(E)$

$= 1 - \frac{4}{11}$

$= \mathbf{\frac{7}{11}}$

4. Estimated number of pangolins living in the nature reserves
= 22 × 5
= **110**

Assumptions made:

(i) No pangolins were born and none died over the one year.

(ii) The likelihood of a tagged pangolin appearing on camera is the same as the likelihood of a non-tagged pangolin appearing on camera.

5. **(i)** Let S be the event that the student prefers soccer.

$P(S) = \frac{\text{area of red sector}}{\text{area of circle}}$

$= \frac{\text{angle of red sector}}{\text{angle of circle}}$

$= \frac{135°}{360°}$

$= \mathbf{\frac{3}{8}}$

(ii) Angle of orange sector = 360° – 135° – 90° – 45°
= 90°

Let R be the event that the student prefers reading.

$P(R) = \frac{\text{area of orange sector}}{\text{area of circle}}$

$= \frac{\text{angle of orange sector}}{\text{angle of circle}}$

$= \frac{90°}{360°}$

$= \mathbf{\frac{1}{4}}$

(iii) Sum of angles of blue and green sectors = 90° + 45°
= 135°

Let E be the event that the student prefers basketball or chess.

$P(E) = \frac{\text{sum of areas of blue and green sectors}}{\text{area of circle}}$

$= \frac{\text{sum of angles of blue and green sectors}}{\text{angle of circle}}$

$= \frac{135°}{360°}$

$= \mathbf{\frac{3}{8}}$

6. **(i)** P(region S) = $\mathbf{\frac{1}{8}}$

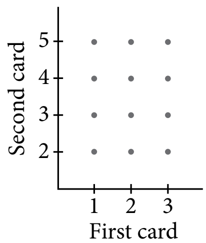
(ii) P(region Q) = $\frac{2}{8}$

$= \mathbf{\frac{1}{4}}$

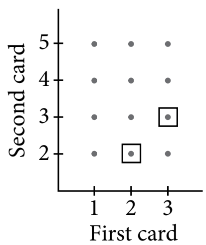
(iii) P(region P or R) = $\frac{3+2}{8}$

$= \mathbf{\frac{5}{8}}$

7. **(i)**



(ii) **(a)**



P(both show same number) = $\frac{2}{12}$

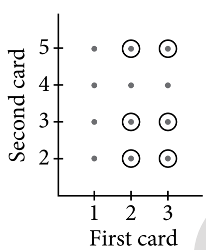
$= \mathbf{\frac{1}{6}}$

(b) P(both show different numbers) = 1 – P(both show same number)

$= 1 - \frac{1}{6}$

$= \mathbf{\frac{5}{6}}$

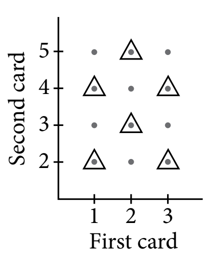
(c)



P(both show prime numbers) = $\frac{6}{12}$

$= \mathbf{\frac{1}{2}}$

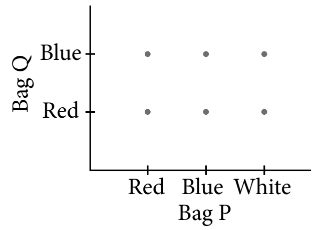
(d)



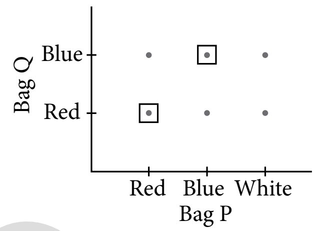
P(exactly one number is a multiple of 2) = $\frac{6}{12}$

$= \mathbf{\frac{1}{2}}$

8. **(i)**



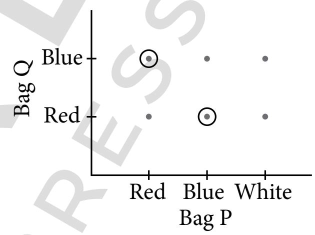
(ii) **(a)**



P(both marbles are of the same colour) = $\frac{2}{6}$

$= \mathbf{\frac{1}{3}}$

(b)



P(marbles are blue and red) = $\frac{2}{6}$

$= \mathbf{\frac{1}{3}}$

(c) P(both marbles are of different colours)

= 1 – P(both marbles are of the same colour)

$= 1 - \frac{1}{3}$

$= \mathbf{\frac{2}{3}}$

9. **(i)**

First number

Second number	+	0	1	2	3	4	5
	0	**0**	**1**	**2**	**3**	**4**	**5**
	1	1	**2**	**3**	4	**5**	**6**
	2	**2**	**3**	**4**	**5**	**6**	**7**
	3	**3**	**4**	**5**	**6**	**7**	**8**
	4	**4**	5	**6**	**7**	**8**	**9**
	5	**5**	**6**	**7**	**8**	**9**	**10**

(ii) Number of possible outcomes = **36**

(iii) **(a)** P(sum is 7) = $\frac{4}{36}$

$= \mathbf{\frac{1}{9}}$

(b) P(sum is a prime number) = $\mathbf{\frac{17}{36}}$

(c) P(sum is not a prime number) = 1 – P(sum is a prime number)

$= 1 - \frac{17}{36}$

$= \mathbf{\frac{19}{36}}$

(d) P(sum is even) = $\frac{18}{36}$

$= \mathbf{\frac{1}{2}}$

(e) P(sum is not even) = 1 – P(sum is even)

$= 1 - \frac{1}{2}$

$= \mathbf{\frac{1}{2}}$

(iv) The number of possible outcomes of obtaining a sum of 8 is 3 while for 7 is 4.
∴ the sum of **7** is more likely to occur because there are more possible outcomes and hence a greater probability of it occurring.

10. (i)

+ (x across, y down)	4	5	6
7	11	**12**	**13**
8	**12**	**13**	**14**
9	**13**	14	**15**

× (x across, y down)	4	5	6
7	**28**	**35**	42
8	**32**	40	**48**
9	**36**	**45**	**54**

(ii) (a) P(sum is a prime number) = $\mathbf{\frac{4}{9}}$

(b) P(sum is greater than 12) = $\frac{6}{9}$

$= \mathbf{\frac{2}{3}}$

(c) P(sum is at most 14) = $\mathbf{\frac{8}{9}}$

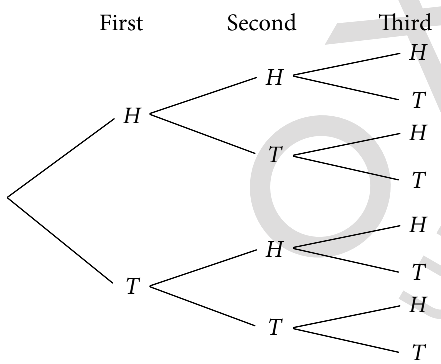
(iii) (a) P(product is odd) = $\mathbf{\frac{2}{9}}$

(b) P(product is even) = $1 - \frac{2}{9}$

$= \mathbf{\frac{7}{9}}$

(c) P(product is at most 40) = $\mathbf{\frac{5}{9}}$

11. (i) For each toss, the possible outcomes are head (H) or tail (T).



(ii) (a) P(three tails) = $\mathbf{\frac{1}{8}}$

(b) P(exactly two tails) = P(HTT, THT, TTH)

$= \mathbf{\frac{3}{8}}$

(c) P(at least two tails) = P(HTT, THT, TTH, TTT)

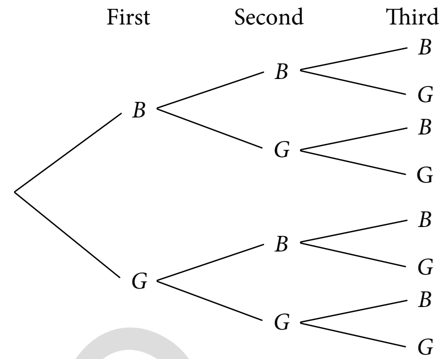
$= \frac{4}{8}$

$= \mathbf{\frac{1}{2}}$

(d) P(no tails) = P(all heads)

$= \mathbf{\frac{1}{8}}$

12. Each baby is either a boy (B) or a girl (G).



(i) P(three baby boys) = $\mathbf{\frac{1}{8}}$

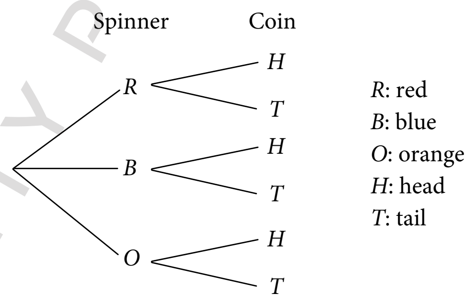
(ii) P(two baby boys and one baby girl)
= P(BBG, BGB, GBB)

$= \mathbf{\frac{3}{8}}$

(iii) P(one baby boy and two baby girls)
= P(BGG, GBG, GGB)

$= \mathbf{\frac{3}{8}}$

13. (i)



(ii) (a) P(red and tail) = $\mathbf{\frac{1}{6}}$

(b) P(blue or orange and head)
= P(blue and head or orange and head)

$= \frac{2}{6}$

$= \mathbf{\frac{1}{3}}$

14. (i) Number of teenagers = 24 + 21 + 25 + 11
= 81

P(teenager plays either cricket or soccer)

$= \frac{n(C \cup S)}{n(\xi)}$

$= \frac{24 + 21 + 25}{81}$

$= \mathbf{\frac{70}{81}}$

(ii) P(teenager only plays cricket) = $\frac{n(C \cup S')}{n(\xi)}$

$= \frac{24}{81}$

$= \mathbf{\frac{8}{27}}$

(iii) P(teenager plays cricket) $= \frac{n(C)}{n(\xi)}$

$= \frac{24 + 21}{81}$

$= \frac{45}{81}$

$= \mathbf{\frac{5}{9}}$

(iv) P(teenager does not play either sport) $= \frac{n((C \cup S)')}{n(\xi)}$

$= \mathbf{\frac{11}{81}}$

15. (i) P(Queen of diamonds) $= \mathbf{\frac{1}{52}}$

(ii) P(a black card) $= \frac{26}{52}$

$= \mathbf{\frac{1}{2}}$

(iii) P(a spade) $= \frac{13}{52}$

$= \mathbf{\frac{1}{4}}$

(iv) P(not a spade) = 1 – P(a spade)

$= 1 - \frac{1}{4}$

$= \mathbf{\frac{3}{4}}$

16. (i) Let S represent the sample space.
Then S = **{11, 14, 16, 41, 44, 46, 61, 64, 66}**.

(ii) (a) Let A be the event that the two-digit number formed is divisible by 3.
Then A = {66}.

$P(A) = \frac{n(A)}{n(S)}$

$= \mathbf{\frac{1}{9}}$

(b) Let B be the event that the two-digit number formed is a perfect square.
Then B = {16, 64}.

$P(B) = \frac{n(B)}{n(S)}$

$= \mathbf{\frac{2}{9}}$

(c) Let C be the event that the two-digit number formed is a prime number.
Then C = {11, 41, 61}.

$P(C) = \frac{n(C)}{n(S)}$

$= \frac{3}{9}$

$= \mathbf{\frac{1}{3}}$

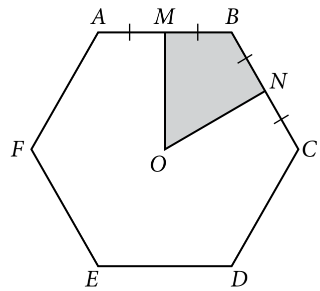
(d) Let D be the event that the two-digit number formed is a composite number.
Then D = {14, 16, 44, 46, 64, 66}.

$P(D) = \frac{n(D)}{n(S)}$

$= \frac{6}{9}$

$= \mathbf{\frac{2}{3}}$

17.



Area of MOB = area of NOB

$= \frac{1}{2} \times$ area of AOB

$= \frac{1}{2} \times (\frac{1}{6} \times$ area of hexagon)

$= \frac{1}{12} \times$ area of hexagon

P(point lies in $MBNO$)

$= \frac{\text{area of } MBNO}{\text{area of hexagon}}$

$= \frac{\text{area of } MOB + \text{area of } NOB}{\text{area of hexagon}}$

$= \frac{\frac{1}{12} \times \text{area of hexagon} + \frac{1}{12} \times \text{area of hexagon}}{\text{area of hexagon}}$

$= \frac{\frac{1}{6} \times \text{area of hexagon}}{\text{area of hexagon}}$

$= \mathbf{\frac{1}{6}}$

18. (i) (a) Total number of possible outcomes = 25

P(sum is 6)

= P(1 and 5, 2 and 4, 3 and 3, 4 and 2, 5 and 1)

$= \frac{5}{25}$

$= \mathbf{\frac{1}{5}}$

(b) P(same number) $= \frac{5}{25}$

$= \mathbf{\frac{1}{5}}$

(c) P(different numbers) = 1 – P(same number)

$= 1 - \frac{1}{5}$

$= \mathbf{\frac{4}{5}}$

(d) P(two different prime numbers)

= P(2 and 3, 2 and 5, 3 and 2, 3 and 5, 5 and 2, 5 and 3)

$= \mathbf{\frac{6}{25}}$

(ii) P(number on the first spinner is less than the number on the second spinner)

= P(1 and 2, 3, 4 or 5, 2 and 3, 4 or 5, 3 and 4 or 5, 4 and 5)

$= \frac{10}{25}$

$= \mathbf{\frac{2}{5}}$

(iii) P(larger of the 2 numbers is 3)

= P(1 and 3, 2 and 3, 3 and 1, 3 and 2)

$= \mathbf{\frac{4}{25}}$

19. (i)

Die

Coin		1	2	3	4	5	6
	H	1	2	3	4	5	6
	T	2	4	6	8	10	12

(ii) (a) P(score is odd) $= \frac{3}{12}$

$= \mathbf{\frac{1}{4}}$

(b) P(score is even) = 1 – P(score is odd)

$= 1 - \frac{1}{4}$

$= \mathbf{\frac{3}{4}}$

(c) P(score is a prime number) $= \frac{4}{12}$

$= \mathbf{\frac{1}{3}}$

(d) P(score is less than or equal to 8) $= \frac{10}{12}$

$= \mathbf{\frac{5}{6}}$

(e) P(score is a multiple of 3) $= \frac{4}{12}$

$= \mathbf{\frac{1}{3}}$

20. (i)

First die

Second die	–	1	2	3	4	5	6
	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

(ii) (a) P(difference is 1) $= \frac{10}{36}$

$= \mathbf{\frac{5}{18}}$

(b) P(difference is non-zero) = 1 – P(difference is 0)

$= 1 - \frac{6}{36}$

$= \mathbf{\frac{5}{6}}$

(c) P(difference is odd) $= \frac{18}{36}$

$= \mathbf{\frac{1}{2}}$

(d) P(difference is a prime number) $= \frac{16}{36}$

$= \mathbf{\frac{4}{9}}$

(e) P(difference is greater than 2) $= \frac{12}{36}$

$= \mathbf{\frac{1}{3}}$

21. (i)

First card	Second card	Sum	Product
1	1	2	1
	2	3	2
	7	8	7
3	1	4	3
	2	5	6
	7	10	21
5	1	6	5
	2	7	10
	7	12	35

(ii) (a) P(both numbers are odd) $= \frac{6}{9}$

$= \mathbf{\frac{2}{3}}$

(b) P(both numbers are prime numbers)

= P(3 and 2, 3 and 7, 5 and 2, 5 and 7)

$= \mathbf{\frac{4}{9}}$

(c) P(sum is greater than 4) $= \frac{6}{9}$

$= \mathbf{\frac{2}{3}}$

(d) P(sum is even) $= \frac{6}{9}$

$= \mathbf{\frac{2}{3}}$

(e) P(product is a prime number) $= \mathbf{\frac{4}{9}}$

(f) P(product is greater than 20) $= \mathbf{\frac{2}{9}}$

(g) P(product is divisible by 7) $= \frac{3}{9}$

$= \mathbf{\frac{1}{3}}$

22. (i) Let E be the event that the selected element is an even number.

Then $E = \{2, 4, 6, 8, 10\}$.

$\therefore P(E) = \frac{n(E)}{n(\xi)}$

$= \frac{5}{10}$

$= \mathbf{\frac{1}{2}}$

(ii) $P((P \cup Q)') = \frac{n((P \cup Q)')}{n(\xi)}$

$= \mathbf{\frac{3}{10}}$

(iii) Let R be the event that the selected element is a prime number.

Then $R = \{2, 3, 5, 7\}$.

$\therefore P(R) = \frac{n(R)}{n(\xi)}$

$= \frac{4}{10}$

$= \mathbf{\frac{2}{5}}$

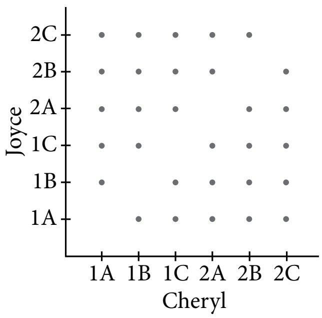
(iv) $P(R \cap Q) = \dfrac{n(R \cap Q)}{n(\xi)}$

$= \dfrac{2}{10}$

$= \mathbf{\dfrac{1}{5}}$

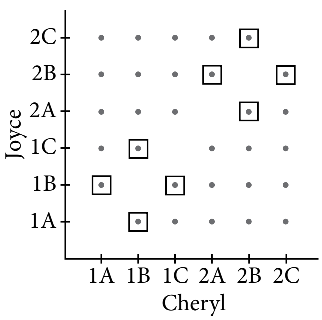
23. Let S represent the sample space.
S = {**(head, 1), (head, 2), (head, 3), (head, 4), (head, 5), (head, 6), (tail, 1), (tail, 2), (tail, 3), (tail, 4), (tail, 5), (tail, 6)**}

24. (i)



Total number of possible outcomes = **30**

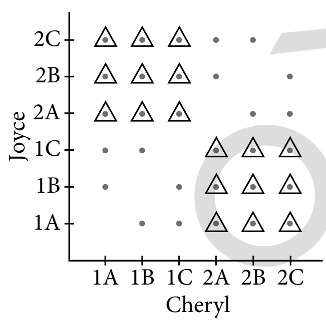
(a)



P(Cheryl and Joyce stay next to each other) $= \dfrac{8}{30}$

$= \mathbf{\dfrac{4}{15}}$

(b)



P(Cheryl and Joyce stay on different storeys) $= \dfrac{18}{30}$

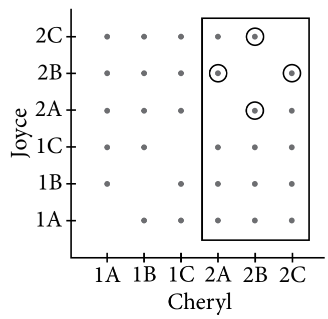
$= \mathbf{\dfrac{3}{5}}$

(c) P(Cheryl and Joyce do not stay next to each other)
= 1 – P(Cheryl and Joyce stay next to each other)

$= 1 - \dfrac{4}{15}$

$= \mathbf{\dfrac{11}{15}}$

(ii)



Since the hotel accepts Cheryl's request to allocate a second storey room to her, the total possible outcomes is now 15.

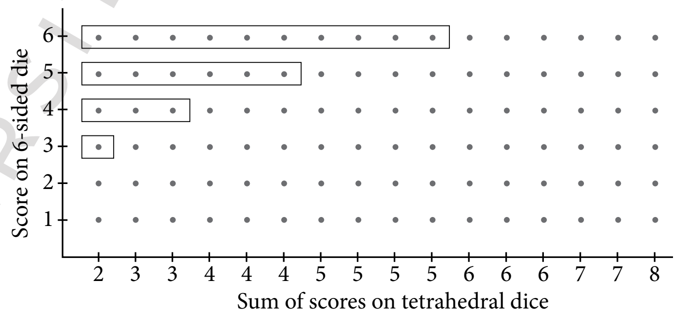
P(Cheryl stays on second storey and is beside Joyce) $= \mathbf{\dfrac{4}{15}}$

25. The possibility diagram below shows the possible sum of the scores on the two tetrahedral dice.
There are a total of 16 possible outcomes.

First tetrahedral die (columns); Second tetrahedral die (rows)

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

The possibility diagram below shows the possible outcomes of the sum of the scores on the two tetrahedral dice and the score on the 6-sided die.
There are a total of 96 possible outcomes.



P(the score on the 6-sided die is greater than the sum of the scores on the two tetrahedral dice)

$= \dfrac{20}{96}$

$= \mathbf{\dfrac{5}{24}}$

26. (i)

$$n(\xi) = 48$$
$$1 + 3 + 8 + 5 + y + 1 + 2x + 3x = 48$$
$$5x + y + 18 = 48$$
$$5x + y = 30 \quad \text{— (1)}$$

$$n(A \cap B) = n(C')$$
$$3 + y = 2x + 1 + 3 + 8$$
$$3 + y = 2x + 12$$
$$2x - y = -9 \quad \text{— (2)}$$

(1) + (2):
$$7x = 21$$
$$\therefore x = \mathbf{3}$$

Substitute $x = 3$ into (1):
$$5(3) + y = 30$$
$$15 + y = 30$$
$$\therefore y = \mathbf{15}$$

(ii) (a) $P(C) = \frac{n(C)}{n(\xi)}$
$$= \frac{5 + y + 1 + 3x}{48}$$
$$= \frac{3x + y + 6}{48}$$
$$= \frac{3(3) + 15 + 6}{48}$$
$$= \frac{30}{48}$$
$$= \mathbf{\frac{5}{8}}$$

(b) $P(A \cup B) = \frac{n(A \cup B)}{n(\xi)}$
$$= \frac{48 - 2x - 3x}{48}$$
$$= \frac{48 - 5x}{48}$$
$$= \frac{48 - 5(3)}{48}$$
$$= \frac{33}{48}$$
$$= \mathbf{\frac{11}{16}}$$

2.3 Addition Law of Probability and mutually exclusive events

Investigation (Mutually exclusive and non-mutually exclusive events)

1. Sample space = {1, 2, 3, 4, 5, 6, 7, 8}

Part 1: Mutually exclusive events

2. A = {2, 3, 5, 7}

$P(A) = \frac{4}{8}$
$= \mathbf{\frac{1}{2}}$

3. B = {4, 8}

$P(B) = \frac{2}{8}$
$= \mathbf{\frac{1}{4}}$

4. No

5. $A \cup B$ = {2, 3, 4, 5, 7, 8}

$P(A \cup B) = \frac{6}{8}$
$= \mathbf{\frac{3}{4}}$

6. $P(A) + P(B) = \frac{1}{2} + \frac{1}{4}$
$= \frac{3}{4}$
$= P(A \cup B)$

Yes, $P(A) + P(B) = P(A \cup B)$. The events A and B are mutually exclusive.

Part 2: Non-mutually exclusive events

7. C = {1, 3, 5, 7}

$P(C) = \frac{4}{8}$
$= \mathbf{\frac{1}{2}}$

8. Yes

9. $A \cup C$ = {1, 2, 3, 5, 7}

$P(A \cup C) = \mathbf{\frac{5}{8}}$

10. $P(A) + P(C) = \frac{1}{2} + \frac{1}{2}$
$= 1$
$\neq P(A \cup C)$

No, $P(A \cup C) \neq P(A) + P(C)$. The events A and C are not mutually exclusive.

Practise Now 7

(i) P(drawing a '7') $= \frac{4}{52}$
$= \mathbf{\frac{1}{13}}$

(ii) Number of picture cards in the pack $= 3 \times 4$
$= 12$

P(drawing a picture card) $= \frac{12}{52}$
$= \mathbf{\frac{3}{13}}$

(iii) Method 1:

P(drawing a '7' or a picture card) = P('7') + P(picture)
$= \frac{1}{13} + \frac{3}{13}$
$= \mathbf{\frac{4}{13}}$

Method 2:

Number of '7' in the pack = 4
Number of picture cards in the pack = 12

P(drawing a '7' or a picture card) $= \frac{16}{52}$
$= \mathbf{\frac{4}{13}}$

(iv) Method 1:

P(drawing neither a '7' nor a picture card)

= 1 – P(drawing a '7' or a picture card)

$= 1 - \frac{4}{13}$

$= \mathbf{\frac{9}{13}}$

Method 2:

Number of cards in the pack that are neither a '7' nor a picture card

= 52 – 4 – 12

= 36

P(drawing neither a '7' nor a picture card) $= \frac{36}{52}$

$= \mathbf{\frac{9}{13}}$

(v) P(drawing a diamond) $= \frac{13}{52}$

$= \mathbf{\frac{1}{4}}$

(vi) Number of '7' in the pack = 4

Number of diamonds in the pack = 13

P(drawing a '7' or a diamond) $= \frac{4 + 13 - 1}{52}$

$= \frac{16}{52}$

$= \mathbf{\frac{4}{13}}$

Practise Now 8

(i) Since only one team can win, the events of each of the teams P, Q, R and S winning are mutually exclusive.

But P(P wins) + P(Q wins) + P(R wins) + P(S wins)

$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840} < 1.$

Therefore, there are more than four teams in the competition.

(ii) (a) P(either P or Q wins) $= \frac{1}{5} + \frac{1}{6}$

$= \mathbf{\frac{11}{30}}$

(b) P(Q, R or S wins) $= \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$= \mathbf{\frac{73}{168}}$

(c) P(Q does not win) = 1 – P(Q wins)

$= 1 - \frac{1}{6}$

$= \mathbf{\frac{5}{6}}$

(d) P(Q, R or S do not win) = 1 – P(Q, R or S wins)

$= 1 - \frac{73}{168}$

$= \mathbf{\frac{95}{168}}$

(iii) The events of each team Q, R or S not winning are not mutually exclusive since there will be more than one team that will not win. This means that the event 'Q does not win' does not exclude the events 'R does not win' and 'S does not win'. Similarly, the event 'R does not win' does not exclude the events 'Q does not win' and 'S does not win', and the event 'S does not win' does not exclude the events 'Q does not win' and 'R does not win'.

Exercise 2B

1. (i) P(number is even) $= \mathbf{\frac{5}{11}}$

(ii) P(number is prime) $= \mathbf{\frac{4}{11}}$

(iii) P(number is either even or prime) $= \frac{5}{11} + \frac{4}{11}$

$= \mathbf{\frac{9}{11}}$

(iv) P(number is neither even nor prime)

= 1 – P(number is either even or prime)

$= 1 - \frac{9}{11}$

$= \mathbf{\frac{2}{11}}$

2. (i) P(red) $= \mathbf{\frac{7}{15}}$

(ii) P(green) $= \frac{5}{15}$

$= \mathbf{\frac{1}{3}}$

(iii) P(either red or green) $= \frac{7}{15} + \frac{5}{15}$

$= \frac{12}{15}$

$= \mathbf{\frac{4}{5}}$

(iv) P(neither red nor green) = 1 – P(either red or green)

$= 1 - \frac{4}{5}$

$= \mathbf{\frac{1}{5}}$

3. (i) P(team wins or loses a match) $= \frac{7}{10} + \frac{2}{15}$

$= \mathbf{\frac{5}{6}}$

(ii) P(team neither wins nor loses a match)

= 1 – P(team wins or loses a match)

$= 1 - \frac{5}{6}$

$= \mathbf{\frac{1}{6}}$

4. (i) P(C does not win) = 1 – P(C wins)

$= 1 - \frac{3}{7}$

$= \mathbf{\frac{4}{7}}$

(ii) P(either A or B wins) $= \frac{1}{14} + \frac{2}{7}$

$= \mathbf{\frac{5}{14}}$

(iii) P(none of these three teams win)

= 1 – P(either one of these three teams win)

$= 1 - \left(\frac{1}{14} + \frac{2}{7} + \frac{3}{7}\right)$

$= 1 - \frac{11}{14}$

$= \mathbf{\frac{3}{14}}$

5. (i) P(either one will win) $= \frac{1}{3} + \frac{1}{8} + \frac{1}{20}$

$= \mathbf{\frac{61}{120}}$

(ii) P(none of them will win) = 1 – P(either one will win)

$= 1 - \frac{61}{120}$

$= \mathbf{\frac{59}{120}}$

(iii) P(Vasi and Waseem will not win)
= 1 – P(Vasi or Waseem wins)
$= 1 - \left(\frac{1}{3} + \frac{1}{8}\right)$
$= 1 - \frac{11}{24}$
$= \mathbf{\frac{13}{24}}$

6. (i) Number of 'U' in the box = 3
P(a 'U') = $\mathbf{\frac{3}{17}}$

(ii) Number of 'E' in the box = 2
P(an 'E') = $\mathbf{\frac{2}{17}}$

(iii) P(a 'U' or an 'E') = P(a 'U') + P(an 'E')
$= \frac{3}{17} + \frac{2}{17}$
$= \mathbf{\frac{5}{17}}$

(iv) Number of consonants in the box = 10
P(a consonant) = $\mathbf{\frac{10}{17}}$

(v) P(a 'U' or a consonant) = P(a 'U') + P(a consonant)
$= \frac{3}{17} + \frac{10}{17}$
$= \mathbf{\frac{13}{17}}$

(vi) Number of 'U' and vowels in the box = 7
P(a 'U' or a vowel) = $\mathbf{\frac{7}{17}}$

7. (i) Number of Kings and Jacks in a pack = 8
P(King or Jack) = $\frac{8}{52}$
$= \mathbf{\frac{2}{13}}$

(ii) P(neither King nor Jack) = 1 – P(King or Jack)
$= 1 - \frac{2}{13}$
$= \mathbf{\frac{11}{13}}$

(iii) Number of Queens in a pack = 4
Number of cards bearing a prime number in a pack = 4 × 4
= 16
P(Queen or prime number) = P(Queen) + P(prime number)
$= \frac{4}{52} + \frac{16}{52}$
$= \mathbf{\frac{5}{13}}$

(iv) Number of cards bearing numbers divisible by 3 = 3 × 4
= 12
Number of cards bearing numbers divisible by 5 = 2 × 4
= 8
P(number divisible by 3 or 5) = $\frac{20}{52}$
$= \mathbf{\frac{5}{13}}$

(v) Number of cards bearing numbers divisible by 2 or 3
= 7 × 4
= 28
P(number divisible by 2 or 3) = $\frac{28}{52}$
$= \mathbf{\frac{7}{13}}$

8. (i) Since only one team can win, the events of each of the teams Alpha, Beta and Gamma winning are mutually exclusive.
But P(Alpha wins) + P(Beta wins) + P(Gamma wins)
$= \frac{4}{15} + \frac{1}{10} + \frac{1}{5} = \frac{17}{30} < 1.$
Therefore, there are more than three teams in the competition.

(ii) (a) P(either Alpha or Gamma wins)
= P(Alpha wins) + P(Gamma wins)
$= \frac{4}{15} + \frac{1}{5}$
$= \mathbf{\frac{7}{15}}$

(b) P(either one will win) = P(Alpha wins) + P(Beta wins) + P(Gamma wins)
$= \frac{4}{15} + \frac{1}{10} + \frac{1}{5}$
$= \mathbf{\frac{17}{30}}$

(c) P(neither Alpha nor Gamma will win)
= 1 – P(either Alpha or Gamma wins)
$= 1 - \frac{7}{15}$
$= \mathbf{\frac{8}{15}}$

(d) P(none of these three teams will win)
= 1 – P(either one will win)
$= 1 - \frac{17}{30}$
$= \mathbf{\frac{13}{30}}$

(iii) The events of Alpha or Gamma not winning are not mutually exclusive since there will be more than one team that will not win. This means that the event 'Alpha does not win' does not exclude the event 'Gamma does not win'. Similarly, the event 'Gamma does not win' does not exclude the event 'Alpha does not win'.

9. (i) Since only one team can win, the events of each of the teams E, F, G and H winning are mutually exclusive.
But P(E wins) + P(F wins) + P(G wins) + P(H wins)
$= \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} = \frac{886}{1155} < 1.$
Therefore, there are more than four teams in the competition.

(ii) (a) P(either E or F wins) = P(E wins) + P(F wins)
$= \frac{1}{3} + \frac{1}{5}$
$= \mathbf{\frac{8}{15}}$

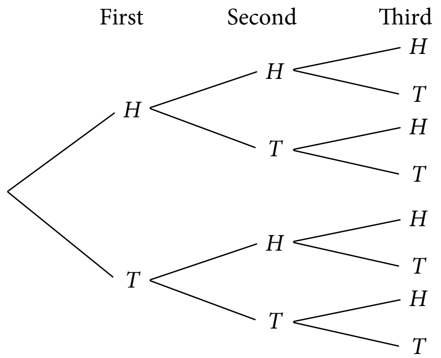
(b) P(F, G or H wins)
= P(F wins) + P(G wins) + P(H wins)
$= \frac{1}{5} + \frac{1}{7} + \frac{1}{11}$
$= \mathbf{\frac{167}{385}}$

(c) P(F does not win) = 1 – P(F wins)
$= 1 - \frac{1}{5}$
$= \mathbf{\frac{4}{5}}$

(d) P(F, G or H do not win) = 1 – P(F, G or H wins)
$= 1 - \frac{167}{385}$
$= \mathbf{\frac{218}{385}}$

(iii) The events of each team F, G or H not winning are not mutually exclusive since there will be more than one team that will not win. This means that the event 'F does not win' does not exclude the events 'G does not win' and 'H does not win'. Similarly, the event 'G does not win' does not exclude the events 'F does not win' and 'H does not win', and the event 'H does not win' does not exclude the events 'F does not win' and 'G does not win'.

10. (i) The possible outcomes of each toss are head (H) or tail (T).



(ii) (a) **Yes**, A and B are mutually exclusive.

(b) **No**, C and D are not mutually exclusive. The outcome THT is favourable to both events.

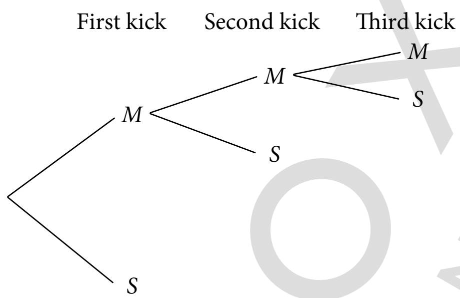
(c) **No**, B and C are not mutually exclusive. The outcomes HTT, THT and TTH are favourable to both events.

(d) **Yes**, A and C are mutually exclusive.

(e) **No**, B and D are not mutually exclusive. The outcome THT is favourable to both events.

(f) **No**, A, B and C are not mutually exclusive. The outcomes HTT, THT and TTH are favourable to B and C.

11. (i) Yasir's possible outcomes are a score (S) or a miss (M).



Total number of outcomes = **4**

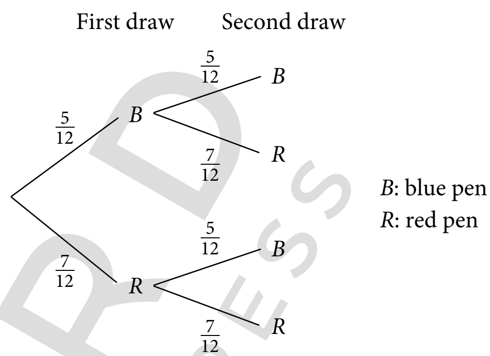
(ii) **No**, A and B are not mutually exclusive. The outcome where the first kick is a miss and the second is a score is favourable to both events.

2.4 Multiplication Law of Probability and independent events

Class Discussion (Choosing an appropriate diagram to represent sample space)

2. It is tedious because there are many possible outcomes and as a result many outcomes to represent in the possibility diagram and in the tree diagram.

Practise Now 9



(i) P(first pen is R) = $\mathbf{\frac{7}{12}}$

(ii) P(second pen is B, given that first pen is R) = $\mathbf{\frac{5}{12}}$

(iii) P(first pen is R and second pen is B) $= \frac{7}{12} \times \frac{5}{12}$

$= \mathbf{\frac{35}{144}}$

(iv) P(second pen is B) = P(BB) + P(RB)

$= \left(\frac{5}{12} \times \frac{5}{12}\right) + \left(\frac{7}{12} \times \frac{5}{12}\right)$

$= \frac{25}{144} + \frac{35}{144}$

$= \mathbf{\frac{5}{12}}$

(v) P(one pen is R and the other is B) = P(BR) + P(RB)

$= \left(\frac{5}{12} \times \frac{7}{12}\right) + \left(\frac{7}{12} \times \frac{5}{12}\right)$

$= \frac{35}{144} + \frac{35}{144}$

$= \mathbf{\frac{35}{72}}$

Practise Now 10

(i) P(chairman is from the 'Technical' department) $= \frac{12}{18}$

$= \frac{2}{3}$

P(chairwoman is from the 'Technical' department) $= \frac{4}{12}$

$= \frac{1}{3}$

P(both chairman and chairwoman are from the 'Technical' department)

$= \frac{2}{3} \times \frac{1}{3}$

$= \mathbf{\frac{2}{9}}$

(ii) P(chairman is from the 'Administrative' department and chairwoman is from the 'Technical' department)

$= \left(1 - \frac{2}{3}\right) \times \frac{1}{3}$

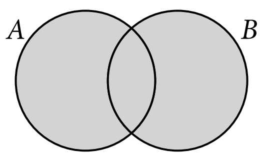
$= \frac{1}{3} \times \frac{1}{3}$

$= \mathbf{\frac{1}{9}}$

Thinking Time (Page 52)

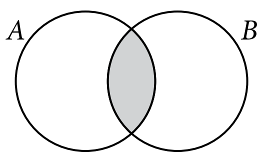
P($A \cup B$) refers to the probability that either event A or event B will occur.

It can be represented by the Venn diagram shown below:



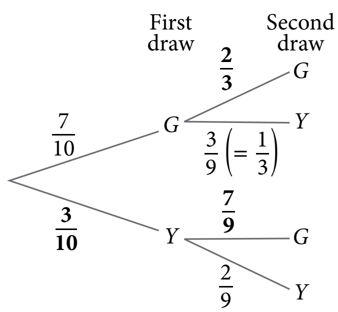
P($A \cap B$) refers to the probability that both events A and B will occur.

It can be represented by the Venn diagram shown below:



Investigation (Dependent events)

1.



G: green marble
Y: yellow marble

2. **(i)** P(second marble is Y, given that first marble is G) = $\mathbf{\frac{1}{3}}$

(ii) P(second marble is Y, given that first marble is Y) = $\mathbf{\frac{2}{9}}$

3. **No**, the probabilities are different.

Yes, the probability of drawing a yellow marble in the second draw depends on the outcome of the first draw. Since the marbles are not replaced, the number of yellow marbles available in the bag in the second draw will vary depending on whether the first draw was of a yellow marble or not. The total number of marbles available in the bag in the second draw will also decrease because of the first draw.

4. P(second marble is Y) = P(GY) + P(YY)

$= \left(\frac{7}{10} \times \frac{1}{3}\right) + \left(\frac{3}{10} \times \frac{2}{9}\right)$

$= \frac{7}{30} + \frac{1}{15}$

$= \mathbf{\frac{3}{10}}$

No, it is not equal to the probabilities in Questions **2(i)** and **(ii)**.

5. **(i)** The probability of event B happening, i.e. P(second marble is Y), depends on the probability of two cases happening, P(GY) and P(YY). Since both the cases of drawing a green marble in the first draw and drawing a yellow marble in the first draw are dependent on the probability of event A happening, then event B is **dependent** on event A.

(ii) **No**.

P(A and B) = P(GY)

$= \frac{7}{10} \times \frac{1}{3}$

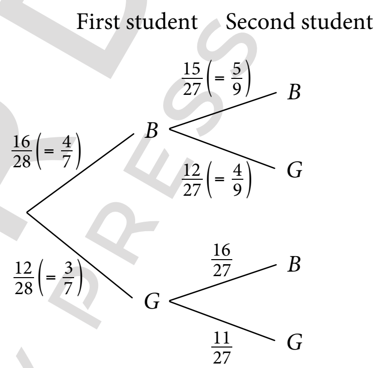
$= \frac{7}{30}$

P(A) × P(B) $= \frac{7}{10} \times \frac{3}{10}$

$= \frac{21}{100}$

Practise Now 11

1. **(i)**



B: boy
G: girl

(ii) Bernard calculated the probability as $\frac{12}{28} \times \frac{12}{28} = \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$.

He did not take into account that there are only 11 girls and 27 students left after the first student was selected.

(iii) **(a)** P(first is B and second is G) = P(BG)

$= \frac{4}{7} \times \frac{4}{9}$

$= \mathbf{\frac{16}{63}}$

(b) P(one is B and one is G) = P(BG) + P(GB)

$= \left(\frac{4}{7} \times \frac{4}{9}\right) + \left(\frac{3}{7} \times \frac{16}{27}\right)$

$= \frac{16}{63} + \frac{16}{63}$

$= \mathbf{\frac{32}{63}}$

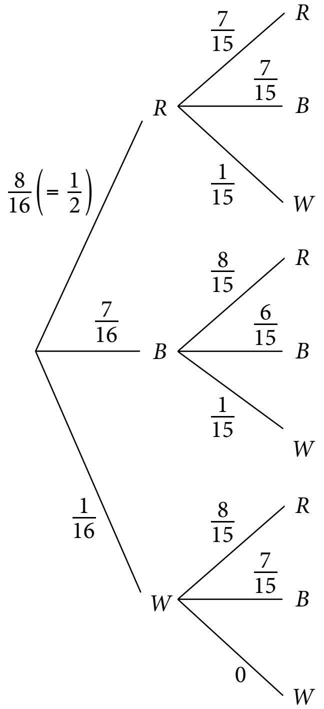
(c) P(at least one G) = 1 − P(both boys)

$= 1 - \left(\frac{4}{7} \times \frac{5}{9}\right)$

$= 1 - \frac{20}{63}$

$= \mathbf{\frac{43}{63}}$

2. **(i)**



R: red ball
B: blue ball
W: white ball

(ii) Cheryl calculated the probability as $\frac{8}{16} \times \frac{7}{15} = \frac{1}{2} \times \frac{7}{15} = \mathbf{\frac{7}{30}}$. She did not take into account that there are two possibilities. The first case is that the first ball is red and the second is blue, and the second case is that the first ball is blue and the second is red.

(iii) **(a)** P(first is *R* and second is *B*) = P(*RB*)

$$= \frac{1}{2} \times \frac{7}{15}$$
$$= \mathbf{\frac{7}{30}}$$

(b) P(one is *R* and one is *B*) = P(*RB*) + P(*BR*)

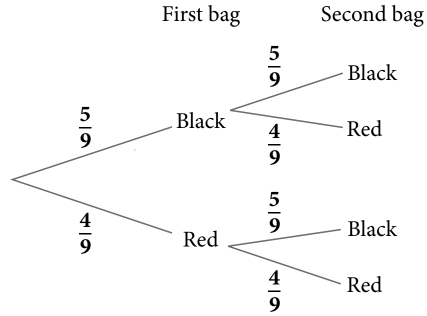
$$= \left(\frac{1}{2} \times \frac{7}{15}\right) + \left(\frac{7}{16} \times \frac{8}{15}\right)$$
$$= \frac{7}{30} + \frac{7}{30}$$
$$= \mathbf{\frac{7}{15}}$$

(c) P(both same colour) = P(*RR*) + P(*BB*)

$$= \left(\frac{1}{2} \times \frac{7}{15}\right) + \left(\frac{7}{16} \times \frac{6}{15}\right)$$
$$= \frac{7}{30} + \frac{7}{40}$$
$$= \mathbf{\frac{49}{120}}$$

Exercise 2C

1. **(i)**



(ii) **(a)** P(black from first) = $\mathbf{\frac{5}{9}}$

(b) P(red from second, given that black from first) = $\mathbf{\frac{4}{9}}$

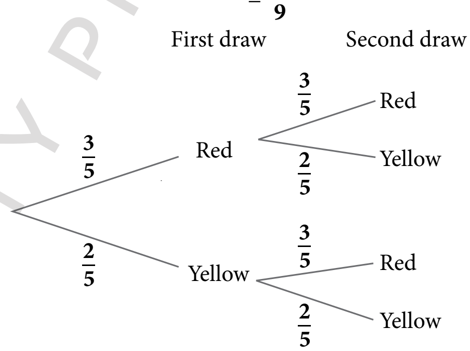
(c) P(black from first and red from second) = $\frac{5}{9} \times \frac{4}{9}$

$$= \mathbf{\frac{20}{81}}$$

(d) P(red from second) = P(black then red) + P(both red)

$$= \left(\frac{5}{9} \times \frac{4}{9}\right) + \left(\frac{4}{9} \times \frac{4}{9}\right)$$
$$= \frac{20}{81} + \frac{16}{81}$$
$$= \mathbf{\frac{4}{9}}$$

2. **(i)**



(ii) **(a)** P(two red balls) = $\frac{3}{5} \times \frac{3}{5}$

$$= \mathbf{\frac{9}{25}}$$

(b) P(one of each colour)

= P(red then yellow) + P(yellow then red)

$$= \left(\frac{3}{5} \times \frac{2}{5}\right) + \left(\frac{2}{5} \times \frac{3}{5}\right)$$
$$= \frac{6}{25} + \frac{6}{25}$$
$$= \mathbf{\frac{12}{25}}$$

(c) P(second is yellow)

= P(red then yellow) + P(both yellow)

$$= \left(\frac{3}{5} \times \frac{2}{5}\right) + \left(\frac{2}{5} \times \frac{2}{5}\right)$$
$$= \frac{6}{25} + \frac{4}{25}$$
$$= \frac{10}{25}$$
$$= \mathbf{\frac{2}{5}}$$

3. **(i)** P(both are punctual) $= \frac{2}{3} \times \frac{7}{8}$

$= \mathbf{\frac{7}{12}}$

(ii) P(Bus A is late and Bus B is punctual) $= \frac{1}{3} \times \frac{7}{8}$

$= \mathbf{\frac{7}{24}}$

(iii) P(exactly one bus is late)

= P(Bus A is late and Bus B is punctual)

+ P(Bus A is punctual and Bus B is late)

$= \left(\frac{1}{3} \times \frac{7}{8}\right) + \left(\frac{2}{3} \times \frac{1}{8}\right)$

$= \frac{7}{24} + \frac{1}{12}$

$= \frac{9}{24}$

$= \mathbf{\frac{3}{8}}$

4. **(i)** P(both break down) = 0.1 × 0.35

= **0.035**

(ii) P(X breaks down but Y does not) = 0.1 × (1 – 0.35)

= 0.1 × 0.65

= **0.065**

(iii) P(exactly one breaks down)

= P(X breaks down and Y does not)

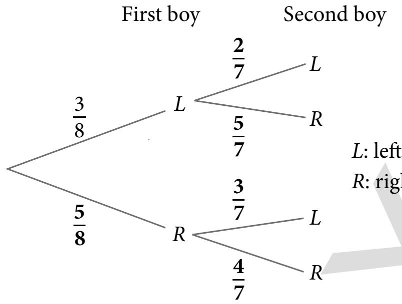
+ P(X does not break down but Y does)

= (0.1× 0.65) + (0.9 × 0.35)

= 0.065 + 0.315

= **0.38**

5. **(i)**



L: left-handed
R: right-handed

(ii) **(a)** P(first is *R* and second is *L*) = P(*RL*)

$= \frac{5}{8} \times \frac{3}{7}$

$= \mathbf{\frac{15}{56}}$

(b) P(one is *R* and the other is *L*) = P(*RL*) + P(*LR*)

$= \left(\frac{5}{8} \times \frac{3}{7}\right) + \left(\frac{3}{8} \times \frac{5}{7}\right)$

$= \frac{15}{56} + \frac{15}{56}$

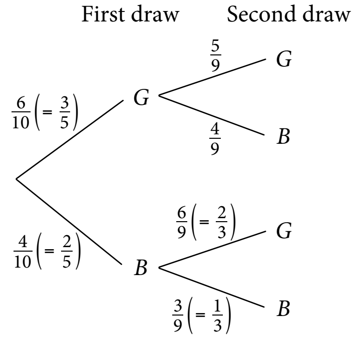
$= \mathbf{\frac{15}{28}}$

(c) P(both are *L*) = P(*LL*)

$= \frac{3}{8} \times \frac{2}{7}$

$= \mathbf{\frac{3}{28}}$

6. **(i)**



G: green card
B: blue card

(ii) Yasir calculated the probability as $\frac{4}{10} \times \frac{4}{10} = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$.

He did not take into account that there are only 3 blue cards and 9 cards left after the first card has been drawn.

(iii) **(a)** P(two green cards) = P(*GG*)

$= \frac{3}{5} \times \frac{5}{9}$

$= \mathbf{\frac{1}{3}}$

(b) P(one card of each colour) = P(*GB*) + P(*BG*)

$= \left(\frac{3}{5} \times \frac{4}{9}\right) + \left(\frac{2}{5} \times \frac{2}{3}\right)$

$= \frac{4}{15} + \frac{4}{15}$

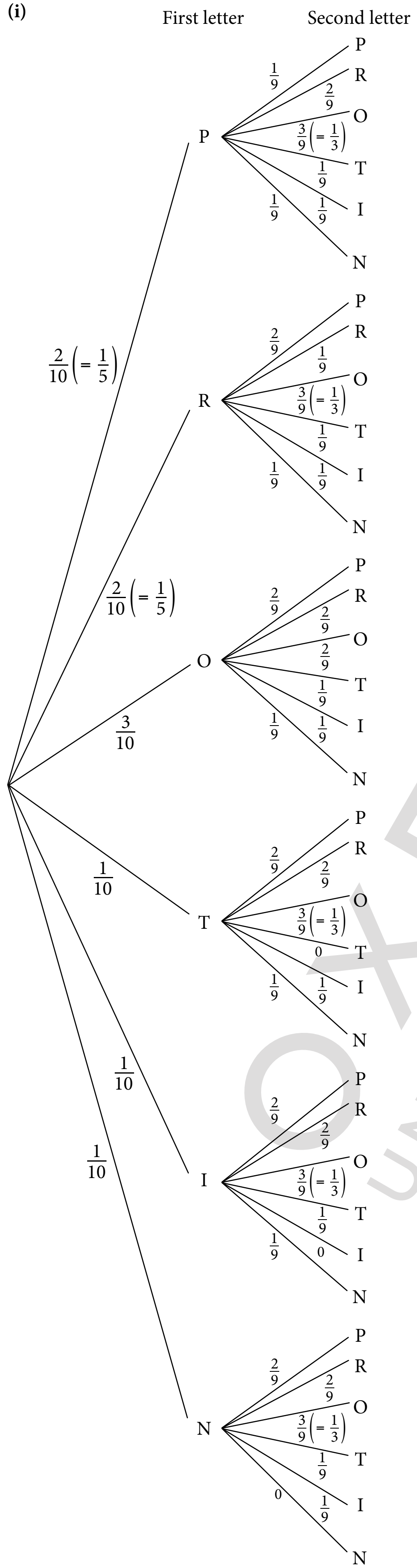
$= \mathbf{\frac{8}{15}}$

(c) P(at least one *B*) = 1 – P(*GG*)

$= 1 - \frac{1}{3}$

$= \mathbf{\frac{2}{3}}$

7. (i)



(ii) Joyce calculated the probability as $\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$. She did not take into account that there are only 2 cards with the letter 'O' and 9 cards left after the first card has been drawn.

(iii) (a) P(first card is 'P' and second card is 'O') $= \frac{1}{5} \times \frac{1}{3}$

$= \mathbf{\frac{1}{15}}$

(b) P(two cards bear the letters 'P' and 'O')

= P(first card is 'P' and second card is 'O')

+ P(first card is 'O' and second card is 'P')

$= \left(\frac{1}{5} \times \frac{1}{3}\right) + \left(\frac{3}{10} \times \frac{2}{9}\right)$

$= \frac{1}{15} + \frac{1}{15}$

$= \mathbf{\frac{2}{15}}$

(c) P(both cards bear the same letter)

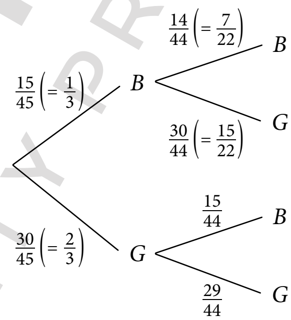
= P(PP) + P(RR) + P(OO)

$= \left(\frac{1}{5} \times \frac{1}{9}\right) + \left(\frac{1}{5} \times \frac{1}{9}\right) + \left(\frac{3}{10} \times \frac{2}{9}\right)$

$= \frac{1}{45} + \frac{1}{45} + \frac{1}{15}$

$= \mathbf{\frac{1}{9}}$

8. (i)



B: boy

G: girl

(ii) Albert calculated the probability as $\frac{15}{45} \times \frac{30}{44} = \frac{1}{3} \times \frac{15}{22} = \frac{5}{22}$. He did not take into account that there are two possibilities. The first case is that the first student is a boy and the second is a girl, and the second case is that the first student is a girl and the second is a boy.

(iii) (a) P(first is G) = $\mathbf{\frac{2}{3}}$

(b) P(second is G, given that first is B) = $\mathbf{\frac{15}{22}}$

(c) P(first is B and second is G) $= \frac{1}{3} \times \frac{15}{22}$

$= \mathbf{\frac{5}{22}}$

(d) P(a boy and girl are selected) = P(BG) + P(GB)

$= \left(\frac{1}{3} \times \frac{15}{22}\right) + \left(\frac{2}{3} \times \frac{15}{44}\right)$

$= \frac{5}{22} + \frac{5}{22}$

$= \mathbf{\frac{5}{11}}$

(e) P(second is G) = P(BG) + P(GG)

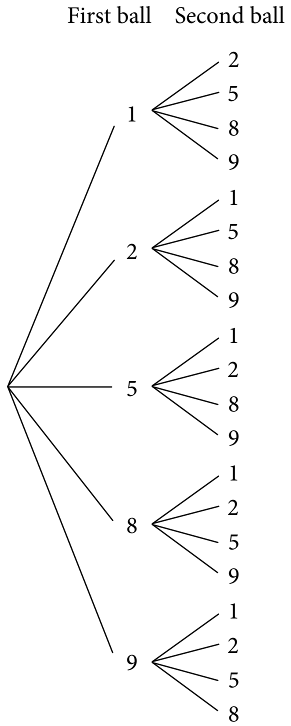
$= \left(\frac{1}{3} \times \frac{15}{22}\right) + \left(\frac{2}{3} \times \frac{29}{44}\right)$

$= \frac{5}{22} + \frac{29}{66}$

$= \mathbf{\frac{2}{3}}$

9. **(i)** P(numbered '8') = $\mathbf{\frac{1}{5}}$

(ii)



(a) P(both numbers are even) = P(2, 8) + P(8, 2)

$= \frac{2}{20}$

$= \mathbf{\frac{1}{10}}$

(b) P(sum of numbers is more than 10)

= P(2, 9) + P(5, 8) + P(5, 9) + P(8, 5) + P(8, 9) + P(9, 2) + P(9, 5) + P(9, 8)

$= \mathbf{\frac{2}{5}}$

(c) P(number on each ball is not a prime number)

= P(1, 8) + P(1, 9) + P(8, 1) + P(8, 9) + P(9, 1) + P(9, 8)

$= \frac{6}{20}$

$= \mathbf{\frac{3}{10}}$

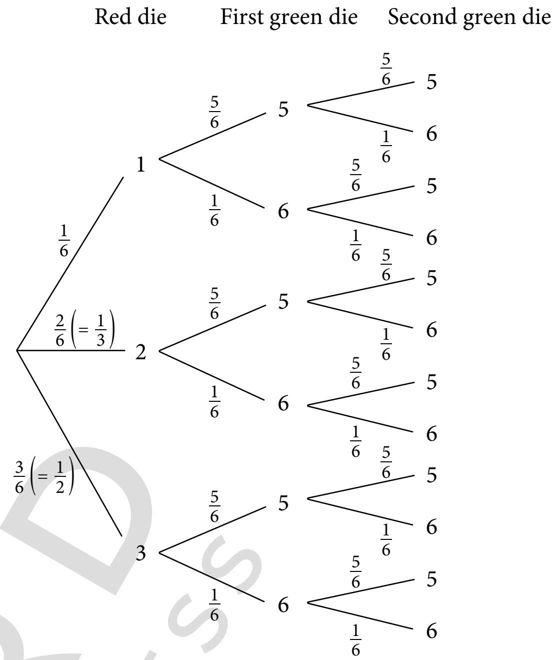
(d) P(only one ball bears an odd number)

= P(1, 2) + P(1, 8) + P(2, 1) + P(2, 5) + P(2, 9) + P(5, 2) + P(5, 8) + P(8, 1) + P(8, 5) + P(8, 9) + P(9, 2) + P(9, 8)

$= \frac{12}{20}$

$= \mathbf{\frac{3}{5}}$

10. **(i)**



(ii) **(a)** P(2, 5, 6) $= \frac{1}{3} \times \frac{5}{6} \times \frac{1}{6}$

$= \mathbf{\frac{5}{108}}$

(b) P(3, 6, 6) $= \frac{1}{2} \times \frac{1}{6} \times \frac{1}{6}$

$= \mathbf{\frac{1}{72}}$

(c) P(exactly two sixes)

= P(1, 6, 6) + P(2, 6, 6) + P(3, 6, 6)

$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{3} \times \frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{2} \times \frac{1}{6} \times \frac{1}{6}\right)$

$= \frac{1}{216} + \frac{1}{108} + \frac{1}{72}$

$= \mathbf{\frac{1}{36}}$

(d) P(sum is 12)

= P(1, 5, 6) + P(1, 6, 5) + P(2, 5, 5)

$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{3} \times \frac{5}{6} \times \frac{5}{6}\right)$

$= \frac{5}{216} + \frac{5}{216} + \frac{25}{108}$

$= \mathbf{\frac{5}{18}}$

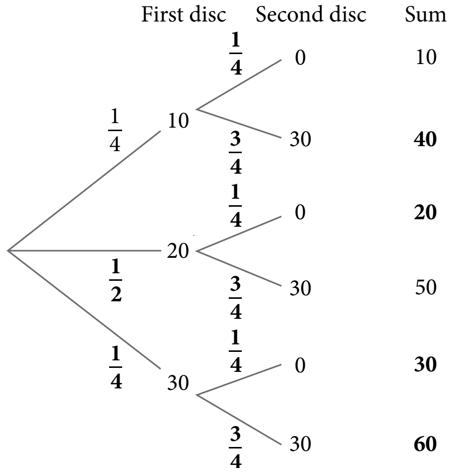
(e) P(sum is divisible by 3) = P(sum is 12) + P(sum is 15)

$= \frac{5}{18}$ + P(3, 6, 6)

$= \frac{5}{18} + \frac{1}{72}$

$= \mathbf{\frac{7}{24}}$

11. (i)



(ii) (a) P(first number is less than or equal to second number obtained)

$= P(10, 30) + P(20, 30) + P(30, 30)$

$= \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{3}{4}\right)$

$= \frac{3}{16} + \frac{3}{8} + \frac{3}{16}$

$= \mathbf{\frac{3}{4}}$

(b) P(second number is 0)

$= P(10, 0) + P(20, 0) + P(30, 0)$

$= \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right)$

$= \frac{1}{16} + \frac{1}{8} + \frac{1}{16}$

$= \mathbf{\frac{1}{4}}$

(iii) (a) P(receives \$2) = P(score is between 10 and 50)

= P(score is 20, 30 or 40)

$= P(10, 30) + P(20, 0) + P(30, 0)$

$= \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right)$

$= \frac{3}{16} + \frac{1}{8} + \frac{1}{16}$

$= \mathbf{\frac{3}{8}}$

(b) P(receives \$5) = P(score is more than 40)

= P(score is 50 or 60)

$= P(20, 30) + P(30, 30)$

$= \left(\frac{1}{2} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{3}{4}\right)$

$= \frac{3}{8} + \frac{3}{16}$

$= \mathbf{\frac{9}{16}}$

(c) P(receives \$2 or \$5) = P(receives \$2) + P(receives \$5)

$= \frac{3}{8} + \frac{9}{16}$

$= \mathbf{\frac{15}{16}}$

(d) P(receives nothing) = 1 – P(receives \$2 or \$5)

$= 1 - \frac{15}{16}$

$= \mathbf{\frac{1}{16}}$

12. (i) P(all three females) $= \frac{18}{40} \times \frac{16}{40} \times \frac{25}{40}$

$= \mathbf{\frac{9}{80}}$

(ii) P(male from front office and female from middle and back)

$= \frac{22}{40} \times \frac{16}{40} \times \frac{25}{40}$

$= \mathbf{\frac{11}{80}}$

(iii) P(exactly one male)

= P(male from only front office) + P(male from only middle office) + P(male from only back office)

$= \left(\frac{22}{40} \times \frac{16}{40} \times \frac{25}{40}\right) + \left(\frac{18}{40} \times \frac{24}{40} \times \frac{25}{40}\right) + \left(\frac{18}{40} \times \frac{16}{40} \times \frac{15}{40}\right)$

$= \frac{11}{80} + \frac{27}{160} + \frac{27}{400}$

$= \mathbf{\frac{299}{800}}$

13. (i) P(all three malfunction) $= 0.03 \times 0.12 \times 0.3$

$= \mathbf{0.001\ 08}$

(ii) P(all three function properly)

$= (1 - 0.03) \times (1 - 0.12) \times (1 - 0.3)$

$= 0.97 \times 0.88 \times 0.7$

$= \mathbf{0.597\ 52}$

(iii) P(at least one pair malfunctions)

= 1 – P(all three function properly)

$= 1 - 0.597\ 52$

$= \mathbf{0.402\ 48}$

(iv) P(exactly two pairs malfunction)

= P(only A and B malfunction) + P(only B and C malfunction) + P(only A and C malfunction)

$= (0.03 \times 0.12 \times 0.7) + (0.97 \times 0.12 \times 0.3) + (0.03 \times 0.88 \times 0.3)$

$= 0.002\ 52 + 0.034\ 92 + 0.007\ 92$

$= \mathbf{0.045\ 36}$

14. (i) (a) P(both black) $= \frac{8}{16} \times \frac{7}{15}$

$= \mathbf{\frac{7}{30}}$

(b) P(one black and one white)

= P(first is black, second is white) + P(first is white, second is black)

$= \left(\frac{8}{16} \times \frac{6}{15}\right) + \left(\frac{6}{16} \times \frac{8}{15}\right)$

$= \frac{1}{5} + \frac{1}{5}$

$= \mathbf{\frac{2}{5}}$

(c) P(both same colour)

= P(both black) + P(both white) + P(both blue)

$= \left(\frac{8}{16} \times \frac{7}{15}\right) + \left(\frac{6}{16} \times \frac{5}{15}\right) + \left(\frac{2}{16} \times \frac{1}{15}\right)$

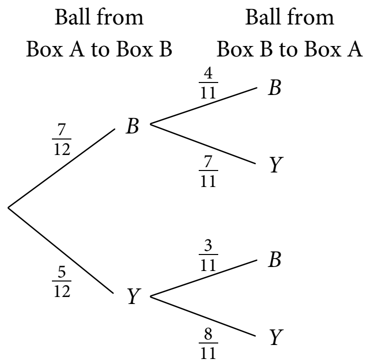
$= \frac{7}{30} + \frac{1}{8} + \frac{1}{120}$

$= \mathbf{\frac{11}{30}}$

(ii) P(all three black) $= \frac{8}{16} \times \frac{7}{15} \times \frac{6}{14}$

$= \mathbf{\frac{1}{10}}$

15. (i)



B: blue ball
Y: yellow ball

(ii) (a) P(Box A has more yellow balls than blue balls) = **0**

(b) P(Box A has exactly 7 blue and 5 yellow balls)

$= \text{P}(BB) + \text{P}(YY)$

$= \left(\frac{7}{12} \times \frac{4}{11}\right) + \left(\frac{5}{12} \times \frac{8}{11}\right)$

$= \frac{7}{33} + \frac{10}{33}$

$= \mathbf{\frac{17}{33}}$

(c) P(twice as many blue balls as yellow balls) = P(YB)

$= \frac{5}{12} \times \frac{3}{11}$

$= \mathbf{\frac{5}{44}}$

16. (i) P(the student was originally from Class A) = $\mathbf{\frac{1}{37}}$

(ii) P(the student is a boy)

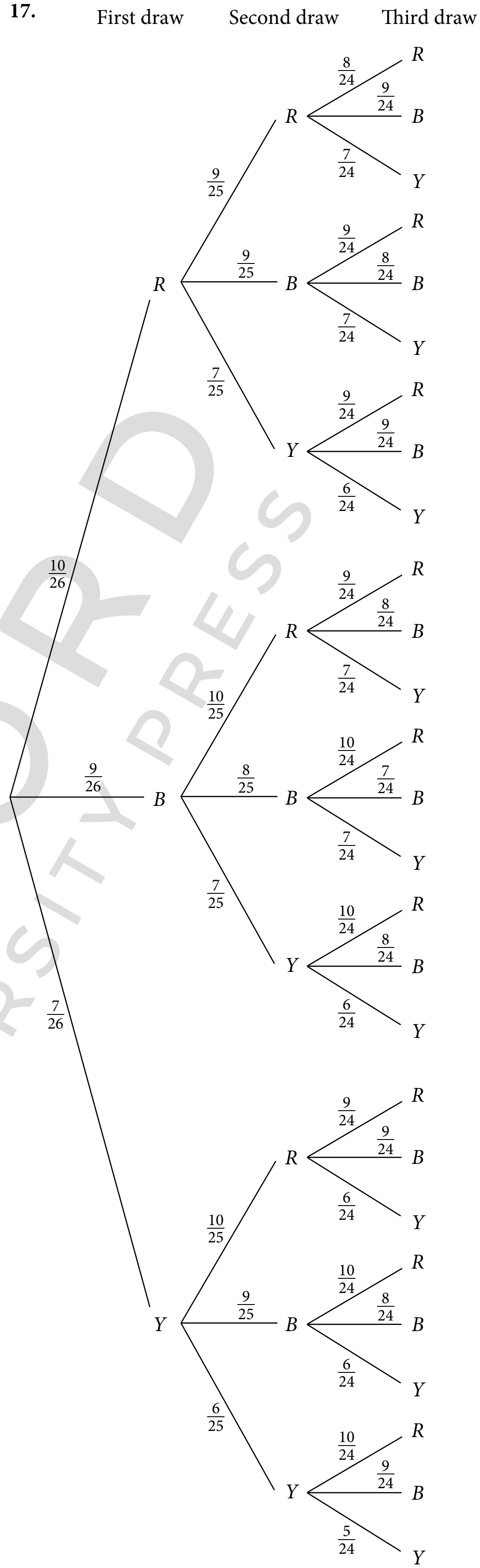
= P(if boy was transferred) + P(if girl was transferred)

$= \left(\frac{18}{35} \times \frac{15}{37}\right) + \left(\frac{17}{35} \times \frac{14}{37}\right)$

$= \frac{54}{259} + \frac{34}{185}$

$= \mathbf{\frac{508}{1295}}$

17.



R: red ball
B: blue ball
Y: yellow ball

(i) P(red, blue, blue) $= \frac{10}{26} \times \frac{9}{25} \times \frac{8}{24}$

$= \mathbf{\frac{3}{65}}$

(ii) P(red, yellow, blue) $= \frac{10}{26} \times \frac{7}{25} \times \frac{9}{24}$

$= \mathbf{\frac{21}{520}}$

(iii) P(three different colours)

= P(RBY) + P(RYB) + P(BRY) + P(BYR) + P(YRB) + P(YBR)

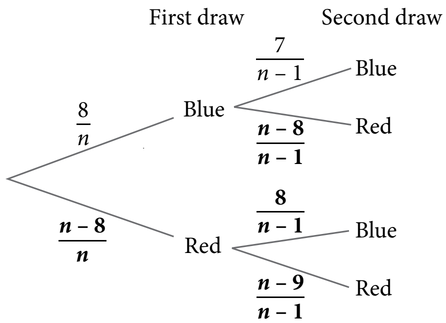
$= \left(\frac{10}{26} \times \frac{9}{25} \times \frac{7}{24}\right) + \left(\frac{10}{26} \times \frac{7}{25} \times \frac{9}{24}\right) + \left(\frac{9}{26} \times \frac{10}{25} \times \frac{7}{24}\right)$

$+ \left(\frac{9}{26} \times \frac{7}{25} \times \frac{10}{24}\right) + \left(\frac{7}{26} \times \frac{10}{25} \times \frac{9}{24}\right) + \left(\frac{7}{26} \times \frac{9}{25} \times \frac{10}{24}\right)$

$= \frac{21}{520} \times 6$

$= \mathbf{\frac{63}{260}}$

18. (i)



(ii) P(two blue marbles) $= \frac{4}{13}$

$\frac{8}{n} \times \frac{7}{n-1} = \frac{4}{13}$

$\frac{56}{n^2 - n} = \frac{4}{13}$

$13 \times 56 = 4(n^2 - n)$

$4n^2 - 4n - 728 = 0$

$n^2 - n - 182 = 0$ (shown)

(iii) $n^2 - n - 182 = 0$

$(n + 13)(n - 14) = 0$

$n + 13 = 0$ or $n - 14 = 0$

$n = -13$ $\quad$ $n = 14$

$\therefore$ ***n* = –13 or 14**

(iv) The solution $n = -13$ must be rejected because n represents the number of marbles, which cannot be negative.

(v) P(one blue and one red marble)

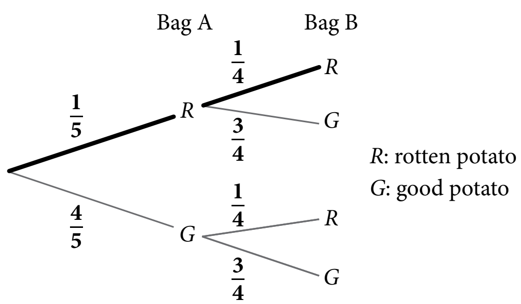
= P(blue then red) + P(red then blue)

$= \left(\frac{8}{14} \times \frac{6}{13}\right) + \left(\frac{6}{14} \times \frac{8}{13}\right)$

$= \frac{24}{91} + \frac{24}{91}$

$= \mathbf{\frac{48}{91}}$

19. (i)



(ii) Yes.

Both events of selecting a potato from each bag are independent of each other, because drawing a rotten potato from Bag A does not affect the chance of drawing a rotten potato from Bag B. Hence when Devi finds the probability of the combined event, she can multiply the respective probabilities of selecting a rotten potato from each bag. This is the Multiplication Law of Probability.

20. P(defective component is the third component tested)

= P(first and second are non-defective and third is defective)

$= \frac{6}{7} \times \frac{5}{6} \times \frac{1}{5}$

$= \mathbf{\frac{1}{7}}$

21. (i) (a) P(game ends on third roll)

= P(first and second are not six and third is six)

$= \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

$= \mathbf{\frac{25}{216}}$

(b) P(game ends on fourth roll)

= P(first three are not six and fourth is six)

$= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

$= \mathbf{\frac{125}{1296}}$

(c) P(game ends by the fourth roll)

= P(game ends on first roll)

+ P(game ends on second roll)

+ P(game ends on third roll)

+ P(game ends on fourth roll)

$= \frac{1}{6} + \left(\frac{5}{6} \times \frac{1}{6}\right) + \frac{25}{216} + \frac{125}{1296}$

$= \frac{1}{6} + \frac{5}{36} + \frac{25}{216} + \frac{125}{1296}$

$= \mathbf{\frac{671}{1296}}$

(ii) (a) P(game ends on third roll)

= P(6, not 6, 6) + P(not 6, 6, 6)

$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$

$= \frac{5}{216} + \frac{5}{216}$

$= \mathbf{\frac{5}{108}}$

(b) P(game ends on third roll and the sum of score is odd)

= P(2 sixes and 1 odd number)

= P(odd number then 2 sixes) + P(6, odd number, 6)

$= \left(\frac{3}{6} \times \frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{3}{6} \times \frac{1}{6}\right)$

$= \frac{1}{72} + \frac{1}{72}$

$= \mathbf{\frac{1}{36}}$

Chapter 3 Statistical Data Analysis

TEACHING NOTES

Suggested Approach

This chapter introduces cumulative frequency curves, quartiles, ranges and scatter diagrams. Before students learn to plot cumulative frequency curves, it would be good for teachers to revise the choice of scales and labelling of scales on both axes. Students are often weak in these areas. Students should be encouraged to draw the curves free hand as well as to use curved rules to assist them.

Similarly, students are also encouraged to practise drawing a line of best fit on a scatter diagram. Teachers may also give examples in real-world contexts which allow students to draw a line of best fit on a scatter diagram and make predictions.

Section 3.1: Cumulative frequency table and curve

Teachers can point out to students that cumulative frequency curves can be used to interpret the data readily; for example, finding the percentage of students who passed the test when given that the passing mark = x marks or the marks above which y% of the students scored and so on.

For more advanced students, teachers can choose to discuss with them how a set of marks can be moderated by changing the passing mark such that p% of students pass, or the range of marks for a certain grade in order for q% of students to obtain that grade etc.

Section 3.2: Median, quartiles, percentiles, range and interquartile Range

From a cumulative frequency curve, median, quartiles and percentiles can be found with ease. However, it should be pointed out to students that the values obtained from the curves are merely estimates. For ungrouped data consisting of an even number of values, say, 40 values, the median is the mean of the two middle values when the values are arranged in order of magnitude, i.e. median = $\frac{a+b}{2}$, where $a = 20^{\text{th}}$ value and $b = 21^{\text{st}}$ value. From the cumulative frequency curve representing the data, the value on the horizontal axis corresponding to the $\left(\frac{40}{2}\right)^{\text{th}} = 20^{\text{th}}$ value is taken to be the estimate of the median. Thus, in general, from the cumulative frequency diagram representing a set of data of n values, the estimates of lower quartile, median, upper quartile and p^{th} percentile are readings corresponding to $\left(\frac{n}{4}\right)^{\text{th}}$, $\left(\frac{n}{2}\right)^{\text{th}}$, $\left(\frac{3n}{4}\right)^{\text{th}}$ and $\left(\frac{pn}{100}\right)^{\text{th}}$ values respectively.

An **average** is a number that summarises a set of data. In many cases, while the average provides sufficient information, it fails to give an indication of whether the data are clustered together or spread across a wide range of values. Interquartile range and range are measures which roughly provide such an indication. There are also other such measures which are known as measures of dispersion.

Teachers should also encourage students to search online for more real-life examples related to cumulative frequency curves.

Section 3.3: Further comparison of data

In this section, students will learn to compare data sets using averages and range. Teachers are also encouraged to use examples in the real-world which will allow students to apply what they will learn in this section to compare data sets. Students should also know the difference between comparing data sets using the interquartile range and the range, as well as the median and the mean.

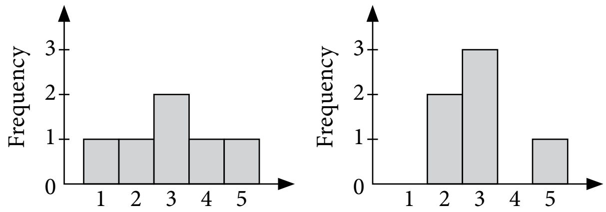
Section 3.4: Scatter diagrams

Teachers can introduce the concept of scatter diagrams by establishing everyday situations with correlations that are either positive or negative, and those which have no correlations at all (see Class Discussion: Scatter diagram with no correlation). Teachers can illustrate how the use of scatter diagrams can verify claims as to the type of correlation two variables have, which might not be as evident when using other statistical diagrams.

It is also important for students to have a rough expectation of the correlation between variables in the real world (see Thinking Time on page 106 of the textbook).

Introductory Problem

1.



2. **No**, because it does not reflect the distribution of the data.

3.1 Cumulative frequency table and curve

Class Discussion (Constructing table of cumulative frequencies)

1.

Number of hours, t	Cumulative frequency
$t \leqslant 2$	3
$t \leqslant 4$	3 + 5 = 8
$\boldsymbol{t \leqslant 6}$	**8 + 16 = 24**
$\boldsymbol{t \leqslant 8}$	**24 + 12 = 36**
$\boldsymbol{t \leqslant 10}$	**36 + 4 = 40**

2. (i) The number of students who used the computer for 6 hours or less is **24**.
   (ii) The number of students who used the computer for more than 8 hours is 40 – 36 = **4**
   (iii) The number of students who used the computer for more than 4 hours but not more than 10 hours is 40 – 8 = **32**
3. The last entry represents the total number of students. It is the maximum possible frequency.
4. (i) **No**, because the tables are based on a "less than or equal" frequency. Since we are unable to find out the number of students who used the computer for exactly 6 hours, we do not know the number of students who used the computer for less than 6 hours.
   (ii) **No**, because we are unable to find out the number of students who used the computer for exactly 8 hours.
   (iii) **No**, because the number of hours is in 2-hour blocks.
5. (i) **Yes**. Number of students who used the computer for less than 6 hours = 14 + 5 + 3 = 22
   (ii) **Yes**. Number of students who used the computer for at least 8 hours = 5
   (iii) **No**. Number of hours is still in 2-hour blocks.

Practise Now 1A

(i)

Length (x mm)	Cumulative frequency
$x < 30$	1
$x < 35$	4
$x < 40$	10
$x < 45$	22
$x < 50$	32
$x < 55$	38
$x < 60$	40

(ii) (a) The number of insects less than 50 mm in length is **32**.
(b) The number of insects at least 45 mm in length is 40 – 22 = **18**.
(c) The number of insects at least 35 mm but less than 50 mm in length is 32 – 4 = **28**.

(iii) The cumulative frequency table is constructed based on a "less than" frequency. Since we do not know the number of insects which are exactly 50 mm in length, it is not possible to calculate the number of insects which are 50 mm or less in length.

(iv) P(length less than 50 mm) = $\frac{32}{40} = \frac{4}{5}$

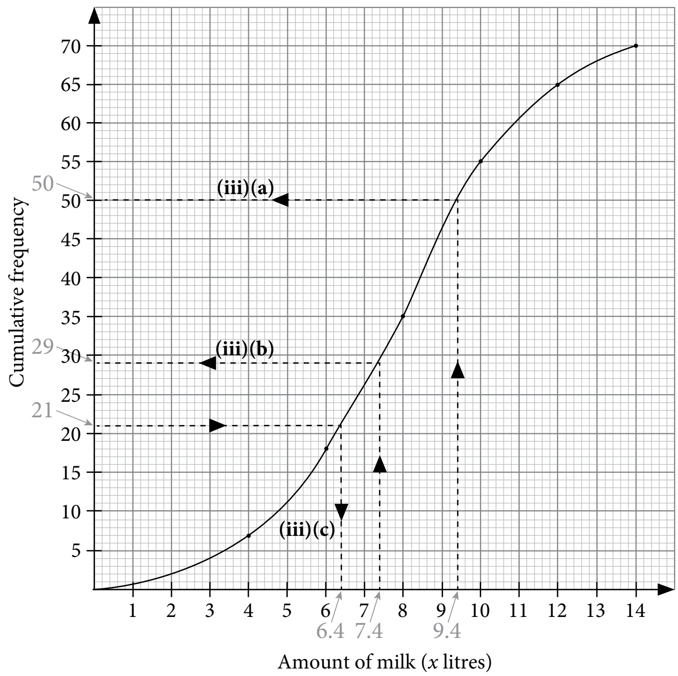
Practise Now 1B

(i)

Amount of milk (x litres)	Number of cows
$x < 4$	7
$x < 6$	18
$x < 8$	**35**
$x < 10$	**55**
$x < 12$	**65**
$x < 14$	**70**

(ii) Scale: 1 cm represents 1 litre on the horizontal axis
1 cm represents 5 cows on the vertical axis

Cumulative frequency curve for the amount of milk produced by cows



(iii) **(a)** From the curve, the number of cows that produced less than 9.4 litres of milk is **50**.

(b) From the curve, the number of cows that produced less than 7.4 litres of milk is 29.
Then 70 – 29 = 41 cows produced at least 7.4 litres of milk.
∴ fraction of cows that produced at least 7.4 litres of milk is $\mathbf{\frac{41}{70}}$.

(c) 70% of cows means $\frac{70}{100} \times 70 = 49$ cows produced less than x litres of milk.
Then 70 – 49 = 21 cows produced at least x litres of milk.
From the curve, x = **6.4**.

(iv) P(both cows produced less than 9.4 litres of milk each)
$= \frac{50}{70} \times \frac{49}{69}$
$= \mathbf{\frac{35}{69}}$

Exercise 3A

1. **(i)**

Marks (m)	Number of students
$m \leqslant 10$	3
$m \leqslant 20$	**15**
$m \leqslant 30$	**24**
$m \leqslant 40$	**35**
$m \leqslant 50$	**52**
$m \leqslant 60$	**71**
$m \leqslant 70$	**91**
$m \leqslant 80$	**105**
$m \leqslant 90$	**115**
$m \leqslant 100$	**120**

(ii) **(a)** Number of students who scored less than or equal to 30 marks = **24**

(b) Number of students who scored more than 80 marks
= 120 – 105
= **15**

(c) Number of students who scored more than 40 marks but not more than 90 marks is 115 – 35 = **80**

(iii) The cumulative frequency table is based on a "less than or equal" frequency. Since it is not possible to find out the number of students who scored exactly 30 marks, it is not possible to state the number of students who scored less than 30 marks.

(iv) Relative frequency = $\frac{15}{120} = \mathbf{\frac{1}{8}}$

2. **(i)**

Daily total rainfall (x mm)	Number of days
$x < 10$	21
$x < 20$	24
$x < 30$	26
$x < 40$	27
$x < 50$	29
$x < 60$	30
$x < 70$	31

(ii) **(a)** The number of days when the rainfall was less than 30 mm is **26**.

(b) The number of days when the rainfall was at least 40 mm is 31 – 27 = **4**

(c) The number of days when the rainfall was at least 20 mm but less than 50 mm is 29 – 24 = **5**

(iii) The cumulative frequency table is based on a "less than" frequency. Since it is not possible to find out the number of days when the rainfall was exactly 60 mm, it is not possible to state the number of days when the rainfall was at most 60 mm.

(iv) Relative frequency = $\mathbf{\frac{4}{31}}$

3. (i)

Number of defects (x)	Cumulative frequency
$x \leqslant 4$	21
$x \leqslant 8$	**39**
$x \leqslant 12$	**53**
$x \leqslant 16$	**59**
$x \leqslant 20$	**62**
$x \leqslant 24$	**64**
$x \leqslant 28$	**65**

(ii) (a) The number of batches graded as Grade 1 is 65 – 62 = **3**

(b) The number of batches graded as Grade 2 is 62 – 53 = **9**

(c) The number of batches graded as Grade 3 is 53 – 21 = **32**

(iii) The cumulative frequency table is based on a "less than or equal to" frequency. Since it is not possible to find out the number of batches which have exactly 16 defects, it is not possible to find out the number of batches which have at least 16 defects.

(iv) Relative frequency = $\frac{65-39}{65} = \frac{26}{65} = \mathbf{\frac{2}{5}}$

4. (i) (a) From the curve, the number of people whose masses are less than or equal to 65 kg is **26**.

(b) From the curve, the number of people whose masses are less than or equal to 68.6 kg is 76.
$\therefore$ the number of people whose masses are more than 68.6 kg is 100 – 76 = **24**

(c) From the curve, the number of people whose masses are less than or equal to 64.4 kg is 19.
Then 100 – 19 = 81 people have masses more than 64.4 kg.
$\therefore$ the percentage of the total number of people whose masses are more than 64.4 kg is $\frac{81}{100} \times 100\% =$ **81%**

(ii) Relative frequency = $\frac{24}{100}$
$= \mathbf{\frac{6}{25}}$

5. (i) (a) From the curve, the number of oranges that contain less than 32 mg of Vitamin C is **180**.

(b) From the curve, the number of oranges that contain less than 26 mg of Vitamin C is 40.
Then 200 – 40 = 160 oranges contain 26 mg or more of Vitamin C.
$\therefore$ fraction of oranges that contain 26 mg or more of Vitamin C is $\frac{160}{200} = \mathbf{\frac{4}{5}}$

(c) 40% of oranges means $\frac{40}{100} \times 200 = 80$ oranges contain at least p mg of Vitamin C.

Then 200 – 80 = 120 oranges contain less than p mg of Vitamin C.
From the curve, p = **29.8**.

(ii) From the curve, the number of oranges that contain less than 28 mg of Vitamin C is 70.

$\therefore$ Relative frequency = $\frac{200-70}{200}$
$= \mathbf{\frac{13}{20}}$

6. (i) (a) From the curve, the number of students who took less than 4 minutes to solve the puzzle is 10.
$\therefore$ the number of students who took at least 4 minutes to solve the puzzle is 40 – 10 = **30**

(b) From the curve, the number of students who took less than 6 minutes to solve the puzzle is 28.
$\therefore$ the percentage of students who took less than 6 minutes to solve the puzzle is $\frac{28}{40} \times 100\% =$ **70%**

(c) $\frac{1}{4}$ of the students means $\frac{1}{4} \times 40 = 10$ students took less than q minutes to solve the puzzle.
From the curve, q = **4**.

(ii) From the curve, the number of students who took less than 10 minutes to solve the puzzle is 38.
Then 40 – 38 = 2 students took at least 10 minutes to solve the puzzle.

$\therefore$ Relative frequency = $\frac{2}{40}$
$= \mathbf{\frac{1}{20}}$

7. (i)

Annual income ($\$x$, in thousands)	Frequency
$0 \leqslant x < 20$	42
$20 \leqslant x < 40$	38
$40 \leqslant x < 60$	22
$60 \leqslant x < 80$	20
$80 \leqslant x < 100$	66
$100 \leqslant x < 120$	12

(ii)

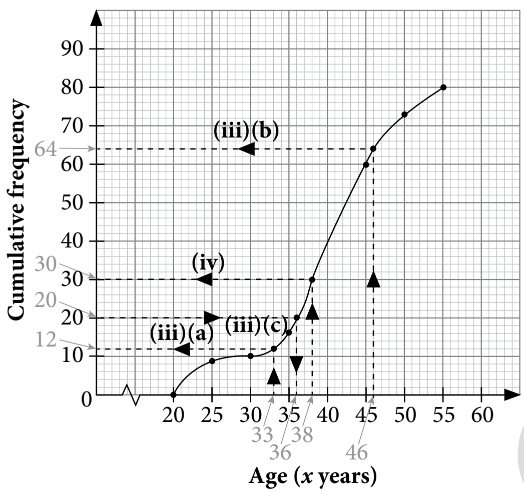
(iii) $\mathbf{80 \leqslant x < 100}$

(iv) Relative frequency = $\frac{60}{200} \times \frac{59}{199}$
$= \mathbf{\frac{177}{1990}}$

8. **(i)**

Age (x years)	Number of workers
$x < 25$	9
$x < 30$	10
$x < 35$	**16**
$x < 40$	**40**
$x < 45$	**61**
$x < 50$	**73**
$x < 55$	**80**

(ii) Scale: 1 cm represents 5 years on the horizontal axis
1 cm represents 10 workers on the vertical axis



(iii) **(a)** From the curve, the number of workers who are younger than 33 years old is **12**.

(b) From the curve, the number of workers who are younger than 46 years old is 64.
Then 80 – 64 = 16 workers are at least 46 years old.
∴ fraction of workers who are at least 46 years old is $\frac{16}{80} = \mathbf{\frac{1}{5}}$.

(c) 75% of workers means $\frac{75}{100} \times 80 = 60$ workers are at least p years old.
Then 80 – 60 = 20 workers are less than p years old.
From the curve, p = **36**.

(iv) From the curve, the number of workers who are younger than 38 years old is 30.

P(the workers are younger than 38 years old) $= \frac{30}{80} \times \frac{29}{79}$
$= \mathbf{\frac{87}{632}}$

9. **(i)** **(a)** From the curve, the number of adults who exercise at most 38 minutes daily is 20.
∴ the number of adults who exercise more than 38 minutes daily is 60 – 20 = **40**

(b) From the curve, the number of adults who exercise at most 35 minutes daily is 16.
∴ percentage of adults who exercise at most 35 minutes daily is $\frac{16}{60} \times 100\% = \mathbf{26\frac{2}{3}\%}$ **or 26.7%** (to 3 s.f.)

(c) $\frac{1}{3}$ of the adults means $\frac{1}{3} \times 60 = 20$ adults exercise more than t minutes daily.
Then 60 – 20 = 40 adults exercise at most t minutes daily.
From the curve, t = **47**.

(ii) From the curve, the number of adults who exercise at most 20 minutes daily is 4.
Fraction of adults who exercise at most 20 minutes daily is $\frac{4}{60} = \frac{1}{15}$.
Therefore the data does not support the claim that one in three adults exercises at most 20 minutes daily.

(iii) From the curve, the number of adults who exercise at most 66 minutes is 59.

P(both adults exercise at most 66 minutes daily) $= \frac{59}{60} \times \frac{58}{59}$
$= \mathbf{\frac{29}{30}}$

(iv)

Time (x minutes)	Number of adults
$x \leqslant 0$	**0**
$x \leqslant 10$	**1**
$x \leqslant 20$	**4**
$\mathbf{x \leqslant 30}$	**11**
$\mathbf{x \leqslant 40}$	**23**
$\mathbf{x \leqslant 50}$	**46**
$\mathbf{x \leqslant 60}$	**56**
$x \leqslant 70$	60

(v) **(a)**

Time (x minutes)	Frequency
$0 < x \leqslant 10$	**1**
$10 < x \leqslant 20$	**3**
$20 < x \leqslant 30$	**7**
$30 < x \leqslant 40$	**12**
$40 < x \leqslant 50$	**23**
$50 < x \leqslant 60$	**10**
$60 < x \leqslant 70$	**4**
$70 < x \leqslant 80$	**0**

(b) Mean amount of time

$$= \frac{(5 \times 1) + (15 \times 3) + (25 \times 7) + (35 \times 12) + (45 \times 23) + (55 \times 10) + (65 \times 4) + (75 \times 0)}{60}$$

$$= \frac{2490}{60}$$

= **41.5 minutes**

(c) The mean is an estimate because we do not know the exact time each adult spends exercising, so we take the mid-value of each range as the representative x value for that particular range.

10. **(i)** **(a)** From the curve, the number of tomatoes with masses at most 56 g is 36.

Then 80 – 36 = 44 tomatoes have masses more than 56 g.

$\therefore$ percentage of Grade A tomatoes is $\frac{44}{80} \times 100\% = \mathbf{55\%}$

(b) 15% of the tomatoes means $\frac{15}{100} \times 80 = 12$ tomatoes are at most y grams.

From the curve, $y = \mathbf{52}$.

(c) Number of tomatoes with masses $52 < x \leqslant 56$ is

$36 - 12 = 24$

$\therefore$ number of Grade B tomatoes is **24**.

(d) From **(i)(a)**, the percentage of Grade A tomatoes is 55%.

Therefore the data does not support his claim.

(e) P(both tomatoes are rated Grade A or Grade B)

$= \frac{68}{80} \times \frac{67}{79}$

$= \mathbf{\frac{1139}{1580}}$

(ii) **(a)**

Mass (x g)	Frequency
$40 < x \leqslant 45$	**2**
$45 < x \leqslant 50$	**6**
$50 < x \leqslant 55$	**18**
$55 < x \leqslant 60$	**44**
$60 < x \leqslant 65$	**10**

(b) Mean mass

$= \frac{(42.5 \times 2) + (47.5 \times 6) + (52.5 \times 18) + (57.5 \times 44) + (62.5 \times 10)}{80}$

$= \frac{4470}{80}$

$= \mathbf{55.875\ g}$

(c) The mean is an estimate because we do not know the exact mass of each tomato, so we take the mid-value of each range as the representative value for that particular range.

11. **(i)**

Number of pull-ups (x)	Frequency
$x \geqslant 0$	**230**
$x \geqslant 6$	**161**
$x \geqslant 8$	**98**
$x \geqslant 10$	**70**
$x \geqslant 12$	**46**
$x \geqslant 16$	**27**
$x \geqslant 20$	13

(ii) The cumulative frequency table is based on a "more than or equal to" frequency. Since it is not possible to find out the number of men who did exactly 8 pull-ups, it is not possible to find out the number of men who did more than 8 pull-ups.

(iii) **(a)** The number of men who achieved the Gold Award is **46**.

(b) The number of men who achieved the Silver Award is $98 - 46 = \mathbf{52}$

(c) The number of men who achieved the Bronze Award is $161 - 98 = \mathbf{63}$

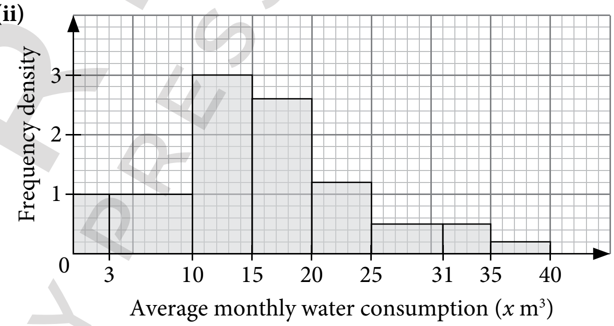
(iv) Relative frequency $= \frac{46}{230}$

$= \mathbf{\frac{1}{5}}$

12. **(i)**

Average monthly water consumption (x m³)	Frequency	Class width	Frequency density $= \frac{\text{Frequency}}{\text{Class width}}$
$0 \leqslant x < 3$	3	3	$3 \div 3 = 1$
$3 \leqslant x < 10$	7	7	$7 \div 7 = 1$
$10 \leqslant x < 15$	15	5	$15 \div 5 = 3$
$15 \leqslant x < 20$	13	5	$13 \div 5 = 2.6$
$20 \leqslant x < 25$	6	5	$6 \div 5 = 1.2$
$25 \leqslant x < 31$	3	6	$3 \div 6 = 0.5$
$31 \leqslant x < 35$	2	4	$2 \div 4 = 0.5$
$35 \leqslant x < 40$	1	5	$1 \div 5 = 0.2$

(ii)



(iii) $\mathbf{10 \leqslant x < 15}$

(iv) Relative frequency $= \frac{11}{50} \times \frac{10}{49}$

$= \mathbf{\frac{11}{245}}$

13. **(i)** **(a)** From the curve, the number of days when the daily maximum temperature exceeded 30 °C is **320**.

(b) From the curve, the number of days when the daily maximum temperature exceeded 32 °C is 155.

Then 365 – 155 = 210 days had a daily maximum temperature of at most 32 °C.

$\therefore$ fraction of days that had a daily maximum of temperature of at most 32 °C is $\frac{210}{365} = \mathbf{\frac{42}{73}}$

(c) If 165 days had a daily maximum temperature of k °C or less, then 365 – 165 = 200 days had a daily maximum temperature that exceeded k °C.

From the curve, $k = \mathbf{31.6}$.

(ii) P(the day had a daily maximum temperature of at most 32.4 °C)

$= \frac{365 - 110}{365}$

$= \mathbf{\frac{51}{73}}$

14. (i) (a) From the curve, the number of bicycles that travelled at a speed of at most 18 km/h is **13**.

(b) From the curve, the number of bicycles that travelled at a speed of at most 25 km/h is 54.
Then 100 – 54 = 46 bicycles travelled at a speed greater than 25 km/h.
∴ fraction of bicycles that travelled at a speed greater than 25 km/h is $\frac{46}{100} = \mathbf{\frac{23}{50}}$

(c) 40% of the bicycles means $\frac{40}{100} \times 100 = 40$ bicycles travelled at a speed less than or equal to v km/h.
From the curve, v = **23**.

(ii) From **(i)(b)**, 46% of the cyclists exceeded the speed limit of 25 km/h. Therefore the data does not support the claim that 50% of cyclists do not keep to the speed limit.

(iii) P(both bicycles exceeded the speed limit) $= \frac{46}{100} \times \frac{45}{99}$
$= \mathbf{\frac{23}{110}}$

(iv)

Speed (u km/h)	Number of bicycles
$u \leqslant 5$	0
$u \leqslant 10$	**3**
$u \leqslant 15$	**8**
$\mathbf{u \leqslant 20}$	**20**
$\mathbf{u \leqslant 25}$	**54**
$\mathbf{u \leqslant 30}$	**76**
$\mathbf{u \leqslant 35}$	**90**
$u \leqslant 40$	**100**

15. (i) **Group B**, because the curve cuts the origin and the start of the curve is the steepest of the three, meaning it has the most people with the least mobile data usage.

(ii) **Group C**, because the curve does not cut the origin and the start of the curve is the gentlest of the three, meaning it has the most number of people having a high mobile data usage.

3.2 Median, quartiles, percentiles, range and interquartile range

Practise Now 2

1. (i) **lower half**: 10 12 14 22 25 | **upper half**: 36 38 44 45 59

Q_1 at 14; Q_2 (**median**) between 25 and 36; Q_3 at 44

Q_1 = **14**

$Q_2 = \frac{25+36}{2}$ = **30.5**

Q_3 = **44**

(ii) Interquartile range $= Q_3 - Q_1$
= 44 – 14
= **30**

(iii) Range = largest value – smallest value
= 59 – 10
= **49**

2. (i) **lower half**: 9 15 16 23 | 26 | **upper half**: 32 35 47 54

Q_1 between 15 and 16; Q_2 (**median**) at 26; Q_3 between 35 and 47

Median = **26**

Lower quartile $= \frac{15+16}{2}$ = **15.5**

Upper quartile $= \frac{35+47}{2}$ = **41**

(ii) Interquartile range = 41 – 15.5 = **25.5**
Range = 54 – 9 = **45**

Thinking Time (Page 80)

No, the position of the first quartile is not always $\frac{n+1}{4}$ or $\frac{1}{4}(n+1)$.

No, the position of the third quartile is not always $\frac{3}{4}(n+1)$.

For Dataset A, $n = 11$, $\frac{1}{4}(n+1) = 3$ and $\frac{3}{4}(n+1) = 9$

For Worked Example 2, $n = 8$, $\frac{1}{4}(n+1) = 2.25$ and $\frac{3}{4}(n+1) = 6.75$

For Practise Now 2 Question 1, $n = 10$, $\frac{1}{4}(n+1) = 2.75$ and $\frac{3}{4}(n+1) = 8.25$

For Practise Now 2 Question 2, $n = 9$, $\frac{1}{4}(n+1) = 2.5$ and $\frac{3}{4}(n+1) = 7.5$

When n is odd, the position of the first quartile is $\frac{1}{4}(n+1)$ and the position of the third quartile is $\frac{3}{4}(n+1)$.

Practise Now 3

Total weekly exercise time (x minutes)	Frequency (f)	Mid-value (x)
$30 < x \leqslant 50$	6	40
$50 < x \leqslant 70$	11	60
$70 < x \leqslant 90$	9	80
$90 < x \leqslant 110$	6	100
$110 < x \leqslant 130$	7	120
$130 < x \leqslant 150$	11	140

Estimated range = 140 – 40
= **100 minutes**

Practise Now 4

(i) From the graph, median score = **75**,
lower quartile = **27**,
upper quartile = **121.5**.

(ii) Interquartile range = 121.5 – 27
= **94.5**

Range = 141 – 6
= **135**

(iii) 10% of total frequency = $\frac{10}{100} \times 120$
= 12
From the graph, the 10[th] percentile = **12**
80% of total frequency = $\frac{80}{100} \times 120$
= 96
From the graph, the 80[th] percentile = **126**

(iv) 60% of the students = $\frac{60}{100} \times 120 = 72$ students scored at least x points.
Then 120 – 72 = 48 students scored less than x points.
From the graph, x = **51**.

Practise Now 5

(i) **(a)** From the graph, median daily revision time = **62 minutes**
(b) From the graph, lower quartile = 50
upper quartile = 68
Interquartile range for School A = 68 – 50
= **18 minutes**

(ii) **(a)** From the graph, median daily revision time = **70 minutes**
(b) From the graph, lower quartile = 56
upper quartile = 79
Interquartile range for School B = 79 – 56
= **23 minutes**

(iii) Students from **School B** spend more time revising daily. The median daily revision time for School B is 70 minutes, which is higher than the median daily revision time of 62 minutes for School A.

(iv) **School A** has a more consistent daily revision time. The interquartile range of 18 minutes for School A is smaller than the interquartile range of 23 minutes for School B.

Exercise 3B

1. **(a)** 2, 4, 5, 6, 7, 8, 10
Range = 10 – 2
= **8**
Lower quartile = **4**
Median = **6**
Upper quartile = **8**
Interquartile range = 8 – 4
= **4**

(b) 51, 54, 63, 64, 66, 70, 72, 80
Range = 80 – 51
= **29**
Lower quartile = $\frac{54+63}{2}$
= **58.5**
Median = $\frac{64+66}{2}$
= **65**
Upper quartile = $\frac{70+72}{2}$
= **71**
Interquartile range = 71 – 58.5
= **12.5**

(c) 9, 10, 14, 16, 18, 22, 27, 32, 40
Range = 40 – 9
= **31**
Lower quartile = $\frac{10+14}{2}$
= **12**
Median = **18**
Upper quartile = $\frac{27+32}{2}$
= **29.5**
Interquartile range = 29.5 – 12
= **17.5**

(d) 2.7, 4.9, 6.7, 8.5, 10.4, 11.8, 13.1, 15.1, 16.7, 22.4
Range = 22.4 – 2.7
= **19.7**
Lower quartile = **6.7**
Median = $\frac{10.4+11.8}{2}$
= **11.1**
Upper quartile = **15.1**
Interquartile range = 15.1 – 6.7
= **8.4**

2. 0, 0, 1, 6, 6, 9, 9, 24, 27, 29
(i) First quartile = **1**
Second quartile = $\frac{6+9}{2}$
= **7.5**
Third quartile = **24**
(ii) Interquartile range = 24 – 1
= **23**
Range = 29 – 0
= **29**

3.

Height (x hours)	Frequency (f)	Mid-value (x)
$100 \leqslant x \leqslant 105$	4	102.5
$105 < x \leqslant 110$	13	107.5
$110 < x \leqslant 115$	10	112.5
$115 < x \leqslant 120$	8	117.5
$120 < x \leqslant 125$	4	122.5
$125 < x \leqslant 130$	1	127.5

Estimated range = 127.5 – 102.5
= **25 hours**

4. **(i)** From the graph, median daily earnings = **$97**
lower quartile = **$88**
upper quartile = **$105**
(ii) Interquartile range = 105 – 88
= $17
Range = 120 – 60
= **$60**
(iii) 20% of total frequency = $\frac{20}{100} \times 300$
= 60
From the graph, the 20[th] percentile = **$85**
90% of total frequency = $\frac{90}{100} \times 300$
= 270
From the graph, the 90[th] percentile = **$110**

5. **(i)** **(a)** From the curve, median height = **35 cm**

(b) From the curve, first quartile = **30 cm**

third quartile = **39 cm**

(c) 30% of total frequency $= \frac{30}{100} \times 60$

$= 18$

From the graph, the 30th percentile = **31 cm**

55% of total frequency $= \frac{55}{100} \times 60$

$= 33$

From the graph, the 55th percentile = **36 cm**

(d) From the graph, number of plants less than or equal to 50 cm is 56.

$\therefore$ number of plants taller than 50 cm is 60 – 56 = **4**

(ii) Number of plants more than 30 cm but less than or equal to 39 cm is 45 – 15 = 30

$\therefore$ P(plant has a height within the interquartile range) $= \frac{30}{60}$

$= \mathbf{\frac{1}{2}}$

6. **(i)** **(a)** From the graph, median daily travelling expenses = **42 cents**

(b) From the graph, median daily travelling expenses = **58 cents**

(ii) **(a)** From the graph, lower quartile = 30 cents

upper quartile = 56 cents

$\therefore$ interquartile range = 56 – 30

= **26 cents**

(b) From the graph, lower quartile = 47 cents

upper quartile = 71 cents

$\therefore$ interquartile range = 74 – 47

= **24 cents**

(iii) Students from **School B** spend more on daily travelling expenses. The median daily travelling expenses is higher for School B than for School A.

7. **(i)** **(a)** From the graph, median amount of plastic waste produced by Factory P = **350 kg**

(b) From the graph, median amount of plastic waste produced by Factory Q = **500 kg**

(ii) **(a)** From the graph, lower quartile = 250 kg

upper quartile = 420 kg

$\therefore$ interquartile range = 420 – 250

= **170 kg**

(b) From the graph, lower quartile = 370 kg

upper quartile = 560 kg

$\therefore$ interquartile range = 560 – 370

= **190 kg**

(iii) **Factory Q** produced more plastic waste. The median amount of plastic waste produced by Factory Q was 500 kg, which is much higher than the plastic waste of 350 kg produced by Factory P.

8. **(i)** **Soil A**

(ii) From the curve, the number of earthworms from Soil A with lengths less than or equal to 60 mm is 320.

So number of earthworms longer than 60 mm is 500 – 320 = 180.

From the curve, the number of earthworms from Soil B with lengths less than or equal to 60 mm is 380.

So number of earthworms longer than 60 mm is 500 – 380 = 120.

$\therefore$ **Soil A** had more 'satisfactory' earthworms.

9. **(i)** Range = 9 – 3

= **6 years**

(ii) Median age $= \frac{5+6}{2}$

= **5.5 years**

(iii) Lower quartile = **4 years**

Upper quartile = **7 years**

(iv) Interquartile range = 7 – 4

= **3 years**

10. **(i)** **(a)** From the curve, lower quartile = **10 minutes**

median = **13 minutes**

upper quartile = **15.25 minute**s

(b) Interquartile range = 15.25 – 10 = **5.25 minutes**

(ii) From the curve, the number of clients who waited for at most 15 minutes = 44

Then 60 – 44 = 16 clients waited for more than 15 minutes.

$\therefore$ percentage of clients who waited for more than 15 minutes at the bank is $\frac{16}{60} \times 100\% =$ **$26\frac{2}{3}$ % or 26.7%** (to 3 s.f.)

(iii) The t-coordinate of the point of intersection of the two cumulative curves is the **median waiting time**. In the original cumulative frequency curve, the number of clients who had waiting time less than or equal to the median time is the same as the number of clients who had waiting time more than the median time. In the second cumulative frequency curve, the number of clients who had waiting time more than the median time is same as the number of clients who had waiting time less than or equal to the median time. Therefore the point of intersection occurs when the frequency is 30 and the t-coordinate is the median waiting time.

11. **(i)** From the graph, lower quartile = **21**

median = **28**

upper quartile = **34**

(ii) Number of people in Group B is **38**.

(iii) From the graph, lower quartile for Group B = 20

upper quartile for Group B = 28.5

$\therefore$ interquartile range for Group B = 28.5 – 20

= **8.5**

(iv) From the graph, the number of people in Group B who can type at most 38 words per minute is 37.

Then 38 – 37 = 1 person can type faster than 38 words per minute.

$\therefore$ percentage of Group B who can type faster than 38 words per minute is $\frac{1}{38} \times 100\% =$ **2.63%** (to 3 s.f.)

(v) **No**. Since the median typing speed for Group A is higher than Group B, the people in Group A generally type faster than Group B. However, the interquartile range for Group A is higher than Group B, which means that Group A is less consistent than Group B.

12. **(a)** From the graph of waiting times on a weekday,
Lower quartile = 27 minutes
Upper quartile = 52 minutes
∴ interquartile range = 52 – 27
= **25 minutes**

(b) From the graph of waiting times on a weekend, the number of people who waited for less than 64 minutes = 600
∴ the number of people who waited for at least 64 minutes on a weekend is 800 – 600 = **200**

(c) From the weekday curve, the number of people who waited for less than 36 minutes is 340.
From the weekend curve, 340 people waited for less than 52 minutes.
∴ David would have waited for **52 minutes** if he were to visit on a weekend.

(d) **(i)** **Weekday**, because the median waiting time of 40 minutes on a weekday is shorter than the median waiting time of 55 minutes on a weekend.

(ii) **Yes**. The median waiting time is longer on a weekend, and the interquartile range of the waiting time on a weekend is 64 – 42 = 22 minutes, which is lower than that on a weekday, indicating a smaller spread and hence more consistent waiting time.

13. **(i)** 16, 21, 23, 37, 50, 53, 65, 80, 81, 100

(a) Range = 100 – 16 = **84**

(b) Median = $\frac{50+53}{2}$ = **51.5**

(c) Lower quartile = 23
Upper quartile = 80
∴ interquartile range = 80 – 23 = **57**

(ii) 79, 99, 103, 114, 121, 171, 198, 200, 235, 308

(a) Range = 308 – 79 = **229**

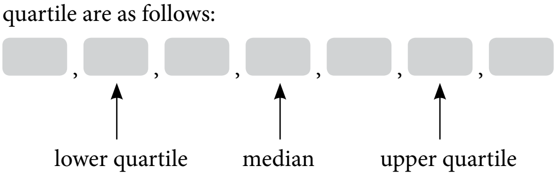
(b) Median = $\frac{121+171}{2}$ = **146**

(c) Lower quartile = 103
Upper quartile = 200
∴ interquartile range = 200 – 103 = **97**

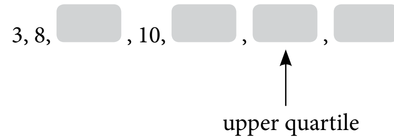
(iii) **City Y**

(iv) City Y has a poorer air quality because its median PSI reading is much higher than City X. City X air quality is generally more consistent because its interquartile range is smaller than City Y.

14. Given that there are 7 integers, arranging them in ascending order, the positions of the lower quartile, median and upper quartile are as follows:



It is given that the median is 10, the lower quartile is 8, and the smallest integer is 3:



Since the mode is 19, it has to occur at least twice, and it must be in the 5th, 6th or 7th position since it is larger than the median. Since it is also given that the largest integer is twice the upper quartile, the largest number cannot be 19.
Thus 19 occupies the 5th and 6th position, of which the 6th position is also the upper quartile. Therefore, the largest number is 38:

3, 8, ☐, 10, 19, 19, 38

Since the mode is 19, no other integers can appear twice. The only integer between 8 and 10 is 9.
∴ the integers are **{3, 8, 9, 10, 19, 19, 38}**.

15. **(i)** **(a)** From the graph, lower quartile = **5.5 hours**
upper quartile = **7 hours**
40th percentile = **6 hours**
80th percentile = **7.2 hours**

(b) Interquartile range = 7 – 5.5
= **1.5 hours**

(ii) From the graph, the number of students who slept at least 6 hours = 24
Then 40 – 24 = 16 students slept for less than 6 hours.
∴ fraction of students who slept for less than 6 hours is $\frac{16}{40} = \mathbf{\frac{2}{5}}$

(iii) Number of students who had a duration of sleep that was between 6 and 7.2 hours is 24 – 8 = 16.
∴ P(both students had a duration of sleep that was between 6 and 7.2 hours) = $\frac{16}{40} \times \frac{15}{39}$
= $\mathbf{\frac{2}{13}}$

(iv) Percentage of the class who had a duration of sleep that was between 6 and 7.2 hours is $\frac{2}{13} \times 100\%$ = 15.4% (to 3 s.f.)
∴ the above data does not support the claim that only 10% of the class had a duration of sleep that was between 6 and 7.2 hours.

3.3 Further comparison of data

Practise Now 6

(i) Mean = $\frac{6 + 9 + 15 + 26 + 10 + 14 + 21 + 3}{8}$
= **13**
Range = 26 – 3
= **23**

(ii) The mean number of grammatical errors made by Bernard is slightly greater than that made by Yasir, indicating that on average, Bernard made more grammatical errors per essay. The number of grammatical errors made by Bernard has a greater range than that made by Yasir, which indicates that there is a greater spread of errors in his essays as compared to Yasir.

Practise Now 7

(i) Rearranging the data in ascending order:

17.1, 17.1, 17.1, 17.7, 17.9, 18.1, 18.3, 18.4, 18.4, 18.9

$$\text{Mean length} = \frac{\text{sum of data values}}{\text{number of data values}} = \frac{179}{10} = \mathbf{17.9\ cm}$$

Total number of data values, $n = 10$

$$\text{Position of median} = \frac{n+1}{2} = \frac{10+1}{2} = 5.5$$

$$\text{Median length} = \frac{17.9 + 18.1}{2} = \mathbf{18\ cm}$$

Modal length = **17.1 cm**

Range = largest value – smallest value
= 18.9 – 17.1
= **1.8 cm**

(ii) Since the mean, median and modal lengths of pencils in box A are all less than the respective mean, median and modal lengths of pencils in box B, the pencils in box A are shorter than that in box B. In addition, the range of the lengths of pencils in box A is greater than the range of the lengths of pencils in box B, which indicates that there is a greater spread in the lengths of pencils in box A.

Exercise 3C

1. **(i)**
$$\text{Mean for Group A} = \frac{4+6+6+7+8+10+11+12}{8} = \mathbf{8\ days}$$

Range for Group A = 12 – 4
= **8 days**

$$\text{Mean for Group B} = \frac{0+1+1+2+3+14+17+25}{8} = \mathbf{7.875\ days}$$

Range for Group B = 25 – 0
= **25 days**

(ii) The mean for both groups are about the same, indicating that on average, the number of days working adults from both groups cooked dinner are about the same.
The range for Group B is much greater than that for Group A, which indicates that there is a much greater spread in the number of days working adults from Group B cooked dinner.

2. **(i)** For the school week,

$$\text{Mean} = \frac{4+13+6+11+13+21+26}{7} = \mathbf{13.4\ minutes}\ \text{(to 3 s.f.)}$$

Range = 26 – 4
= **22 minutes**

(ii) For the week during the school holidays,

$$\text{Mean} = \frac{22+14+7+12+27+5+14}{7} = \mathbf{14.4\ minutes}\ \text{(to 3 s.f.)}$$

Range = 27 – 5
= **22 minutes**

(iii) The mean time taken by Shaha to make breakfast in the week during the school holidays is slightly greater than in the school week, indicating that on average, Shaha took a longer time to make breakfast in the week during school holidays.
The range of time taken by Shaha to make breakfast in the school week is the same as that in the week during the school holidays, which indicates that the spread in the time taken in both weeks are the same.

3. **(i)** For Train A,

$$\text{Mean} = \frac{(2\times3)+(3\times2)+(4\times5)+(5\times12)+(6\times10)+(7\times6)+(8\times1)+(9\times1)}{8} = \mathbf{5.275\ minutes}$$

Range = 9 – 2
= **7 minutes**

For Train B,

$$\text{Mean} = \frac{(2\times4)+(3\times3)+(4\times9)+(5\times9)+(6\times7)+(7\times5)+(8\times3)+(9\times0)}{8} = \mathbf{4.975\ minutes}$$

Range = 8 – 2
= **6 minutes**

(ii) **Train B** because the range for Train B is lower, indicating that there is a smaller spread of time taken, which means it arrives late more consistently.

(iii) **Train B** because the mean time it takes to arrive after the scheduled time is less, indicating that on average, it arrives closer to the scheduled time compared to Train A.

4. **(i)** **(a)** Estimated mean waiting time

$$= \frac{(5\times21)+(11\times23)+(27\times25)+(13\times27)+(4\times29)}{5+11+27+13+4} = \mathbf{25\ minutes}$$

(b) Estimated range of the waiting time = 29 – 21
= **8 minutes**

(ii) The mean for both hospitals are the same, indicating that on average, the waiting times at both hospitals are the same.
The range of the waiting time at Hospital B is greater than that for Hospital A, which indicates that there is a greater spread in the waiting times at Hospital B.

5. (i) (a) For City A,

Estimated mean temperature

$$= \frac{(0 \times 37.5) + (4 \times 42.5) + (12 \times 47.5) + (23 \times 52.5) + (8 \times 57.5) + (3 \times 62.5)}{0 + 4 + 12 + 23 + 8 + 3}$$

= **51.9 °C**

For City B,

Estimated mean temperature

$$= \frac{(2 \times 37.5) + (14 \times 42.5) + (16 \times 47.5) + (10 \times 52.5) + (5 \times 57.5) + (3 \times 62.5)}{2 + 14 + 16 + 10 + 5 + 3}$$

= **48.6 °C**

(b) For City A,

Estimated range = 62.5 – 42.5

= **20 °C**

For City B,

Estimated range = 62.5 – 37.5

= **25 °C**

(c) Position of median $= \frac{50 + 1}{2}$

= 25.5

For City A,

Class interval where median lies: $\mathbf{50 \leqslant x < 55}$

For City B,

Class interval where median lies: $\mathbf{45 \leqslant x < 50}$

(d) For City A,

Model class: $\mathbf{50 \leqslant x < 55}$

For City B,

Model class: $\mathbf{45 \leqslant x < 50}$

(ii) **City A** is warmer on the whole because its estimated mean temperature and median temperature is greater than City B, indicating that on average, the temperature in City A is higher.

(iii) **City B** has a greater variability in its daily temperature because the estimated range of temperatures in City B is greater than that in City A, indicating a wider spread of temperatures.

6. (i) Position of median $= \frac{12 + 1}{2}$

= 6.5

For Class P,

Mean reading time $= \frac{(0 \times 4) + (1 \times 2) + (2 \times 1) + (3 \times 0) + (4 \times 3) + (5 \times 2)}{4 + 2 + 1 + 0 + 3 + 2}$

= **2.17 hours** (to 3 s.f.)

Median reading time $= \frac{1 + 2}{2}$

= **1.5 hours**

Modal reading time = **0 hour**

Range of reading times = 5 – 0

= **5 hours**

For Class Q,

Mean reading time $= \frac{(0 \times 0) + (1 \times 1) + (2 \times 0) + (3 \times 2) + (4 \times 5) + (5 \times 4)}{0 + 1 + 0 + 2 + 5 + 4}$

= **3.92 hours** (to 3 s.f.)

Median reading time = **4 hours**

Modal reading time = **4 hours**

Range of reading times = 5 – 1

= **4 hours**

(ii) **Class Q** because its range of reading times is smaller than that of Class P.

(iii) **Class Q** because its mean, median and modal reading times are greater than that of Class P.

7. (i) **Yes**, because the data is taken from equal number of students per school.

So if we find the sum of masses of the 100 students for each school and add them up, then take the mean of a total of 200 students, we get $\frac{\sum fx + \sum fy}{200} = \frac{100\bar{x} + 100\bar{y}}{200} = \frac{\bar{x} + \bar{y}}{2}$.

(Since the mean mass of each student from School R and School S are $\bar{x} = \frac{\sum fx}{100}$ and $\bar{y} = \frac{\sum fy}{100}$ respectively, $\sum fx = 100\bar{x}$ and $\sum fy = 100\bar{y}$.)

(ii) **No**, because the combined range is 80 kg – 45 kg = 35 kg, which is not equal to the sum of respective range of 70 kg – 45 kg = 25 kg and 80 kg – 40 kg = 40 kg.

(iii) Combined mean

$$= \frac{(7 \times 40) + (26 \times 45) + (60 \times 50) + (34 \times 55) + (25 \times 60) + (33 \times 65) + (10 \times 70) + (1 \times 75) + (4 \times 80)}{200}$$

$$= \frac{11\,060}{200}$$

= **55.3 kg**

Combined range = 80 – 40

= **40 kg**

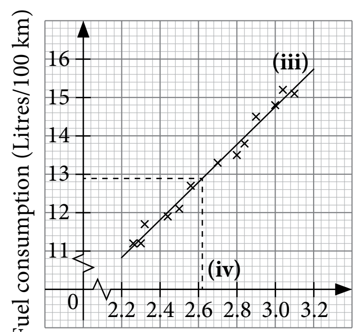
3.4 Scatter diagrams

Thinking Time (Page 102)

Yes, the equation of the line of best fit can similarly be found by obtaining two points on the line to find the gradient and the y-intercept, which was covered in Book 3.

Practise Now 8

(i)



(ii) **Strong positive correlation**

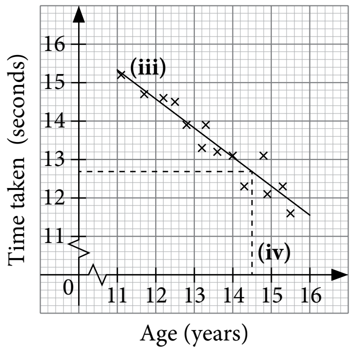
(iii) The line of best fit is drawn with approximately half of the plots lying above the line and the other half lying below the line of best fit.

(iv) Using the line of best fit on the scatter diagram, a truck with a total mass of 2.62 tons is estimated to need **12.9 litres** of petrol.

(v) **No**, it would be unreliable since the mass of 4.8 tons lies outside of the range as the truck did not carry more than 3.1 tons of mass in total.

Practise Now 9

(i)



(ii) **Strong negative correlation**

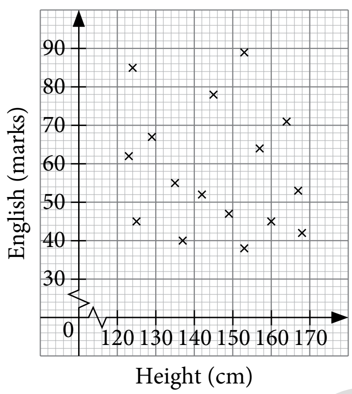
(iii) The line of best fit is drawn with approximately half of the plots lying above the line and the other half lying below the line of best fit.

(iv) Using the line of best fit on the scatter diagram, a pupil aged 14.5 years is expected to complete the 100 m race in **12.7 seconds**.

(v) **No**, it would be unreliable since the age of 17 years lies outside of the range as no pupil older than 16 years old took part in the race.

Class Discussion (Scatter diagram with no correlation)

1.



2. There is no correlation between the height of the pupils and the number of marks they scored in the English examination, as no clear trend can be observed.

Thinking Time (Page 106)

1. **Positive**. This is true in most cases for students preparing for an examination.
2. **Positive**. More cars on the road generally will cause more traffic accidents.
3. **No correlation**. There is no relationship between the house number and the weight of an occupant.
4. **Positive**. The amount of interest one earns depends on the principal he has.
5. **No correlation**. The price of a shirt usually varies by design, and shirts of the same design can also have small variations, such as colour.
6. **Negative**. The amount of food one can consume depends on its accumulated quantity; the smaller the cupcake, the more cupcakes that can be eaten.
7. **No correlation**. Human beings generally have 10 fingers and the height of a man does not affect the number of fingers he has.
8. **Positive**. It is assumed that the sprinter competes in the same league.

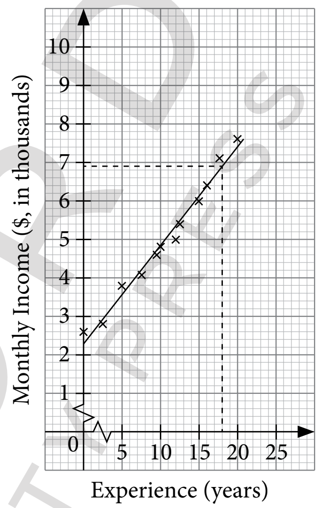
Journal Writing (Page 106)

Teachers can get students to come up with a list of variables, and assess their understanding. A few of these answers can be used for a class discussion to further consolidate what was covered in this section.

Exercise 3D

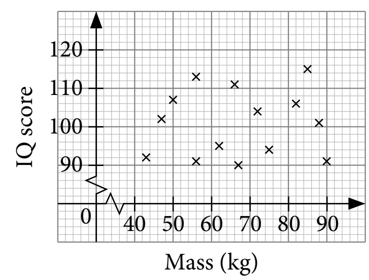
1. **(a)** **Strong negative correlation**
(b) **Moderate positive correlation**
(c) **Strong positive correlation**
(d) **No correlation**

2.



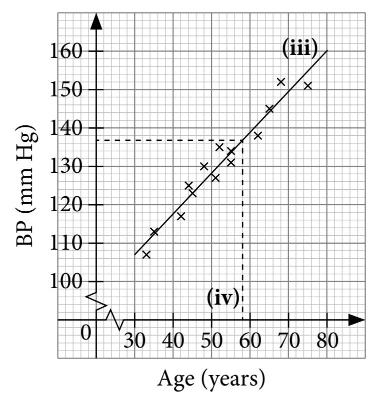
Using the line of best fit on the scatter diagram, the estimated monthly income earned by an individual with 18 years of experience is **$6900**.

3. **(i)**



(ii) **No correlation**

4. **(i)**



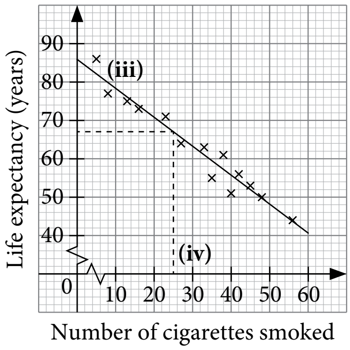
(ii) **Strong positive correlation**

(iii) The line of best fit is drawn with approximately half of the plots lying above the line and the other half lying below the line of best fit.

(iv) Using the line of best fit on the scatter diagram, the blood pressure of a 58-year-old lady could be **137 mm Hg**.

(v) **No**, it would be unreliable because the age of 86 years lies outside of the range of the data and that the data involved only women.

5. **(i)**



(ii) **Strong negative correlation**

(iii) The line of best fit is drawn with approximately half of the plots lying above the line and the other half lying below the line of best fit.

(iv) Using the line of best fit on the scatter diagram, the life expectancy of a man who smoked 25 cigarettes per day could be **67 years**.

(v) **No**, it would be unreliable because the data collected consists only smokers.

Chapter 4 Vectors

TEACHING NOTES

Suggested Approach

This topic could be difficult for the average-ability students. It would be easier for the students to understand this topic if vector diagrams on the Cartesian plane are used to explain the different concepts. Teachers should give students more examples to practise in each section as this will help them to master the concepts well.

Section 4.1: Vectors in two dimensions

As students might have difficulties comprehending the concept of vectors, teachers should help them distinguish between scalar and vector quantities (see Introductory Problem on page 112 of the textbook). Opportunities should also be given to students to discuss on more examples of scalars and vectors. Students should be able to explain why their examples are scalars or vectors. With that, teachers should spend some time to go through the different representations of vectors as well as how a vector can be described using a column vector with the aid of a Cartesian plane.

Section 4.2: Addition of vectors

In this section, introduce the idea of vector addition using the scenario on page 123 of the textbook. With the use of diagrams, students should better understand both the Triangle Law and Parallelogram Law of Vector Addition. A possible misconception that students might have is to simply add the magnitudes of vectors when finding the magnitude of the resultant vector. For example, on page 123 of the textbook, they might find the distance between P and R by taking the sum of 2.1 km and 1.9 km, which is 4 km. As such, teachers may address this misconception by using Worked Example 4 to help students understand that $|a + b|$ is not equal to $|a| + |b|$ in general.

Some students might also have misconceptions about the zero vector. Teachers are encouraged to go through examples of zero vectors in real-world contexts with students to help them understand (see Class Discussion: The zero vector).

Section 4.3: Vector subtraction

For this section, introduce the idea of vector subtraction using the addition of negative vectors or Triangle Law of Vector Subtraction with the aid of diagrams. Students should be given more practice to draw the resultant vector from examples of vector subtraction as this will help them to better understand the concepts.

Section 4.4: Scalar multiples of a vector

For scalar multiplication, a diagram to illustrate the concept should be used to help students understand. Students should be given time to think about how a positive or negative value of k affects a vector $k\mathbf{a}$, given a vector $\mathbf{a}$ (see Thinking Time on page 142 of the textbook).

Section 4.5: Expression of a vector in terms of two other vectors

Recap that the sum or difference of two vectors is also a vector. Hence, any vector can be expressed as the sum or difference of two other vectors. Teachers should encourage students to use squared papers or graph papers to draw the required vector as it allows better visualisation of how it can be formed using the sum or difference of the other vectors.

Section 4.6: Position vectors

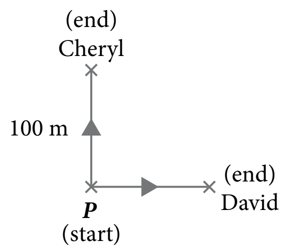
Teachers should explain that the position vector of a point must have a fixed starting point, which is usually, but not limited to, the origin O on a Cartesian plane. Hence, students should understand that any vector $\overrightarrow{AB}$ on the Cartesian plane can be expressed in terms of the position vectors of A and B. They should also know that the vector $\overrightarrow{AB}$ can be regarded as a movement from A to B, which is known as the translation vector.

Section 4.7: Applications of vectors

In this section, students should be given opportunities to discuss real-life examples of resultant vectors (see Class Discussion: Real-life applications of the resultant vector of two vectors or of the difference between two vectors). Students will also solve geometric problems with what they have learnt in this chapter. Teachers may highlight to students that to find the ratio of the area of two triangles, they must be able to find a relationship between the triangles, for example, triangles with a common height, or similar triangles (students will learn about areas of similar triangles in Chapter 12). There could also be an intermediate triangle that connects the two triangles.

Introductory Problem

1.



2. **No**. They both walked in different directions.
3. We need to know the **direction** of the distance travelled.
4. The displacement of David from P is 100 m in the east direction.
5. Examples of scalars: mass, temperature, volume, energy and time
   Examples of vectors: weight, force, momentum and electric field

4.1 Vectors in two dimensions

Practise Now 1A

$\overrightarrow{AB} = \begin{pmatrix} \mathbf{-3} \\ \mathbf{4} \end{pmatrix}$ $\qquad |\overrightarrow{AB}| = \sqrt{(-3)^2 + 4^2}$

$= \sqrt{25}$

$= \mathbf{5}$ **units**

$\mathbf{c} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{-2} \end{pmatrix}$ $\qquad |\mathbf{c}| = \sqrt{(-2)^2 + (-2)^2}$

$= \sqrt{8}$

$= \mathbf{2.83}$ **units** (to 3 s.f.)

$\overrightarrow{DE} = \begin{pmatrix} \mathbf{5} \\ \mathbf{0} \end{pmatrix}$ $\qquad |\overrightarrow{DE}| = \sqrt{5^2 + 0^2}$

$= \sqrt{25}$

$= \mathbf{5}$ **units**

$\mathbf{f} = \begin{pmatrix} \mathbf{0} \\ \mathbf{-3} \end{pmatrix}$ $\qquad |\mathbf{f}| = \sqrt{(0)^2 + (-3)^2}$

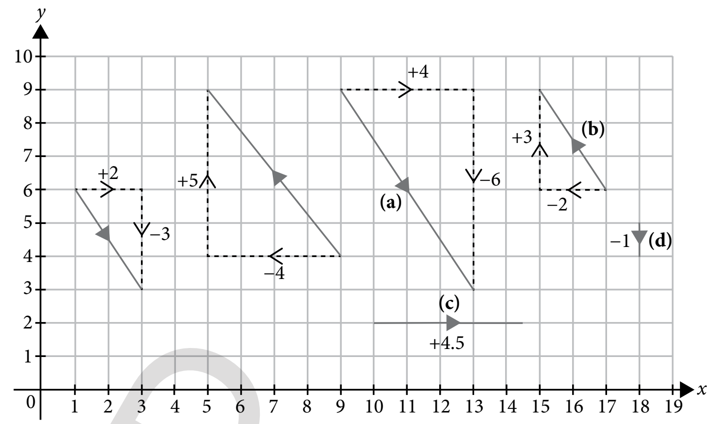
$= \sqrt{9}$

$= \mathbf{3}$ **units**

Class Discussion (Equal vectors)

1. **(i)** The magnitudes of vectors **a** and **b** are equal.
   We can confirm that their magnitudes are equal by measuring their lengths or by calculating their lengths using Pythagoras' Theorem.
   (ii) Vectors **a** and **b** have the same direction.
   We can confirm that their directions are equal by calculating their gradient using $\dfrac{\text{vertical change}}{\text{horizontal change}}$.
   (iii) The x- and y-components of vectors **a** and **b** are the same.
2. **(i)** The magnitudes of vectors **a** and **c** are the same.
   (ii) Vectors **a** and **c** are in opposite directions.
   (iii) The x- and y-components of vectors **a** and **c** have the same absolute value but opposite signs.
3. Vectors **a** and **d** have the same magnitude but different directions.
4. Vectors **a** and **e** have the same direction, but the magnitude of vector **a** is 2 times the magnitude of vector **e**.

Practise Now 1B



Thinking Time (Page 116)

(a) **(i)** **No**, they are not equal.

(ii) The magnitude of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is half that of $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$, and both $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ have the same direction. Both vectors are parallel.

(b) **(i)** **No**, they are not equal.

(ii) The magnitudes of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ are equal, but both are in opposite directions. Both vectors are parallel, and they are each a negative vector of the other.

Practise Now 2

(i) **(a)** $\begin{pmatrix} x + 2 \\ 4 - y \end{pmatrix} = \begin{pmatrix} 10 - x \\ y - 5 \end{pmatrix}$

$x + 2 = 10 - x$ and $4 - y = y - 5$

$2x = 8 \qquad 2y = 9$

$x = 4 \qquad y = 4\frac{1}{2}$

$\therefore$ $\mathbf{x = 4}$ **and** $\mathbf{y = 4\frac{1}{2}}$

(b) $\mathbf{a} = \begin{pmatrix} x + 2 \\ 4 - y \end{pmatrix}$

$= \begin{pmatrix} 4 + 2 \\ 4 - 4\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -\frac{1}{2} \end{pmatrix}$

Negative of **a** = −**a**

$= -\begin{pmatrix} 6 \\ -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-6} \\ \mathbf{\frac{1}{2}} \end{pmatrix}$

(c) $|\mathbf{a}| = \sqrt{6^2 + \left(-\frac{1}{2}\right)^2}$

$= \sqrt{36 + \frac{1}{4}}$

$= \sqrt{\frac{145}{4}}$ units

Since $\mathbf{b} = \mathbf{a} = \begin{pmatrix} 6 \\ -\frac{1}{2} \end{pmatrix}$, then $|\mathbf{b}| = \sqrt{6^2 + \left(-\frac{1}{2}\right)^2}$

$= \sqrt{\frac{145}{4}}$ units

$\therefore |\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{145}{4}}$ units (shown)

(ii) (a)

$$|\mathbf{a}| = |\mathbf{b}|$$
$$\sqrt{(x+2)^2 + (4-y)^2} = \sqrt{(10-x)^2 + (y-5)^2}$$
$$(x+2)^2 + (4-y)^2 = (10-x)^2 + (y-5)^2$$
$$x^2 + 4x + 4 + 16 - 8y + y^2 = 100 - 20x + x^2 + y^2 - 10y + 25$$
$$4x - 8y + 20 = -20x - 10y + 125$$
$$2y = 105 - 24x$$
$$y = \mathbf{\frac{105 - 24x}{2}}$$

(b) Although they have the same magnitude, **a** and **b** may have different directions. In $y = \frac{105 - 24x}{2}$, x and y have no fixed values.

For example, if $x = 0$, $y = \frac{105 - 0}{2} = 52\frac{1}{2}$

Then $\mathbf{a} = \begin{pmatrix} x+2 \\ 4-y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 10-x \\ y-5 \end{pmatrix}$

$= \begin{pmatrix} 0+2 \\ 4-52\frac{1}{2} \end{pmatrix}$ $\quad = \begin{pmatrix} 10-0 \\ 52\frac{1}{2}-5 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -48\frac{1}{2} \end{pmatrix}$ $\quad = \begin{pmatrix} 10 \\ 47\frac{1}{2} \end{pmatrix}$

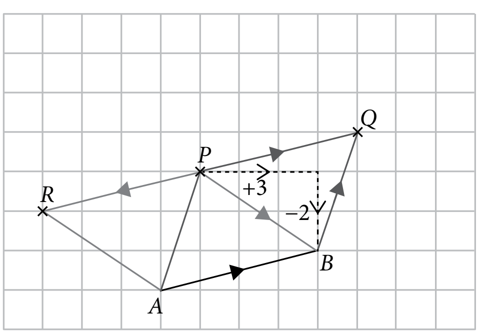
$|\mathbf{a}| = \sqrt{2^2 + \left(-48\frac{1}{2}\right)^2}$ and $|\mathbf{b}| = \sqrt{10^2 + \left(47\frac{1}{2}\right)^2}$

$= \sqrt{4 + \frac{9409}{4}}$ $\quad = \sqrt{100 + \frac{9025}{4}}$

$= \sqrt{\frac{9425}{4}}$ units $\quad = \sqrt{\frac{9425}{4}}$ units

$\therefore |\mathbf{a}| = |\mathbf{b}|$ but $\mathbf{a} \neq \mathbf{b}$ in this case.

Practise Now 3



(i) $\overrightarrow{PB} = \begin{pmatrix} \mathbf{3} \\ \mathbf{-2} \end{pmatrix}$

(ii) $\overrightarrow{BQ} = \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \end{pmatrix}$

(iii) $\overrightarrow{PR} = \begin{pmatrix} \mathbf{-4} \\ \mathbf{-1} \end{pmatrix}$

(iv) $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude but $\overrightarrow{PQ} \neq \overrightarrow{PR}$ because they do not have the same direction.

Exercise 4A

1. **(a)** Magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \sqrt{3^2 + 4^2}$

$= \sqrt{25}$

= 5 units

(b) Magnitude of $\begin{pmatrix} -5 \\ 12 \end{pmatrix} = \sqrt{(-5)^2 + 12^2}$

$= \sqrt{169}$

= 13 units

(c) Magnitude of $\begin{pmatrix} -7 \\ -2 \end{pmatrix} = \sqrt{(-7)^2 + (-2)^2}$

$= \sqrt{53}$

= 7.28 units (to 3 s.f.)

(d) Magnitude of $\begin{pmatrix} 0 \\ -6\frac{1}{2} \end{pmatrix} = \sqrt{0^2 + \left(-6\frac{1}{2}\right)^2}$

$= \sqrt{\frac{169}{4}}$

= 6.5 units

(e) Magnitude of $\begin{pmatrix} 8 \\ 0 \end{pmatrix} = \sqrt{8^2 + 0^2}$

$= \sqrt{64}$

= 8 units

2. **(a)** $-\begin{pmatrix} 12 \\ -7 \end{pmatrix} = \begin{pmatrix} \mathbf{-12} \\ \mathbf{7} \end{pmatrix}$

(b) $-\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix}$

(c) $-\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} \mathbf{-4} \\ \mathbf{-8} \end{pmatrix}$

(d) $-\begin{pmatrix} -3 \\ -1.2 \end{pmatrix} = \begin{pmatrix} \mathbf{3} \\ \mathbf{1.2} \end{pmatrix}$

(e) $-\begin{pmatrix} 0 \\ 3\frac{1}{4} \end{pmatrix} -= \begin{pmatrix} \mathbf{0} \\ \mathbf{-3\frac{1}{4}} \end{pmatrix}$

3. $\begin{pmatrix} a \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ a + 2b \end{pmatrix}$

$a = -2$ and $3 = a + 2b$

$2b = 3 - (-2)$

$b = \frac{5}{2}$

$= 2\frac{1}{2}$

$\therefore$ ***a* = −2 and *b* = $\mathbf{2\frac{1}{2}}$**

4. **(i)** $|\overrightarrow{AB}| = \sqrt{7^2 + 0^2}$

$= \sqrt{49}$

$= \mathbf{7}$ **units**

(ii) Since $ABCD$ is a parallelogram, AB // DC and AD // BC.

(a) $\overrightarrow{DC} = \overrightarrow{AB}$

$= \begin{pmatrix} \mathbf{7} \\ \mathbf{0} \end{pmatrix}$

(b) $\overrightarrow{DA} = -\overrightarrow{AD}$

$= -\overrightarrow{BC}$

$= -\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{3} \\ \mathbf{-4} \end{pmatrix}$

5. $\overrightarrow{AB} = \begin{pmatrix} \mathbf{-4} \\ 3 \end{pmatrix}$ $\quad |\overrightarrow{AB}| = \sqrt{(-4)^2 + 3^2}$

$= \sqrt{25}$

$= \mathbf{5}$ **units**

$\overrightarrow{CD} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-2} \end{pmatrix}$ $\quad |\overrightarrow{CD}| = \sqrt{1^2 + (-2)^2}$

$= \sqrt{5}$

$= \mathbf{2.24}$ **units** (to 3 s.f.)

$\mathbf{p} = \begin{pmatrix} \mathbf{3} \\ \mathbf{3} \end{pmatrix}$ $\quad |\mathbf{p}| = \sqrt{3^2 + 3^2}$

$= \sqrt{18}$

$= \mathbf{4.24}$ **units** (to 3 s.f.)

$\mathbf{q} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{-1} \end{pmatrix}$ $\quad |\mathbf{q}| = \sqrt{(-2)^2 + (-1)^2}$

$= \sqrt{5}$

$= \mathbf{2.24}$ **units** (to 3 s.f.)

$\overrightarrow{RS} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{0} \end{pmatrix}$ $\quad |\overrightarrow{RS}| = \sqrt{(-2)^2 + 0^2}$

$= \sqrt{4}$

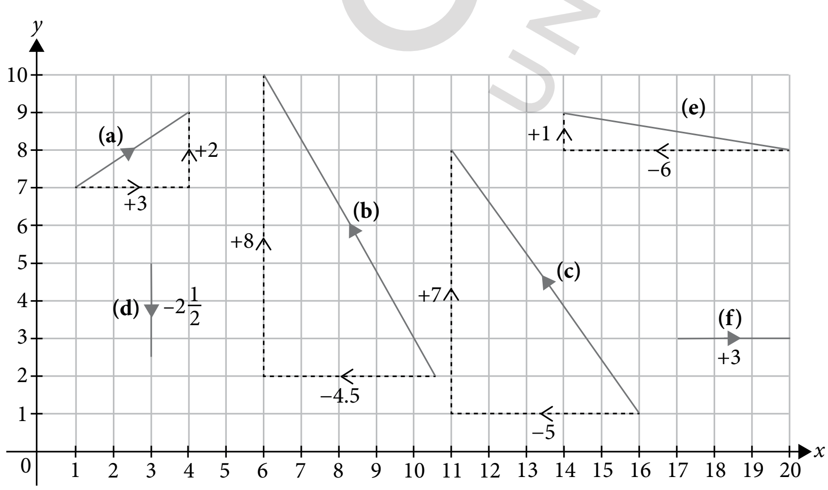
$= \mathbf{2}$ **units**

$\overrightarrow{TU} = \begin{pmatrix} \mathbf{0} \\ \mathbf{4} \end{pmatrix}$ $\quad |\overrightarrow{TU}| = \sqrt{0^2 + 4^2}$

$= \sqrt{16}$

$= \mathbf{4}$ **units**

6.



7. **(i)** **(a)** $\begin{pmatrix} x-3 \\ 2-y \end{pmatrix} = \begin{pmatrix} 5-x \\ y-9 \end{pmatrix}$

$x - 3 = 5 - x$ and $2 - y = y - 9$

$2x = 8 \qquad 2y = 11$

$x = 4 \qquad y = 5\frac{1}{2}$

$\therefore \mathbf{x = 4}$ **and** $\mathbf{y = 5\frac{1}{2}}$

(b) $\mathbf{a} = \begin{pmatrix} x-3 \\ 2-y \end{pmatrix}$

$= \begin{pmatrix} 4-3 \\ 2-5\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -3\frac{1}{2} \end{pmatrix}$

Negative of $\mathbf{a} = -\mathbf{a}$

$= -\begin{pmatrix} 1 \\ -3\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-1} \\ \mathbf{3\frac{1}{2}} \end{pmatrix}$

(c) $|\mathbf{a}| = \sqrt{1^2 + \left(-3\frac{1}{2}\right)^2}$

$= \sqrt{1 + \left(-\frac{7}{2}\right)^2}$

$= \sqrt{1 + \frac{49}{4}}$

$= \sqrt{\frac{53}{4}}$ units

Since $\mathbf{b} = \mathbf{a} = \begin{pmatrix} 1 \\ -3\frac{1}{2} \end{pmatrix}$, then $|\mathbf{b}| = \sqrt{1^2 + \left(-3\frac{1}{2}\right)^2}$

$= \sqrt{\frac{53}{4}}$ units

$\therefore |\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{53}{4}}$ units (shown)

(ii) **(a)** $|\mathbf{a}| = |\mathbf{b}|$

$\sqrt{(x-3)^2 + (2-y)^2} = \sqrt{(5-x)^2 + (y-9)^2}$

$(x-3)^2 + (2-y)^2 = (5-x)^2 + (y-9)^2$

$x^2 - 6x + 9 + 4 - 4y + y^2 = 25 - 10x + x^2 + y^2 - 18y + 81$

$-6x - 4y + 13 = -10x - 18y + 106$

$14y = 93 - 4x$

$\mathbf{y = \frac{93 - 4x}{14}}$

(b) Although they have the same magnitude, **a** and **b** may have different directions. In $y = \frac{93 - 4x}{14}$, x and y have no fixed values.

For example, if $x = 0$, $y = \frac{93 - 0}{14} = 6\frac{9}{14}$

Then $\mathbf{a} = \begin{pmatrix} x - 3 \\ 2 - y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 - x \\ y - 9 \end{pmatrix}$

$= \begin{pmatrix} 0 - 3 \\ 2 - 6\frac{9}{14} \end{pmatrix}$ $= \begin{pmatrix} 5 - 0 \\ 6\frac{9}{14} - 9 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ -4\frac{9}{14} \end{pmatrix}$ $= \begin{pmatrix} 5 \\ -2\frac{5}{14} \end{pmatrix}$

$|\mathbf{a}| = \sqrt{(-3)^2 + \left(-4\frac{9}{14}\right)^2}$ and $|\mathbf{b}| = \sqrt{5^2 + \left(-2\frac{5}{14}\right)^2}$

$= \sqrt{9 + \frac{4225}{196}}$ $= \sqrt{25 + \frac{1089}{196}}$

$= \sqrt{\frac{5989}{196}}$ units $= \sqrt{\frac{5989}{196}}$ units

$\therefore$ $|\mathbf{a}| = |\mathbf{b}|$ but $\mathbf{a} \neq \mathbf{b}$ in this case.

8. **(i)** $|\overrightarrow{AB}| = \sqrt{(-3)^2 + 4^2}$

$= \sqrt{25}$

$= 5$ units

$|\overrightarrow{CD}| = \sqrt{0^2 + 5^2}$

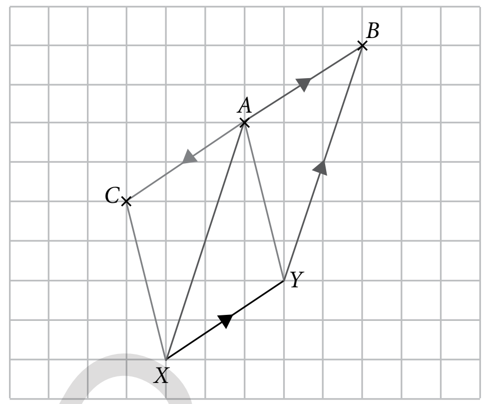
$= \sqrt{25}$

$= 5$ units

$\therefore$ $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ (shown)

(ii) $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ only indicates that the magnitudes of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal. From the column vectors given, they have different directions.

9.



(i) $\overrightarrow{AY} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-4} \end{pmatrix}$

(ii) $\overrightarrow{YB} = \begin{pmatrix} \mathbf{2} \\ \mathbf{6} \end{pmatrix}$

(iii) $\overrightarrow{AC} = \begin{pmatrix} \mathbf{-3} \\ \mathbf{-2} \end{pmatrix}$

(iv) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have the same magnitude but $\overrightarrow{AB} \neq \overrightarrow{AC}$ because they do not have the same direction.

10. $|\mathbf{a}| = 7$

$\sqrt{n^2 + (-3)^2} = 7$

$\sqrt{n^2 + 9} = 7$

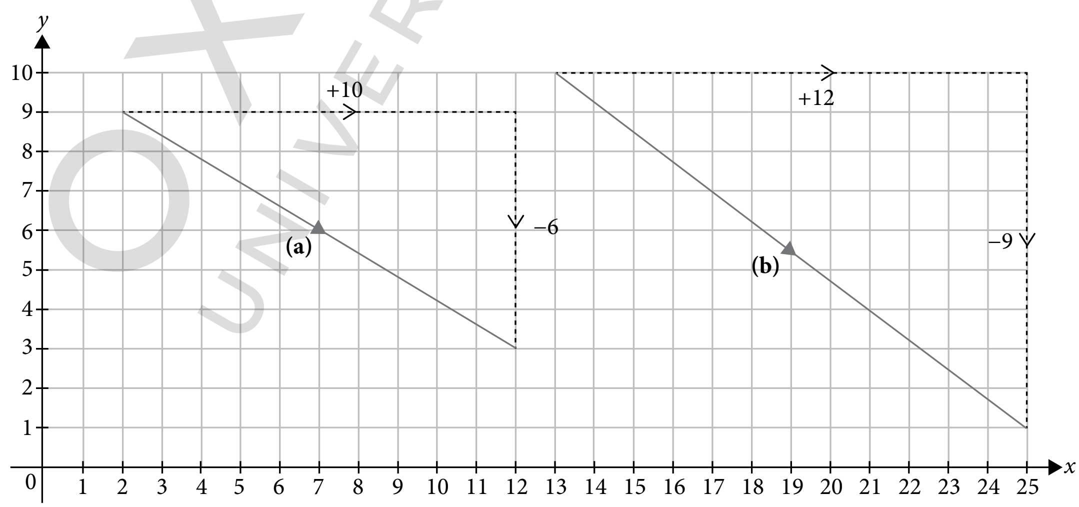
$n^2 + 9 = 49$

$n^2 = 40$

$n = \pm\sqrt{40}$

$\therefore$ the possible values of n are $\mathbf{\sqrt{40}}$ **and** $\mathbf{-\sqrt{40}}$.

11.



(a) Two times of $\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \end{pmatrix}$

(b) Three times of the negative of $\begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ -9 \end{pmatrix}$

12. $\begin{pmatrix} 13s \\ 4t \end{pmatrix} = \begin{pmatrix} 6t + 20 \\ 18 - 7s \end{pmatrix}$

$$13s = 6t + 20$$
$$s = \frac{6t + 20}{13} \quad \text{— (1)}$$
$$4t = 18 - 7s$$
$$7s + 4t = 18 \quad \text{— (2)}$$

Substitute (1) into (2):

$$7\left(\frac{6t + 20}{13}\right) + 4t = 18$$
$$42t + 140 + 52t = 234$$
$$94t = 94$$
$$t = 1$$

Substitute $t = 1$ into (1):

$$s = \frac{6(1) + 20}{13}$$
$$= 2$$

$\therefore$ **$s = 2$ and $t = 1$**

13. **(i)** **(a)** $\overrightarrow{AB} = \begin{pmatrix} -2.5 \\ -5 \end{pmatrix}$ and $\overrightarrow{IJ} = \begin{pmatrix} -2.5 \\ -5 \end{pmatrix}$.

Hence, they have the same magnitude and direction.

$\therefore \overrightarrow{AB} = \overrightarrow{IJ}$

(b) $\overrightarrow{DC}$ **and** $\overrightarrow{HG}$

(ii) **(a)** $\overrightarrow{JA}$, $\overrightarrow{GD}$ **and** $\overrightarrow{FE}$

(b) $\overrightarrow{GF}$, $\overrightarrow{KJ}$ **and** $\overrightarrow{LA}$

(c) $\overrightarrow{AD}$, $\overrightarrow{JG}$ **and** $\overrightarrow{IH}$

(d) $\overrightarrow{EG}$

(iii) $\overrightarrow{AG} \neq \overrightarrow{DJ}$ because they have different directions.

(iv) $\overrightarrow{BD} \neq \overrightarrow{HJ}$ because they have opposite directions.

(v) **(a)** $\overrightarrow{CB}$

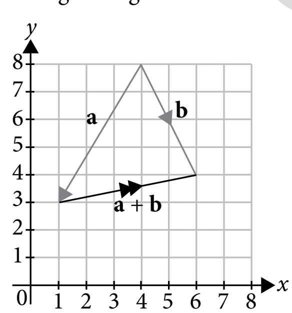
(b) $\overrightarrow{GD}$

(c) $\overrightarrow{FG}$

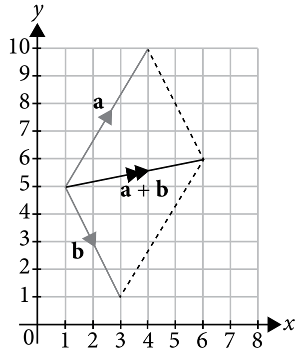
4.2 Addition of vectors

Practise Now 4

1. **(i)** Using Triangle Law of Vector Addition:



Using Parallelogram Law of Vector Addition:



(ii) From the diagram, $\mathbf{a} = \begin{pmatrix} \mathbf{3} \\ \mathbf{5} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-4} \end{pmatrix}$ and

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} \mathbf{5} \\ \mathbf{1} \end{pmatrix}$.

(iii) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$$= \begin{pmatrix} 3 + 2 \\ 5 + (-4) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{5} \\ \mathbf{1} \end{pmatrix}$$

(iv) $|\mathbf{a}| = \sqrt{3^2 + 5^2}$

$$= \mathbf{\sqrt{34}\ units}$$

$|\mathbf{b}| = \sqrt{2^2 + (-4)^2}$

$$= \mathbf{\sqrt{20}\ units}$$

$|\mathbf{a} + \mathbf{b}| = \sqrt{5^2 + 1^2}$

$$= \mathbf{\sqrt{26}\ units}$$

(v) **No**.

$|\mathbf{a} + \mathbf{b}| = \sqrt{26}$

$= 5.10$ units (to 3 s.f.)

$|\mathbf{a}| + |\mathbf{b}| = \sqrt{34} + \sqrt{20}$

$= 10.3$ units (to 3 s.f.)

$\therefore |\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$

2. **(a)** $\begin{pmatrix} 6 \\ 9 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 + 2 \\ 9 + 3 \end{pmatrix}$

$$= \begin{pmatrix} \mathbf{8} \\ \mathbf{12} \end{pmatrix}$$

(b) $\begin{pmatrix} 8 \\ -3 \end{pmatrix} + \begin{pmatrix} -10 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 - 10 + 4 \\ -3 - 5 + 8 \end{pmatrix}$

$$= \begin{pmatrix} \mathbf{2} \\ \mathbf{0} \end{pmatrix}$$

Practise Now 5

(i) $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

(ii) $\overrightarrow{SR} + \overrightarrow{PS} = \overrightarrow{PS} + \overrightarrow{SR}$

$$= \overrightarrow{PR}$$

(iii) $\overrightarrow{PR} + \overrightarrow{RS} + \overrightarrow{SQ} = \overrightarrow{PS} + \overrightarrow{SQ}$

$$= \overrightarrow{PQ}$$

Class Discussion (The zero vector)

1. $\overrightarrow{PR} + \overrightarrow{RP}$ represents the entire journey of the boat, which is a zero displacement of the boat from Changi Jetty (P).

2. Since there is no overall change in position, $\overrightarrow{PR} + \overrightarrow{RP} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Practise Now 6

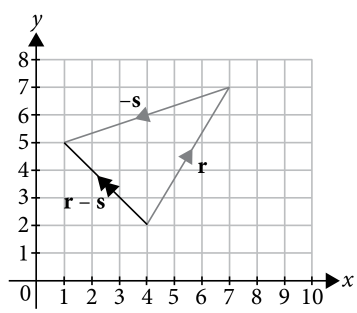
(a) $\begin{pmatrix} 8 \\ -1 \end{pmatrix} + \begin{pmatrix} -8 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$

(b) $\begin{pmatrix} -6 \\ -7 \end{pmatrix} + \begin{pmatrix} \mathbf{6} \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

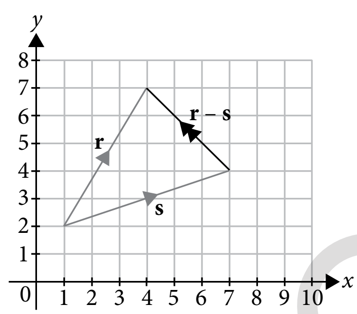
4.3 Vector subtraction

Practise Now 7

(i) Using Addition of negative vector:



Using Triangle Law of Vector Subtraction:



(ii) From the diagram, $\mathbf{r} = \begin{pmatrix} \mathbf{3} \\ \mathbf{5} \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} \mathbf{6} \\ \mathbf{2} \end{pmatrix}$ and $\mathbf{r} - \mathbf{s} = \begin{pmatrix} \mathbf{-3} \\ \mathbf{3} \end{pmatrix}$.

(iii) $\mathbf{r} - \mathbf{s} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 3-6 \\ 5-2 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-3} \\ \mathbf{3} \end{pmatrix}$

(iv) $|\mathbf{r}| = \sqrt{3^2 + 5^2}$

$= \boldsymbol{\sqrt{34}}$ **units**

$|\mathbf{s}| = \sqrt{6^2 + 2^2}$

$= \boldsymbol{\sqrt{40}}$ **units**

$|\mathbf{r} - \mathbf{s}| = \sqrt{(-3)^2 + 3^2}$

$= \boldsymbol{\sqrt{18}}$ **units**

(v) **No.**

$|\mathbf{r} - \mathbf{s}| = \sqrt{18}$

$= 4.24$ units (to 3 s.f.)

$|\mathbf{r}| - |\mathbf{s}| = \sqrt{34} - \sqrt{40}$

$= -0.494$ units (to 3 s.f.)

$\therefore |\mathbf{r} - \mathbf{s}| \neq |\mathbf{r}| - |\mathbf{s}|$

Thinking Time (Page 132)

$|\mathbf{a} - \mathbf{b}| = |\mathbf{a}| - |\mathbf{b}|$ if and only if **a** and **b** are parallel vectors in the same direction and $|\mathbf{a}| > |\mathbf{b}|$.

If **a** and **b** are parallel vectors in the same direction and $|\mathbf{a}| > |\mathbf{b}|$,

$\mathbf{b} = k\mathbf{a}$ where $0 < k < 1$.

$|\mathbf{a} - \mathbf{b}| = |\mathbf{a} - k\mathbf{a}|$

$= |(1 - k)\mathbf{a}|$

$= (1 - k)|\mathbf{a}|$

$|\mathbf{a}| - |\mathbf{b}| = |\mathbf{a}| - |k\mathbf{a}|$

$= |\mathbf{a}| - k|\mathbf{a}|$

$= (1 - k)|\mathbf{a}|$

Practise Now 8

(i) **b – a**

(ii) **a – b**

(iii) **m – n**

(iv) **v + w**

(v) –**w** + (–**v**) = **–w – v**

Practise Now 9

(i) $\overrightarrow{PR} = \overrightarrow{OQ}$

$= \mathbf{q}$

(ii) $\overrightarrow{RQ} = \overrightarrow{PO}$

$= -\overrightarrow{OP}$

$= \mathbf{-p}$

(iii) **Method 1:** Parallelogram Law of Vector Addition

$\overrightarrow{OR} = \mathbf{p} + \mathbf{q}$

Method 2: Triangle Law of Vector Addition

$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= \mathbf{p} + \mathbf{q}$

(iv) $\overrightarrow{RO} = -\overrightarrow{OR}$

$= -(\mathbf{p} + \mathbf{q})$

$= \mathbf{-p - q}$

(v) **Method 1:** Triangle Law of Vector Subtraction

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$

Method 2: Triangle Law of Vector Addition

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$= -\overrightarrow{OP} + \overrightarrow{OQ}$

$= -\mathbf{p} + \mathbf{q}$

$= \mathbf{q} - \mathbf{p}$

(vi) **Method 1:** Triangle Law of Vector Subtraction

$\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$

Method 2: Negative vector

$\overrightarrow{QP} = -\overrightarrow{PQ}$

$= -(\mathbf{q} - \mathbf{p})$

$= \mathbf{p} - \mathbf{q}$

Practise Now 10

(a) $\overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{\overrightarrow{AC}}$

(b) $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{CA}$

$= \overrightarrow{CA} + \overrightarrow{AB}$

$= \boldsymbol{\overrightarrow{CB}}$

(c) **Not possible** to simplify $\overrightarrow{AB} - \overrightarrow{BC}$.

(d) $\overrightarrow{PQ} - \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{RP}$

$= \overrightarrow{RP} + \overrightarrow{PQ}$

$= \boldsymbol{\overrightarrow{RQ}}$

(e) $\overrightarrow{PQ} - \overrightarrow{RQ} = \overrightarrow{PQ} + \overrightarrow{QR}$

$= \boldsymbol{\overrightarrow{PR}}$

(f) $\overrightarrow{PQ} + \overrightarrow{RP} - \overrightarrow{RS} = \overrightarrow{RP} + \overrightarrow{PQ} - \overrightarrow{RS}$

$= \overrightarrow{RQ} - \overrightarrow{RS}$

$= \overrightarrow{RQ} + \overrightarrow{SR}$

$= \overrightarrow{SR} + \overrightarrow{RQ}$

$= \boldsymbol{\overrightarrow{SQ}}$

Practise Now 11

(a) (i) $\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ -6 \end{pmatrix} = \begin{pmatrix} 1-8 \\ 3-(-6) \end{pmatrix}$

$= \begin{pmatrix} 1-8 \\ 3+6 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-7} \\ \mathbf{9} \end{pmatrix}$

(ii) $\begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \end{pmatrix} = \begin{pmatrix} -2-(-5)+(-6) \\ -3-4+7 \end{pmatrix}$

$= \begin{pmatrix} -2+5-6 \\ -3-4+7 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-3} \\ \mathbf{0} \end{pmatrix}$

(b) (i) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} 13 \\ -10 \end{pmatrix}$

$\therefore$ ***x* = 13 and *y* = –10**

(ii) $\begin{pmatrix} 3 \\ y \end{pmatrix} - \begin{pmatrix} x \\ -9 \end{pmatrix} = \begin{pmatrix} 4 \\ x \end{pmatrix}$

$\begin{pmatrix} 3-x \\ y+9 \end{pmatrix} = \begin{pmatrix} 4 \\ x \end{pmatrix}$

$3 - x = 4$ — (1)

$y + 9 = x$ — (2)

From (1), $x = 3 - 4$

$= -1$

Substitute $x = -1$ into (2):

$y + 9 = -1$

$y = -1 - 9$

$= -10$

$\therefore$ ***x* = –1 and *y* = –10**

Exercise 4B

1. (a) $\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 2+7 \\ 4+5 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{9} \\ \mathbf{9} \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-4 \\ -5+1 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-1} \\ \mathbf{-4} \end{pmatrix}$

(c) $\begin{pmatrix} -9 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ -8 \end{pmatrix} + \begin{pmatrix} -3 \\ 7 \end{pmatrix} = \begin{pmatrix} -9+0-3 \\ -2-8+7 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-12} \\ \mathbf{-3} \end{pmatrix}$

2. (i) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} -7 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$\mathbf{b} + \mathbf{a} = \begin{pmatrix} -7 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$\therefore \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(ii) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \left[\begin{pmatrix} -7 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -5 \end{pmatrix}\right]$

$= \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} -6 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$\therefore (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

3. (i) $\overrightarrow{LM} + \overrightarrow{MN} = \boldsymbol{\overrightarrow{LN}}$

(ii) $\overrightarrow{PN} + \overrightarrow{LP} = \overrightarrow{LP} + \overrightarrow{PN}$

$= \boldsymbol{\overrightarrow{LN}}$

(iii) $\overrightarrow{LN} + \overrightarrow{NM} + \overrightarrow{MP} = \overrightarrow{LM} + \overrightarrow{MP}$

$= \boldsymbol{\overrightarrow{LP}}$

4. (a) $\begin{pmatrix} 12 \\ -6 \end{pmatrix} + \begin{pmatrix} -12 \\ 6 \end{pmatrix} = \begin{pmatrix} 12-12 \\ -6+6 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ 7 \end{pmatrix} + \begin{pmatrix} -5 \\ -7 \end{pmatrix} = \begin{pmatrix} 5-5 \\ 7-7 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$

(c) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} x-x \\ y-y \end{pmatrix}$

$= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$

5. (a) $\begin{pmatrix} 9 \\ \mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{-9} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} -3 \\ \mathbf{-7} \end{pmatrix} + \begin{pmatrix} \mathbf{3} \\ 7 \end{pmatrix} = \mathbf{0}$

(c) $\begin{pmatrix} \boldsymbol{q} \\ p \end{pmatrix} + \begin{pmatrix} -q \\ \boldsymbol{-p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6. (a) $\overrightarrow{AB} + \overrightarrow{BA} = \mathbf{0}$

(b) $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \overrightarrow{PR} + \overrightarrow{RP}$
$= \mathbf{0}$

(c) $\overrightarrow{MN} + \overrightarrow{LM} + \overrightarrow{NL} = \overrightarrow{MN} + \overrightarrow{NL} + \overrightarrow{LM}$
$= \overrightarrow{ML} + \overrightarrow{LM}$
$= \mathbf{0}$

7. (i) **p – q**
(ii) **q – p**
(iii) **b – a**
(iv) **a + b**
(v) **s – r**
(vi) **r + s**
(vii) –(**m** + **n**) = **–m – n**
(viii) **n – m**

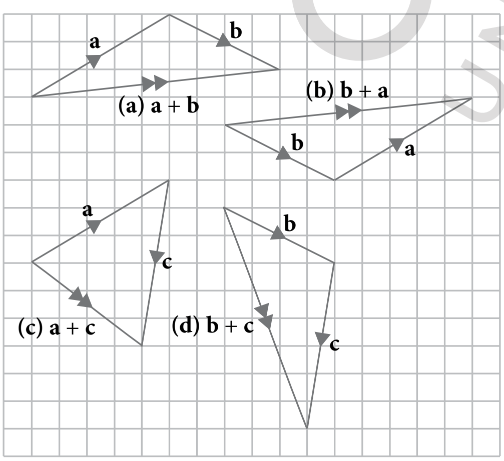
8. (a) $\begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5-3 \\ 4-2 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{2} \\ \mathbf{2} \end{pmatrix}$

(b) $\begin{pmatrix} -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1+3 \\ 3+4 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{2} \\ \mathbf{7} \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 2+5-7 \\ 3-2+3 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{0} \\ \mathbf{4} \end{pmatrix}$

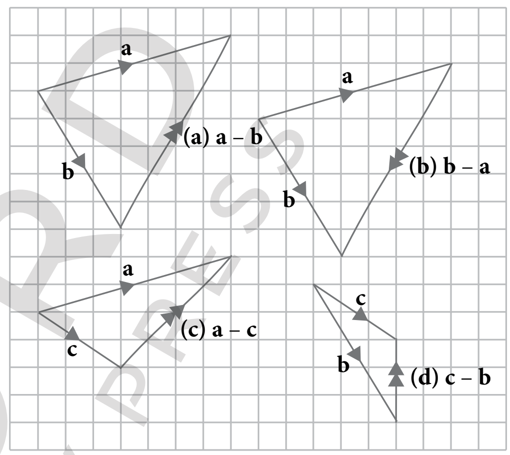
(d) $\begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 4+2-3 \\ 7-5+6 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{3} \\ \mathbf{8} \end{pmatrix}$

9.



10. (i) $\overrightarrow{PT} + \overrightarrow{TR} = \boldsymbol{\overrightarrow{PR}}$
(ii) $\overrightarrow{SQ} + \overrightarrow{QR} = \boldsymbol{\overrightarrow{SR}}$
(iii) $\overrightarrow{TR} + \overrightarrow{ST} = \overrightarrow{ST} + \overrightarrow{TR}$
$= \boldsymbol{\overrightarrow{SR}}$
(iv) $\overrightarrow{SQ} + \overrightarrow{QT} = \boldsymbol{\overrightarrow{ST}}$
(v) $\overrightarrow{SQ} + \overrightarrow{QR} + \overrightarrow{PS} = \overrightarrow{SR} + \overrightarrow{PS}$
$= \overrightarrow{PS} + \overrightarrow{SR}$
$= \boldsymbol{\overrightarrow{PR}}$
(vi) $\overrightarrow{RQ} + \overrightarrow{QT} + \overrightarrow{TP} + \overrightarrow{PS} = \overrightarrow{RT} + \overrightarrow{TP} + \overrightarrow{PS}$
$= \overrightarrow{RP} + \overrightarrow{PS}$
$= \boldsymbol{\overrightarrow{RS}}$

11.



12. (i) $\overrightarrow{RT} = \overrightarrow{OS}$
$= \mathbf{s}$
(ii) $\overrightarrow{TS} = \overrightarrow{RO}$
$= -\overrightarrow{OR}$
$= \mathbf{-r}$
(iii) $\overrightarrow{OT} = \overrightarrow{OR} + \overrightarrow{RT}$
$= \overrightarrow{OR} + \overrightarrow{OS}$
$= \mathbf{r + s}$
(iv) $\overrightarrow{RS} = \mathbf{s - r}$
(v) $\overrightarrow{SR} = -\overrightarrow{RS}$
$= -(\mathbf{s} - \mathbf{r})$
$= \mathbf{r - s}$

13. (a) $\overrightarrow{RS} + \overrightarrow{ST} = \boldsymbol{\overrightarrow{RT}}$
(b) $\overrightarrow{RS} - \overrightarrow{RT} = \overrightarrow{RS} + \overrightarrow{TR}$
$= \overrightarrow{TR} + \overrightarrow{RS}$
$= \boldsymbol{\overrightarrow{TS}}$
(c) $\overrightarrow{RT} - \overrightarrow{RS} = \overrightarrow{RT} + \overrightarrow{SR}$
$= \overrightarrow{SR} + \overrightarrow{RT}$
$= \boldsymbol{\overrightarrow{ST}}$
(d) **Not possible** to simplify $\overrightarrow{RS} - \overrightarrow{ST}$.
(e) $\overrightarrow{RS} - \overrightarrow{TS} = \overrightarrow{RS} + \overrightarrow{ST}$
$= \boldsymbol{\overrightarrow{RT}}$
(f) $\overrightarrow{RS} + \overrightarrow{TR} - \overrightarrow{TU} = \overrightarrow{TR} + \overrightarrow{RS} - \overrightarrow{TU}$
$= \overrightarrow{TS} - \overrightarrow{TU}$
$= \overrightarrow{TS} + \overrightarrow{UT}$
$= \overrightarrow{UT} + \overrightarrow{TS}$
$= \boldsymbol{\overrightarrow{US}}$

14. **(a)** $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 7+3 \\ -5-2 \end{pmatrix}$$
$$= \begin{pmatrix} 10 \\ -7 \end{pmatrix}$$

$\therefore$ **$x = 10$ and $y = -7$**

(b) $\begin{pmatrix} 3 \\ y \end{pmatrix} - \begin{pmatrix} x \\ -8 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$

$$\begin{pmatrix} 3-x \\ y+8 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$$

$3 - x = -6$ and $y + 8 = 9$

$x = 9$ $\quad$ $y = 1$

$\therefore$ **$x = 9$ and $y = 1$**

(c) $\begin{pmatrix} y \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ 2x \end{pmatrix} = \begin{pmatrix} 6 \\ x \end{pmatrix}$

$$\begin{pmatrix} y-4 \\ 3+2x \end{pmatrix} = \begin{pmatrix} 6 \\ x \end{pmatrix}$$

$y - 4 = 6$ and $3 + 2x = x$

$y = 10$ $\quad$ $x = -3$

$\therefore$ **$x = -3$ and $y = 10$**

(d) $\begin{pmatrix} 2x \\ 5 \end{pmatrix} - \begin{pmatrix} y-3 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ 3y \end{pmatrix}$

$$\begin{pmatrix} 2x-y+3 \\ 5+10 \end{pmatrix} = \begin{pmatrix} 4 \\ 3y \end{pmatrix}$$
$$\begin{pmatrix} 2x-y+3 \\ 15 \end{pmatrix} = \begin{pmatrix} 4 \\ 3y \end{pmatrix}$$

$2x - y + 3 = 4$

$2x - y = 1$ — (1)

$15 = 3y$ — (2)

From (2), $y = 5$

Substitute $y = 5$ into (1):

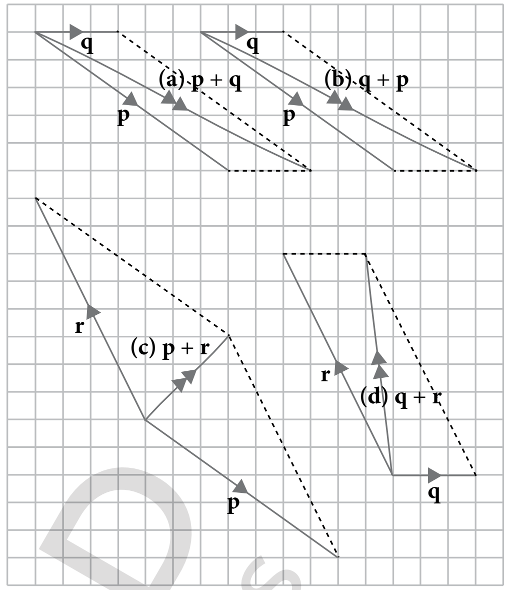
$2x - 5 = 1$

$2x = 6$

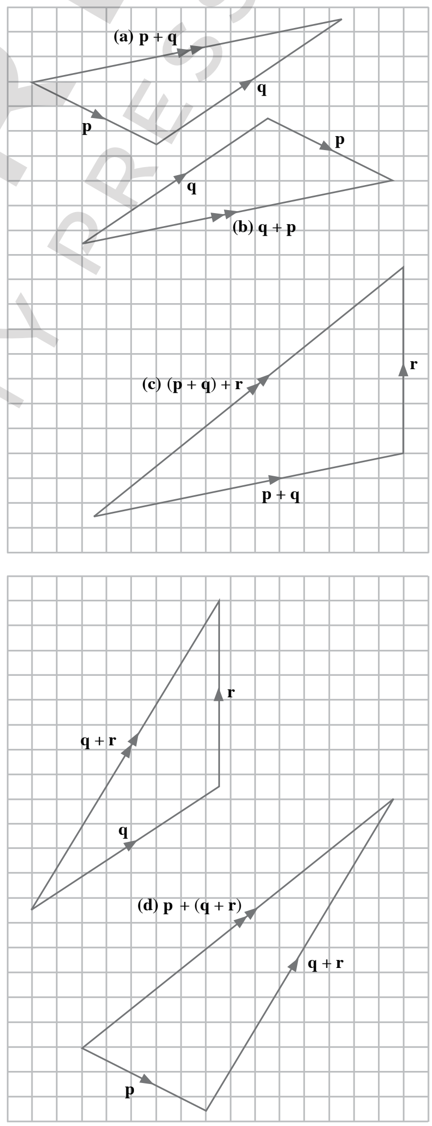
$x = 3$

$\therefore$ **$x = 3$ and $y = 5$**

15.



16. **(i)**



(ii) **Yes, p + q = q + p.**
Vector addition is commutative.

(iii) **Yes, (p + q) + r = p + (q + r).**
Vector addition is associative.

17. (i) (a) $\overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR}$

$= \boldsymbol{\overrightarrow{PR}}$

(b) $\overrightarrow{RO} - \overrightarrow{QO} = \overrightarrow{RO} + \overrightarrow{OQ}$

$= \boldsymbol{\overrightarrow{RQ}}$

(c) $\overrightarrow{PR} - \overrightarrow{SR} + \overrightarrow{SQ} = \overrightarrow{PR} + \overrightarrow{RS} + \overrightarrow{SQ}$

$= \overrightarrow{PS} + \overrightarrow{SQ}$

$= \boldsymbol{\overrightarrow{PQ}}$

(ii) (a) $\overrightarrow{SR} = \overrightarrow{PQ}$

$= \mathbf{a}$

(b) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$= \overrightarrow{PQ} + \overrightarrow{PS}$

$= \mathbf{a} + \mathbf{b}$

(c) $\overrightarrow{SQ} = \overrightarrow{SP} + \overrightarrow{PQ}$

$= -\overrightarrow{PS} + \overrightarrow{PQ}$

$= -\mathbf{b} + \mathbf{a}$

$= \mathbf{a} - \mathbf{b}$

18. (a) $\overrightarrow{SK} + \mathbf{u} = \mathbf{0}$

$\therefore \mathbf{u} = \boldsymbol{\overrightarrow{KS}}$

(b) $\overrightarrow{SP} + \overrightarrow{PQ} + \mathbf{u} = \mathbf{0}$

$\overrightarrow{SQ} + \mathbf{u} = \mathbf{0}$

$\therefore \mathbf{u} = \boldsymbol{\overrightarrow{QS}}$

(c) $\overrightarrow{PS} + \overrightarrow{SK} + \overrightarrow{KR} = \mathbf{u}$

$\overrightarrow{PK} + \overrightarrow{KR} = \mathbf{u}$

$\overrightarrow{PR} = \mathbf{u}$

$\therefore \mathbf{u} = \boldsymbol{\overrightarrow{PR}}$

(d) $\overrightarrow{PK} + (-\overrightarrow{SK}) = \mathbf{u}$

$\overrightarrow{PK} - \overrightarrow{SK} = \mathbf{u}$

$\overrightarrow{PK} + \overrightarrow{KS} = \mathbf{u}$

$\overrightarrow{PS} = \mathbf{u}$

$\therefore \mathbf{u} = \boldsymbol{\overrightarrow{PS}}$

(e) $\overrightarrow{PS} + (-\overrightarrow{RS}) = \mathbf{u}$

$\overrightarrow{PS} - \overrightarrow{RS} = \mathbf{u}$

$\overrightarrow{PS} + \overrightarrow{SR} = \mathbf{u}$

$\overrightarrow{PR} = \mathbf{u}$

$\therefore \mathbf{u} = \boldsymbol{\overrightarrow{PR}}$

(f) $\overrightarrow{PQ} + \overrightarrow{QR} + (-\overrightarrow{PR}) = \mathbf{u}$

$\overrightarrow{PR} - \overrightarrow{PR} = \mathbf{u}$

$\therefore \mathbf{u} = \mathbf{0}$

4.4 Scalar multiples of a vector

Thinking Time (Page 142)

If k is positive, vectors **a** and **b** are parallel and in the same direction.
If k is negative, vectors **a** and **b** are parallel but in opposite directions.

Practise Now 12

1. (a) (i) Since $\begin{pmatrix} 6 \\ -9 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, then $\begin{pmatrix} 6 \\ -9 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ are **parallel**.

(ii) If $\begin{pmatrix} 14 \\ 18 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ 9 \end{pmatrix}$ are parallel, then there must be a value of k that satisfies $\begin{pmatrix} 14 \\ 18 \end{pmatrix} = k\begin{pmatrix} -7 \\ 9 \end{pmatrix}$.

$14 = k(-7)$, i.e. $k = -2$

$18 = k(9)$, i.e. $k = 2$

But $-2 \neq 2$.

$\therefore \begin{pmatrix} 14 \\ 18 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ 9 \end{pmatrix}$ are **not parallel**.

(iii) Since $\begin{pmatrix} -3 \\ 6 \end{pmatrix} = -\frac{3}{4}\begin{pmatrix} 4 \\ -8 \end{pmatrix}$, then $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -8 \end{pmatrix}$ are **parallel**.

(b) A vector in the same direction as $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ is

$2 \times \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} \mathbf{8} \\ \mathbf{-6} \end{pmatrix}$.

A vector in the opposite direction of $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ is

$-\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} \mathbf{-4} \\ \mathbf{3} \end{pmatrix}$.

2. Since $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ p \end{pmatrix}$ are parallel vectors, then

$\begin{pmatrix} 4 \\ -3 \end{pmatrix} = k\begin{pmatrix} 12 \\ p \end{pmatrix}$.

$4 = 12k$, i.e. $k = \frac{4}{12}$

$= \frac{1}{3}$

$-3 = kp$

$= \frac{1}{3}p$

$\therefore p = -3(3)$

$= \mathbf{-9}$

Practise Now 13

1. (i) $\mathbf{u} + 3\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -6 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-5} \\ \mathbf{5} \end{pmatrix}$

(ii) $3\mathbf{u} - 2\mathbf{v} - \mathbf{w} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 3+4-3 \\ 6-2+4 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{4} \\ \mathbf{8} \end{pmatrix}$

2. $2\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 2x+2 \\ 2y-3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$2x + 2 = 5$ and $2y - 3 = 3$

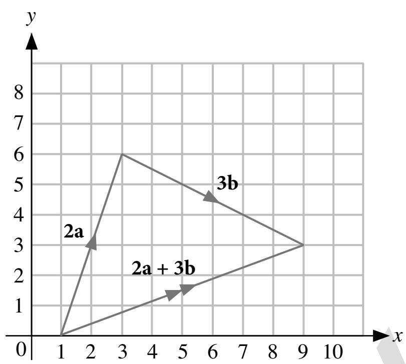
$2x = 3$ $\quad$ $2y = 6$

$x = 1.5$ $\quad$ $y = 3$

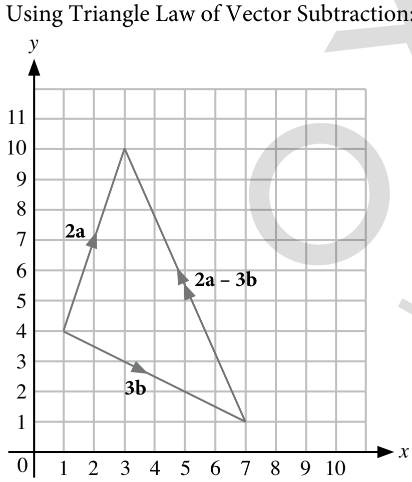
$\therefore$ ***x* = 1.5 and *y* = 3**

Class Discussion (Graphical representation of vectors)

1. **(i)** Using Triangle Law of Vector Addition:



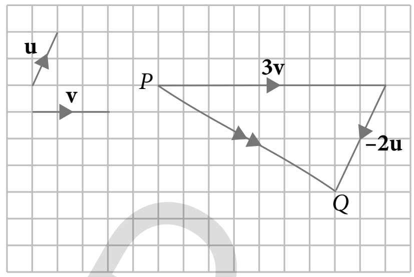
(ii) Using Triangle Law of Vector Subtraction:



4.5 Expression of a vector in terms of two other vectors

Class Discussion (Expressing a vector in terms of two other vectors)

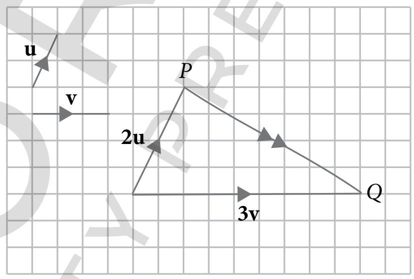
By Triangle Law of Vector Addition:



$\overrightarrow{PQ} = 3\mathbf{v} + (-2\mathbf{u})$

$= 3\mathbf{v} - 2\mathbf{u}$

By Triangle Law of Vector Subtraction:



$\overrightarrow{PQ} = 3\mathbf{v} - 2\mathbf{u}$

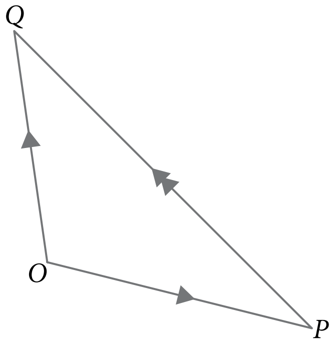
4.6 Position vectors

Practise Now 14

(a) The position vector of P is $\overrightarrow{OP} = \begin{pmatrix} \mathbf{8} \\ \mathbf{-2} \end{pmatrix}$.

The position vector of Q is $\overrightarrow{OQ} = \begin{pmatrix} \mathbf{-1} \\ \mathbf{7} \end{pmatrix}$.

Method 1: Using Vector Subtraction



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{pmatrix} -1 \\ 7 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{-9} \\ \mathbf{9} \end{pmatrix}$$

Method 2: Using Vector Addition

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$
$$= -\begin{pmatrix} 8 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{-9} \\ \mathbf{9} \end{pmatrix}$$

(b) **Method 1:** Using Vector Subtraction

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$
$$\overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$
$$\overrightarrow{OB} - \begin{pmatrix} 6 \\ -7 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$
$$\overrightarrow{OB} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 6 \\ -7 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$\therefore$ coordinates of B are **(2, –2)**.

Method 2: Using Vector Addition

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$
$$\overrightarrow{OB} = \begin{pmatrix} 6 \\ -7 \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$\therefore$ coordinates of B are **(2, –2)**.

Exercise 4C

1. **(a)** Since $\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -8 \\ 4 \end{pmatrix}$, then $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -8 \\ 4 \end{pmatrix}$ are **parallel**.

(b) If $\begin{pmatrix} 9 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 18 \\ 21 \end{pmatrix}$ are parallel, then there must be a value of k that satisfies $\begin{pmatrix} 9 \\ 7 \end{pmatrix} = k\begin{pmatrix} 18 \\ 21 \end{pmatrix}$.

$9 = k(18)$, i.e. $k = \frac{1}{2}$

$7 = k(21)$, i.e. $k = \frac{1}{3}$

But $\frac{1}{2} \neq \frac{1}{3}$.

$\therefore$ $\begin{pmatrix} 9 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 18 \\ 21 \end{pmatrix}$ are **not parallel**.

(c) Since $\begin{pmatrix} 6 \\ -8 \end{pmatrix} = -2\begin{pmatrix} -3 \\ 4 \end{pmatrix}$, then $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ are **parallel**.

2. **(a)** A vector in the same direction as $\begin{pmatrix} 8 \\ -7 \end{pmatrix}$ is

$2 \times \begin{pmatrix} 8 \\ -7 \end{pmatrix} = \begin{pmatrix} \mathbf{16} \\ \mathbf{-14} \end{pmatrix}$.

A vector in the opposite direction of $\begin{pmatrix} 8 \\ -7 \end{pmatrix}$ is

$-3\begin{pmatrix} 8 \\ -7 \end{pmatrix} = \begin{pmatrix} \mathbf{-24} \\ \mathbf{21} \end{pmatrix}$.

(b) A vector in the same direction as $\begin{pmatrix} 3 \\ 9 \end{pmatrix}$ is $\frac{1}{3} \times \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \end{pmatrix}$.

A vector in the opposite direction of $\begin{pmatrix} 3 \\ 9 \end{pmatrix}$ is

$-\frac{2}{3}\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{-6} \end{pmatrix}$.

(c) A vector in the same direction as $\begin{pmatrix} -6 \\ -2 \end{pmatrix}$ is

$\frac{1}{2} \times \begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} \mathbf{-3} \\ \mathbf{-1} \end{pmatrix}$.

A vector in the opposite direction of $\begin{pmatrix} -6 \\ -2 \end{pmatrix}$ is

$-\frac{1}{2}\begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} \mathbf{3} \\ \mathbf{1} \end{pmatrix}$.

3. **(a)**
$$\mathbf{p} + 2\mathbf{q} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 6 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 5 + 12 \\ 2 - 6 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{17} \\ \mathbf{-4} \end{pmatrix}$$

(b) $3\mathbf{p} - \frac{1}{2}\mathbf{q} = 3\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 6 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 15 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ -1.5 \end{pmatrix}$

$= \begin{pmatrix} 15 - 3 \\ 6 + 1.5 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{12} \\ \mathbf{7\frac{1}{2}} \end{pmatrix}$

(c) $4\mathbf{p} - 3\mathbf{q} + \mathbf{r} = 4\begin{pmatrix} 5 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 6 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 20 \\ 8 \end{pmatrix} - \begin{pmatrix} 18 \\ -9 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 20 - 18 - 3 \\ 8 + 9 - 4 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-1} \\ \mathbf{13} \end{pmatrix}$

4. **(a)** The position vector of A is $\overrightarrow{OA} = \begin{pmatrix} \mathbf{4} \\ \mathbf{7} \end{pmatrix}$.

(b) The position vector of B is $\overrightarrow{OB} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{5} \end{pmatrix}$.

(c) The position vector of C is $\overrightarrow{OC} = \begin{pmatrix} \mathbf{6} \\ \mathbf{-1} \end{pmatrix}$.

(d) The position vector of D is $\overrightarrow{OD} = \begin{pmatrix} \mathbf{-4} \\ \mathbf{-9} \end{pmatrix}$.

5. **(i)** $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$= -\overrightarrow{OP} + \overrightarrow{OQ}$

$= -\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-1} \\ \mathbf{-2} \end{pmatrix}$

(ii) $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{0} \\ \mathbf{7} \end{pmatrix}$

(iii) $\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$

$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{1} \\ \mathbf{-5} \end{pmatrix}$

(iv) $\overrightarrow{PR} = -\overrightarrow{RP}$

$= -\begin{pmatrix} 1 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-1} \\ \mathbf{5} \end{pmatrix}$

6. **(a)** Since $\begin{pmatrix} 6 \\ -3 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} -4 \\ 2 \end{pmatrix}$, then $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ are **parallel**.

(b) Since $\begin{pmatrix} -5 \\ 15 \end{pmatrix} = \frac{5}{3}\begin{pmatrix} -3 \\ 9 \end{pmatrix}$, then $\begin{pmatrix} -5 \\ 15 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 9 \end{pmatrix}$ are **parallel**.

(c) If $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ are parallel, then there must be a value of k that satisfies $\begin{pmatrix} 7 \\ -8 \end{pmatrix} = k\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

$7 = k(2)$, i.e. $k = 3\frac{1}{2}$

$-8 = k(-3)$, i.e. $k = 2\frac{2}{3}$

But $3\frac{1}{2} \neq 2\frac{2}{3}$.

$\therefore \begin{pmatrix} 7 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ are **not parallel**.

7. **(a)** Since $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 20 \\ p \end{pmatrix}$ are parallel vectors, then

$\begin{pmatrix} 5 \\ 2 \end{pmatrix} = k\begin{pmatrix} 20 \\ p \end{pmatrix}$.

$5 = 20k$, i.e. $k = \frac{5}{20}$

$= \frac{1}{4}$

$2 = kp$

$= \frac{1}{4}p$

$\therefore p = 2(4)$

$= \mathbf{8}$

(b) Since $\begin{pmatrix} h \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -9 \end{pmatrix}$ are parallel vectors, then

$\begin{pmatrix} h \\ 12 \end{pmatrix} = k\begin{pmatrix} 3 \\ -9 \end{pmatrix}$.

$12 = -9k$

$k = \frac{12}{-9}$

$= -1\frac{1}{3}$

$\therefore h = 3k$

$= 3\left(-1\frac{1}{3}\right)$

$= \mathbf{-4}$

8. **(a)**

$\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore$ **$x = 2$ and $y = 1$**

(b) $4\mathbf{u} + \mathbf{v} = 2\begin{pmatrix} \frac{1}{2} \\ 9 \end{pmatrix}$

$$4\begin{pmatrix} 2 \\ y \end{pmatrix} + \begin{pmatrix} x \\ 2 \end{pmatrix} = 2\begin{pmatrix} \frac{1}{2} \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 4y \end{pmatrix} + \begin{pmatrix} x \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 18 \end{pmatrix}$$

$$\begin{pmatrix} 8 + x \\ 4y + 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 18 \end{pmatrix}$$

$8 + x = 1$ and $4y + 2 = 18$

$x = -7$ $\qquad$ $4y = 16$

$y = 4$

$\therefore$ **$x = -7$ and $y = 4$**

(c) $5\mathbf{p} - 2\mathbf{q} = \begin{pmatrix} 3 \\ 23 \end{pmatrix}$

$$5\begin{pmatrix} x \\ 5 \end{pmatrix} - 2\begin{pmatrix} 6 \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 23 \end{pmatrix}$$

$$\begin{pmatrix} 5x \\ 25 \end{pmatrix} - \begin{pmatrix} 12 \\ 2y \end{pmatrix} = \begin{pmatrix} 3 \\ 23 \end{pmatrix}$$

$$\begin{pmatrix} 5x - 12 \\ 25 - 2y \end{pmatrix} = \begin{pmatrix} 3 \\ 23 \end{pmatrix}$$

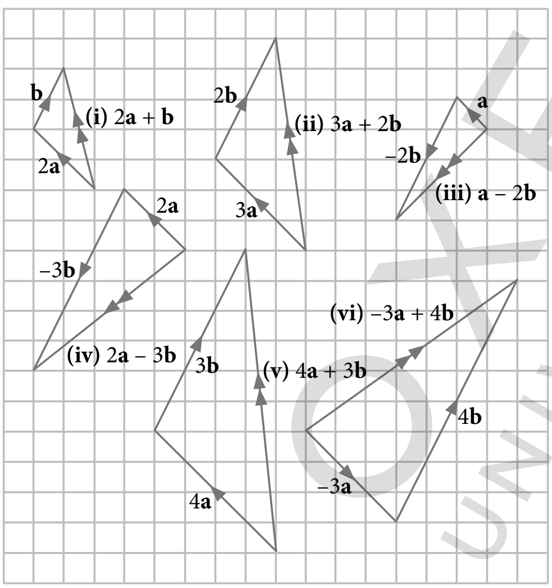
$5x - 12 = 3$ and $25 - 2y = 23$

$5x = 15$ $\qquad$ $2y = 2$

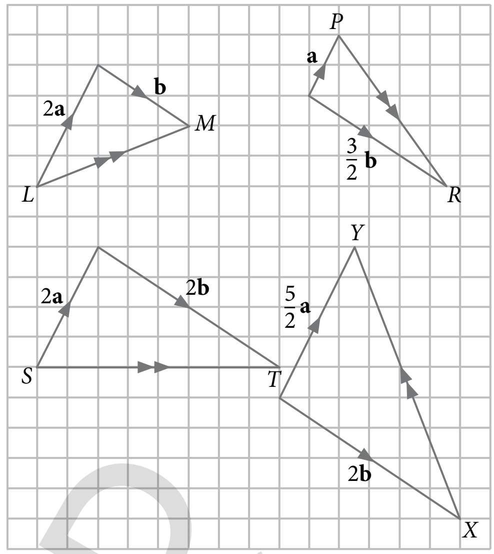
$x = 3$ $\qquad$ $y = 1$

$\therefore$ **$x = 3$ and $y = 1$**

9.



10.



$\overrightarrow{LM} = \mathbf{2a + b}$

$\overrightarrow{PR} = \mathbf{\frac{3}{2}b - a}$

$\overrightarrow{ST} = \mathbf{2a + 2b}$

$\overrightarrow{XY} = \mathbf{\frac{5}{2}a - 2b}$

11. $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

$$\overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$\overrightarrow{OB} - \begin{pmatrix} -3 \\ 8 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 4 \end{pmatrix}$$

$\therefore$ coordinates of B are **(–5, 4)**.

12. **(i)** $\overrightarrow{CD} = \frac{2}{3}\overrightarrow{AB}$

$$= \frac{2}{3}\begin{pmatrix} 9 \\ -15 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{6} \\ \mathbf{-10} \end{pmatrix}$$

(ii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\begin{pmatrix} 9 \\ -15 \end{pmatrix} = \overrightarrow{OB} - \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} 9 \\ -15 \end{pmatrix} + \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -8 \end{pmatrix}$$

$\therefore$ coordinates of B are **(7, –8)**.

(iii) $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$\overrightarrow{OC} = \overrightarrow{OD} - \overrightarrow{CD}$

$$= \begin{pmatrix} 8 \\ -5 \end{pmatrix} - \begin{pmatrix} 6 \\ -10 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$\therefore$ coordinates of C are **(2, 5)**.

13. If $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ are parallel vectors, there must be a unique value k such that $\begin{pmatrix} a \\ b \end{pmatrix} = k\begin{pmatrix} c \\ d \end{pmatrix}$.

$a = kc$, i.e. $k = \frac{a}{c}$

$b = kd$, i.e. $k = \frac{b}{d}$

$\therefore \frac{a}{c} = \frac{b}{d}$

14. $\mathbf{u} = \begin{pmatrix} -15 \\ 8 \end{pmatrix}$

$|\mathbf{u}| = \sqrt{(-15)^2 + 8^2}$
$= \sqrt{289}$
$= 17$ units

Since $\mathbf{u} = k\mathbf{v}$, $|\mathbf{u}| = |k\mathbf{v}| = k|\mathbf{v}|$
$|\mathbf{u}| = k|\mathbf{v}|$
$17 = k(51)$
$\therefore k = \mathbf{\frac{1}{3}}$
$\mathbf{u} = k\mathbf{v}$
$\mathbf{v} = \frac{1}{k}\mathbf{u}$
$= 3\begin{pmatrix} -15 \\ 8 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{-45} \\ \mathbf{24} \end{pmatrix}$

15. (i) $2\overrightarrow{AB} + 5\overrightarrow{CD} = 2\begin{pmatrix} -3 \\ 5 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 4 \end{pmatrix}$
$= \begin{pmatrix} -6 \\ 10 \end{pmatrix} + \begin{pmatrix} 5 \\ 20 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{-1} \\ \mathbf{30} \end{pmatrix}$

(ii) If $\overrightarrow{EF}$ is parallel to $\overrightarrow{AB}$, there must be a value of x that satisfies $\begin{pmatrix} k \\ 7.5 \end{pmatrix} = x\begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

$7.5 = 5x$
$x = 1.5$
$\therefore k = -3x$
$= -3(1.5)$
$= \mathbf{-4.5}$

(iii) Since $\overrightarrow{PQ} = \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \frac{1}{4}\overrightarrow{CD}$, then $\begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ are parallel.

$\therefore \overrightarrow{PQ}$ is parallel to $\overrightarrow{CD}$.

16. (i) $\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$
$= \begin{pmatrix} t \\ 6 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{t + 3} \\ \mathbf{4} \end{pmatrix}$

(ii) If $\overrightarrow{LM}$ is parallel to $\mathbf{p} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$, then there must be a value of k that satisfies $\begin{pmatrix} t + 3 \\ 4 \end{pmatrix} = k\begin{pmatrix} 8 \\ 1 \end{pmatrix}$.

$4 = k(1)$
$k = 4$
$t + 3 = k(8)$
$\therefore t = 4(8) - 3$
$= \mathbf{29}$

(iii) If $|\overrightarrow{LM}| = |\mathbf{p}|$, $\sqrt{(t+3)^2 + 4^2} = \sqrt{8^2 + 1^2}$
$(t + 3)^2 + 4^2 = 8^2 + 1^2$
$t^2 + 6t + 9 + 16 = 64 + 1$
$t^2 + 6t - 40 = 0$
$(t - 4)(t + 10) = 0$
$t - 4 = 0$ or $t + 10 = 0$
$t = 4$ $\quad t = -10$

$\therefore$ the two possible values of t are **4 or −10**.

17. (i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$
$\overrightarrow{OQ} = \overrightarrow{PQ} + \overrightarrow{OP}$
$= \begin{pmatrix} 8 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 10 \\ -5 \end{pmatrix}$

$\therefore$ coordinates of Q are **(10, −5)**.

(ii) gradient of $PQ = \frac{-5 - (-3)}{10 - 2}$
$= \frac{-2}{8}$
$= -\frac{1}{4}$

(iii) If $\overrightarrow{PQ} = \begin{pmatrix} x \\ y \end{pmatrix}$, gradient of $PQ = \boldsymbol{\frac{y}{x}}$.

(iv) If the gradient of PQ is $\frac{y}{x}$, $\boldsymbol{\overrightarrow{PQ} = k\begin{pmatrix} x \\ y \end{pmatrix}}$, where k is a constant.

4.7 Applications of vectors

Class Discussion (Real-life applications of the resultant vector of two vectors or of the difference between two vectors)

An airplane flies 200 km north then 300 km east.
An object is experiencing two perpendicular forces $F_1 = 50$ N and $F_2 = 30$ N.
A plane is heading due north with an air speed of 400 km/h when it is blown off course by a wind of 100 km/h from the northeast.

Practise Now 15

Since *ABCD* is a parallelogram, then

$$\overrightarrow{DC} = \overrightarrow{AB}$$

$$\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{OC} - \begin{pmatrix} 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\overrightarrow{OC} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -9 \end{pmatrix}$$

∴ coordinates of *C* are **(1, –9)**.

Practise Now 16

(i) **(a)** **Method 1:** Parallelogram Law of Vector Addition

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$$
$$= \mathbf{8a + 4b}$$

Method 2: Triangle Law of Vector Addition

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$= \overrightarrow{AB} + \overrightarrow{AD}$$
$$= \mathbf{8a + 4b}$$

(b) $DF = \frac{1}{4} DC$

$$\overrightarrow{DF} = \frac{1}{4}\overrightarrow{DC}$$
$$= \frac{1}{4}\overrightarrow{AB}$$
$$= \frac{1}{4}(8\mathbf{a})$$
$$= \mathbf{2a}$$

(c) $\overrightarrow{CF} = \overrightarrow{CD} + \overrightarrow{DF}$

$$= -\overrightarrow{AB} + 2\mathbf{a}$$
$$= -8\mathbf{a} + 2\mathbf{a}$$
$$= \mathbf{-6a}$$

(d) $\triangle EDF$ and $\triangle BCF$ are similar.

$$\frac{EF}{BF} = \frac{DF}{CF}$$
$$= \frac{1}{3}$$
$$EF = \frac{1}{3}BF$$
$$\overrightarrow{EF} = \frac{1}{3}\overrightarrow{FB}$$
$$= \frac{1}{3}(\overrightarrow{FC} + \overrightarrow{CB})$$
$$= \frac{1}{3}(-\overrightarrow{CF} - \overrightarrow{AD})$$
$$= \frac{1}{3}(6\mathbf{a} - 4\mathbf{b})$$
$$= \mathbf{\frac{2}{3}(3a - 2b)}$$

(ii) $\dfrac{\text{Area of }\triangle EDF}{\text{Area of }\triangle ADF} = \dfrac{\frac{1}{2} \times DE \times h}{\frac{1}{2} \times AD \times h}$, where h is the common height of $\triangle EDF$ and $\triangle ADF$.

$$= \frac{DE}{AD}$$
$$= \frac{DE}{BC}$$
$$= \mathbf{\frac{1}{3}}$$

Practise Now 17

(i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \mathbf{3q - 9p}$$

$$\overrightarrow{OR} = \frac{1}{3}\overrightarrow{OP}$$
$$= \frac{1}{3}(9\mathbf{p})$$
$$= 3\mathbf{p}$$

$$\overrightarrow{OS} = \frac{1}{2}\overrightarrow{OQ}$$
$$= \frac{1}{2}(3\mathbf{q})$$
$$= \frac{3}{2}\mathbf{q}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$
$$= \frac{3}{2}\mathbf{q} - 3\mathbf{p}$$
$$= \mathbf{\frac{3}{2}(q - 2p)}$$

(ii) $\overrightarrow{RT} = \frac{1}{4}\overrightarrow{RQ}$

$$\overrightarrow{OT} - \overrightarrow{OR} = \frac{1}{4}(\overrightarrow{OQ} - \overrightarrow{OR})$$
$$\overrightarrow{OT} - 3\mathbf{p} = \frac{1}{4}(3\mathbf{q} - 3\mathbf{p})$$
$$\overrightarrow{OT} = \frac{3}{4}\mathbf{q} - \frac{3}{4}\mathbf{p} + 3\mathbf{p}$$
$$= \frac{9}{4}\mathbf{p} + \frac{3}{4}\mathbf{q}$$
$$= \frac{3}{4}(3\mathbf{p} + \mathbf{q})$$

$$\therefore \overrightarrow{TS} = \overrightarrow{OS} - \overrightarrow{OT}$$
$$= \frac{3}{2}\mathbf{q} - \frac{3}{4}(3\mathbf{p} + \mathbf{q})$$
$$= \frac{6}{4}\mathbf{q} - \frac{9}{4}\mathbf{p} - \frac{3}{4}\mathbf{q}$$
$$= \frac{3}{4}\mathbf{q} - \frac{9}{4}\mathbf{p}$$
$$= \mathbf{\frac{3}{4}(q - 3p)}$$

Practise Now 18

(i) Let $\overrightarrow{QA} = \mathbf{a}$ and $\overrightarrow{QB} = \mathbf{b}$.

Then $\overrightarrow{AB} = \overrightarrow{QB} - \overrightarrow{QA}$

$$= \mathbf{b} - \mathbf{a}$$

$$\overrightarrow{QP} = 2\overrightarrow{QA}$$
$$= 2\mathbf{a}$$
$$\overrightarrow{QR} = 2\overrightarrow{QB}$$
$$= 2\mathbf{b}$$
$$\overrightarrow{PR} = \overrightarrow{QR} - \overrightarrow{QP}$$
$$= 2\mathbf{b} - 2\mathbf{a}$$
$$= 2(\mathbf{b} - \mathbf{a})$$
$$= 2\overrightarrow{AB}$$

Since $\overrightarrow{PR} = 2\overrightarrow{AB}$, then *PR* is parallel to *AB* and *PR* = 2*AB*. (shown)

(ii) Using the same reasoning in part **(i)** for $\triangle SPR$, we can show that $\overrightarrow{PR} = 2\overrightarrow{DC}$.

Since $\overrightarrow{PR} = 2\overrightarrow{AB} = 2\overrightarrow{DC}$, then $\overrightarrow{AB} = \overrightarrow{DC}$.

∴ *AB* is parallel to *DC* and *AB* = *DC*, i.e. *ABCD* is a parallelogram. (shown)

Exercise 4D

1. Since *ABCD* is a parallelogram, then

$$\overrightarrow{DC} = \overrightarrow{AB}$$

$$\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{OC} - \begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{OC} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 11 \end{pmatrix}$$

∴ coordinates of *C* are **(9, 11)**.

2. **(i)** $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CB}$

$= \frac{1}{2}\overrightarrow{DA}$

$= -\frac{1}{2}\overrightarrow{AD}$

$= \mathbf{-\frac{1}{2}\, q}$

(ii) $\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD}$

$= \mathbf{p - q}$

(iii) $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$

$= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$

$= \mathbf{p + \frac{1}{2}\, q}$

(iv) $\overrightarrow{MD} = \overrightarrow{CD} - \overrightarrow{CM}$

$= -\overrightarrow{AB} - (-\frac{1}{2}\,\mathbf{q})$

$= \mathbf{\frac{1}{2}\, q - p}$

3. **(i)** $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$

$= \overrightarrow{BD} + \frac{1}{3}\overrightarrow{BD}$

$= \frac{4}{3}\overrightarrow{BD}$

$= \mathbf{\frac{4}{3}\, q}$

(ii) $\overrightarrow{AD} = \overrightarrow{BD} - \overrightarrow{BA}$

$= \mathbf{q - p}$

(iii) $\overrightarrow{CA} = \overrightarrow{BA} - \overrightarrow{BC}$

$= \mathbf{p - \frac{4}{3}\, q}$

4. **(i)** $\overrightarrow{MR} = \overrightarrow{MQ} + \overrightarrow{QR}$

$= \overrightarrow{PM} + \overrightarrow{PS}$

$= 2\mathbf{b} + \mathbf{a}$

$= \mathbf{a + 2b}$

(ii) $\overrightarrow{RN} = \overrightarrow{RS} + \overrightarrow{SN}$

$= \overrightarrow{QP} + \frac{1}{3}\overrightarrow{SR}$

$= -\overrightarrow{PQ} + \frac{1}{3}\overrightarrow{PQ}$

$= -\frac{2}{3}\overrightarrow{PQ}$

$= -\frac{2}{3}(2\,\overrightarrow{PM})$

$= -\frac{4}{3}(2\mathbf{b})$

$= \mathbf{-\frac{8}{3}\, b}$

(iii) $\overrightarrow{NM} = \overrightarrow{NR} + \overrightarrow{RM}$

$= -\overrightarrow{RN} + (-\overrightarrow{MR})$

$= \frac{8}{3}\mathbf{b} - (\mathbf{a} + 2\mathbf{b})$

$= \mathbf{\frac{2}{3}\, b - a}$

5. $\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB}$

$= \frac{1}{2}\overrightarrow{OA} - \overrightarrow{OB}$

$= \mathbf{\frac{1}{2}\, a - b}$

6. **(i)** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \mathbf{b - a}$

(ii) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \overrightarrow{AB} + (-\overrightarrow{CB})$

$= \overrightarrow{AB} - \frac{3}{2}\overrightarrow{AC}$

$\frac{5}{2}\overrightarrow{AC} = \overrightarrow{AB}$

$\overrightarrow{AC} = \frac{2}{5}\overrightarrow{AB}$

$= \mathbf{\frac{2}{5}\,(b - a)}$

(iii) $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \mathbf{a} + \frac{2}{5}(\mathbf{b} - \mathbf{a})$

$= \mathbf{a} + \frac{2}{5}\mathbf{b} - \frac{2}{5}\mathbf{a}$

$= \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$

$= \mathbf{\frac{1}{5}\,(3a + 2b)}$

7. **(i)** $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$= \mathbf{v - u}$

(ii) $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$

$= \mathbf{\frac{1}{2}\, u}$

(iii) $\overrightarrow{AN} = \frac{1}{2}\overrightarrow{AC}$

$= \mathbf{\frac{1}{2}\, v}$

(iv) $\overrightarrow{MN} = \overrightarrow{AN} - \overrightarrow{AM}$

$= \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$

$= \mathbf{\frac{1}{2}\,(v - u)}$

$\overrightarrow{BC}$ and $\overrightarrow{MN}$ are **parallel** vectors because $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$.

8. **(a)** If *PQRS* is a parallelogram, then

$$\overrightarrow{PS} = \overrightarrow{QR}$$

$$\overrightarrow{OS} - \overrightarrow{OP} = \overrightarrow{OR} - \overrightarrow{OQ}$$

$$\overrightarrow{OS} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OS} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

∴ coordinates of *S* are **(2, 2)**.

(b) If $PRQS$ is a parallelogram, then

$$\overrightarrow{QS} = \overrightarrow{RP}$$
$$\overrightarrow{OS} - \overrightarrow{OQ} = \overrightarrow{OP} - \overrightarrow{OR}$$
$$\overrightarrow{OS} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$
$$\overrightarrow{OS} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

∴ coordinates of S are **(0, –2)**.

9. (i) $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$
$= \mathbf{v - u}$

(ii) $\overrightarrow{BE} = \frac{2}{5}\overrightarrow{BC}$
$= \mathbf{\frac{2}{5}(v - u)}$

(iii) $\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$
$= \mathbf{v} + \frac{3}{2}\mathbf{u}$
$= \mathbf{\frac{3}{2}u + v}$

(iv) $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$
$= \mathbf{u} + \frac{2}{5}(\mathbf{v} - \mathbf{u})$
$= \mathbf{u} + \frac{2}{5}\mathbf{v} - \frac{2}{5}\mathbf{u}$
$= \frac{3}{5}\mathbf{u} + \frac{2}{5}\mathbf{v}$
$= \mathbf{\frac{1}{5}(3u + 2v)}$

(v) $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CD}$
$= -\mathbf{u} + \mathbf{v} + \frac{3}{2}\mathbf{u}$
$= \mathbf{\frac{1}{2}u + v}$

10. (i) $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$
$= 15\mathbf{b} - 15\mathbf{a}$
$= \mathbf{15(b - a)}$

(ii) $\overrightarrow{PA} = \overrightarrow{PR} + \overrightarrow{RA}$
$= \overrightarrow{PR} + (-\frac{3}{4}\overrightarrow{PR})$
$= \frac{1}{4}\overrightarrow{PR}$
$= \mathbf{\frac{15}{4}(b - a)}$

(iii) $\overrightarrow{OA} = \overrightarrow{OP} + \overrightarrow{PA}$
$= 15\mathbf{a} + \frac{15}{4}(\mathbf{b} - \mathbf{a})$
$= 15\mathbf{a} + \frac{15}{4}\mathbf{b} - \frac{15}{4}\mathbf{a}$
$= \frac{45}{4}\mathbf{a} + \frac{15}{4}\mathbf{b}$
$= \mathbf{\frac{15}{4}(3a + b)}$

(iv) $\overrightarrow{OB} = \overrightarrow{OP} + \overrightarrow{PB}$
$= \overrightarrow{OP} + \frac{1}{3}\overrightarrow{PQ}$
$= \overrightarrow{OP} + \frac{1}{3}\overrightarrow{OR}$
$= 15\mathbf{a} + \frac{1}{3}(15\mathbf{b})$
$= 15\mathbf{a} + 5\mathbf{b}$
$= \mathbf{5(3a + b)}$

11. (i) $\overrightarrow{PC} = \overrightarrow{OC} - \overrightarrow{OP}$
$= \overrightarrow{OQ} + \overrightarrow{QC} - \overrightarrow{OP}$
$= \overrightarrow{OQ} + \frac{3}{2}\overrightarrow{OQ} - \overrightarrow{OP}$
$= \frac{5}{2}\overrightarrow{OQ} - \overrightarrow{OP}$
$= \frac{5}{2}(8\mathbf{q}) - 8\mathbf{p}$
$= 20\mathbf{q} - 8\mathbf{p}$
$= \mathbf{4(5q - 2p)}$

(ii) $\overrightarrow{PB} = \overrightarrow{PC} + \overrightarrow{CB}$
$= \overrightarrow{PC} - 3\overrightarrow{PB}$
$4\overrightarrow{PB} = \overrightarrow{PC}$
$= 4(5\mathbf{q} - 2\mathbf{p})$
$\overrightarrow{PB} = \mathbf{5q - 2p}$

(iii) $\overrightarrow{OB} = \overrightarrow{OP} + \overrightarrow{PB}$
$= 8\mathbf{p} + (5\mathbf{q} - 2\mathbf{p})$
$= \mathbf{6p + 5q}$

(iv) $\overrightarrow{QB} = \overrightarrow{OB} - \overrightarrow{OQ}$
$= (6\mathbf{p} + 5\mathbf{q}) - 8\mathbf{q}$
$= 6\mathbf{p} - 3\mathbf{q}$
$= \mathbf{3(2p - q)}$

12. (i) (a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$
$$= \begin{pmatrix} 2 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1} \\ \mathbf{-3} \end{pmatrix}$$

(b) $\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS}$
$$= \begin{pmatrix} -1 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$$

(c) $\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR}$
$$= \begin{pmatrix} 2 \\ 8 \end{pmatrix} - \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{3} \\ \mathbf{1} \end{pmatrix}$$

(d) $\overrightarrow{TQ} = \overrightarrow{OQ} - \overrightarrow{OT}$
$$= \begin{pmatrix} 2 \\ 8 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{6} \\ \mathbf{2} \end{pmatrix}$$

(ii) $\overrightarrow{RQ} = \frac{1}{2}\overrightarrow{TQ}$

$\therefore \frac{RQ}{TQ} = \mathbf{\frac{1}{2}}$

13. (i) (a) $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$
$= \mathbf{p} - \mathbf{q}$

(b) $\overrightarrow{QS} = \frac{2}{5}\overrightarrow{QP}$
$= \mathbf{\frac{2}{5}(p - q)}$

(c) $\overrightarrow{OS} = \overrightarrow{OQ} + \overrightarrow{QS}$
$= \mathbf{q} + \frac{2}{5}(\mathbf{p} - \mathbf{q})$
$= \mathbf{q} + \frac{2}{5}\mathbf{p} - \frac{2}{5}\mathbf{q}$
$= \frac{2}{5}\mathbf{p} + \frac{3}{5}\mathbf{q}$
$= \mathbf{\frac{1}{5}(2p + 3q)}$

(d) $\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS}$
$= \frac{3}{2}\overrightarrow{OQ} - \overrightarrow{OS}$
$= \frac{3}{2}\mathbf{q} - (\frac{2}{5}\mathbf{p} + \frac{3}{5}\mathbf{q})$
$= \frac{9}{10}\mathbf{q} - \frac{2}{5}\mathbf{p}$
$= \mathbf{\frac{1}{10}(9q - 4p)}$

(ii) (a) $\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$
$= \overrightarrow{OS} - \frac{2}{3}\overrightarrow{OP}$
$= \frac{1}{5}(2\mathbf{p} + 3\mathbf{q}) - \frac{2}{3}\mathbf{p}$
$= \frac{2}{5}\mathbf{p} + \frac{3}{5}\mathbf{q} - \frac{2}{3}\mathbf{p}$
$= \frac{3}{5}\mathbf{q} - \frac{4}{15}\mathbf{p}$
$= \frac{1}{15}(9\mathbf{q} - 4\mathbf{p})$
$= \frac{2}{3}\left[\frac{1}{10}(9\mathbf{q} - 4\mathbf{p})\right]$
$= \frac{2}{3}\overrightarrow{ST}$ (shown)

(b) Since $\overrightarrow{RS} = \frac{2}{3}\overrightarrow{ST}$, RS is parallel to ST.
$\therefore$ **the points R, S and T are collinear and $RS : ST = 2 : 3$.**

14. (i) $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$
$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{2} \\ \mathbf{8} \end{pmatrix}$

(ii) $\overrightarrow{AM} = \overrightarrow{BM} - \overrightarrow{BA}$
$= \frac{1}{2}\overrightarrow{BC} + \overrightarrow{AB}$
$= \frac{1}{2}\begin{pmatrix} 2 \\ 8 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix}$
$= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix}$
$= \begin{pmatrix} \mathbf{5} \\ \mathbf{-1} \end{pmatrix}$

(iii) If $ABCD$ is a parallelogram, then
$\overrightarrow{AD} = \overrightarrow{BC}$
$\overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{BC}$
$\overrightarrow{OD} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$
$\overrightarrow{OD} = \begin{pmatrix} 2 \\ 8 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ 10 \end{pmatrix}$
$\therefore$ coordinates of D are **(3, 10)**.

15. (i) (a) $\overrightarrow{SA} = \frac{2}{3}\overrightarrow{SR}$
$= \frac{2}{3}\overrightarrow{PQ}$
$= \mathbf{\frac{2}{3}b}$

(b) $\overrightarrow{QB} = \frac{2}{3}\overrightarrow{PS}$
$= \mathbf{\frac{2}{3}a}$

(c) $\overrightarrow{PB} = \overrightarrow{PQ} + \overrightarrow{QB}$
$= \mathbf{b} + \frac{2}{3}\mathbf{a}$
$= \mathbf{\frac{2}{3}a + b}$

(d) $\overrightarrow{QS} = \overrightarrow{PS} - \overrightarrow{PQ}$
$= \mathbf{a - b}$

(e) $\overrightarrow{BA} = \overrightarrow{BR} + \overrightarrow{RA}$
$= \frac{1}{2}\overrightarrow{QB} + \frac{1}{2}\overrightarrow{AS}$
$= \frac{1}{2}\overrightarrow{QB} - \frac{1}{2}\overrightarrow{SA}$
$= \frac{1}{2}(\frac{2}{3}\mathbf{a}) - \frac{1}{2}(\frac{2}{3}\mathbf{b})$
$= \frac{1}{3}\mathbf{a} - \frac{1}{3}\mathbf{b}$
$= \mathbf{\frac{1}{3}(a - b)}$

(ii) $\frac{BA}{QS} = \mathbf{\frac{1}{3}}$

16. (i) (a) $\overrightarrow{RS} = \overrightarrow{PS} - \overrightarrow{PR}$
$= 5\mathbf{b} - (3\mathbf{a} + 12\mathbf{b})$
$= 5\mathbf{b} - 3\mathbf{a} - 12\mathbf{b}$
$= \mathbf{-3a - 7b}$

(b) $\overrightarrow{RT} = \overrightarrow{RP} + \overrightarrow{PT}$
$= -\overrightarrow{PR} + \frac{1}{3}\overrightarrow{PR}$
$= -(3\mathbf{a} + 12\mathbf{b}) + \frac{1}{3}(3\mathbf{a} + 12\mathbf{b})$
$= -\frac{2}{3}(3\mathbf{a} + 12\mathbf{b})$
$= \mathbf{-2a - 8b}$

(c) $\overrightarrow{RQ} = \overrightarrow{PQ} - \overrightarrow{PR}$
$= (4\mathbf{a} + \mathbf{b}) - (3\mathbf{a} + 12\mathbf{b})$
$= \mathbf{a - 11b}$

(ii) $\overrightarrow{QT} = \overrightarrow{RT} - \overrightarrow{RQ}$
$= (-2\mathbf{a} - 8\mathbf{b}) - (\mathbf{a} - 11\mathbf{b})$
$= -2\mathbf{a} - 8\mathbf{b} - \mathbf{a} + 11\mathbf{b}$
$= -3\mathbf{a} + 3\mathbf{b}$
$= 3(\mathbf{b} - \mathbf{a})$ (shown)

(iii) $\overrightarrow{QS} = \overrightarrow{PS} - \overrightarrow{PQ}$
$= 5\mathbf{b} - (4\mathbf{a} + \mathbf{b})$
$= 5\mathbf{b} - 4\mathbf{a} - \mathbf{b}$
$= 4\mathbf{b} - 4\mathbf{a}$
$= \mathbf{4(b - a)}$

(iv) (a) $\frac{QT}{QS} = \mathbf{\frac{3}{4}}$

(b) $\frac{\text{Area of } \triangle PQT}{\text{Area of } \triangle PQS} = \frac{\frac{1}{2} \times QT \times h}{\frac{1}{2} \times QS \times h}$, where h is the common height of $\triangle PQT$ and $\triangle PQS$.
$= \frac{QT}{QS}$
$= \mathbf{\frac{3}{4}}$

(c) $\frac{\text{Area of } \triangle PQT}{\text{Area of } \triangle PQR} = \frac{\frac{1}{2} \times PT \times h}{\frac{1}{2} \times PR \times h}$, where h is the common height of $\triangle PQT$ and $\triangle PQR$.
$= \frac{PT}{PR}$
$= \frac{1}{3}$

$\therefore \frac{\text{Area of } \triangle RQT}{\text{Area of } \triangle PQR} = \frac{2}{3}$

$\therefore \frac{\text{Area of } \triangle PQT}{\text{Area of } \triangle RQT} = \frac{1}{3} \div \frac{2}{3}$
$= \mathbf{\frac{1}{2}}$

17. (i) (a) $\overrightarrow{OT} = \overrightarrow{OC} + \overrightarrow{CT}$
$= \overrightarrow{OC} - \overrightarrow{TC}$
$= \mathbf{q} - 3(\mathbf{p} - \mathbf{q})$
$= \mathbf{q} - 3\mathbf{p} + 3\mathbf{q}$
$= \mathbf{4q - 3p}$

(b) $\overrightarrow{AT} = \overrightarrow{AC} + \overrightarrow{CT}$
$= (\overrightarrow{OC} - \overrightarrow{OA}) + (-\overrightarrow{TC})$
$= \mathbf{q} - \mathbf{p} - 3(\mathbf{p} - \mathbf{q})$
$= \mathbf{q} - \mathbf{p} - 3\mathbf{p} + 3\mathbf{q}$
$= \mathbf{4q - 4p}$

(c) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \overrightarrow{OA} + \overrightarrow{OC}$
$= \mathbf{p + q}$

(d) $\overrightarrow{BT} = \overrightarrow{OT} - \overrightarrow{OB}$
$= (4\mathbf{q} - 3\mathbf{p}) - (\mathbf{p} + \mathbf{q})$
$= 4\mathbf{q} - 3\mathbf{p} - \mathbf{p} - \mathbf{q}$
$= \mathbf{3q - 4p}$

(e) $\overrightarrow{TR} = \overrightarrow{TB} + \overrightarrow{BR}$
$= -\overrightarrow{BT} + \frac{1}{4}\overrightarrow{BT}$
$= -\frac{3}{4}\overrightarrow{BT}$
$= -\frac{3}{4}(3\mathbf{q} - 4\mathbf{p})$
$= \mathbf{\frac{3}{4}(4p - 3q)}$

(ii) $\overrightarrow{CR} = \overrightarrow{CB} + \overrightarrow{BR}$
$= \overrightarrow{OA} + (\overrightarrow{BT} + \overrightarrow{TR})$
$= \mathbf{p} + 3\mathbf{q} - 4\mathbf{p} + 3\mathbf{p} - \frac{9}{4}\mathbf{q}$
$= (3 - \frac{9}{4})\mathbf{q}$
$= \mathbf{\frac{3}{4}q}$ (shown)

(iii) $\frac{CR}{OC} = \mathbf{\frac{3}{4}}$

18. (i) Let $\overrightarrow{BA} = \mathbf{a}$ and $\overrightarrow{BD} = \mathbf{b}$.
$\overrightarrow{AD} = \overrightarrow{BD} - \overrightarrow{BA}$
$= \mathbf{b} - \mathbf{a}$
$\overrightarrow{BP} = \frac{1}{2}\overrightarrow{BA}$
$= \frac{1}{2}\mathbf{a}$
$\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BD}$
$= \frac{1}{2}\mathbf{b}$
$\overrightarrow{PQ} = \overrightarrow{BQ} - \overrightarrow{BP}$
$= \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$
$= \frac{1}{2}(\mathbf{b} - \mathbf{a})$
$= \frac{1}{2}\overrightarrow{AD}$

Since $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AD}$, then PQ is parallel to AD and $PQ = \frac{1}{2}AD$. (shown)

(ii) Using the same reasoning in part **(i)** for $\triangle CSR$, we can show that $SR = \frac{1}{2}AD$.
Since $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AD}$ and $\overrightarrow{SR} = \frac{1}{2}\overrightarrow{AD}$, then $\overrightarrow{PQ} = \overrightarrow{SR}$.
$\therefore$ PQ is parallel to SR and $PQ = SR$, i.e. $PQRS$ is a parallelogram. (shown)

Chapter 5 Relations and Functions

TEACHING NOTES

Suggested Approach

In this chapter, students learn the important concept of functions. This chapter takes the approach of progressing from relations to functions. As a start, teachers may use the example of a fruit juicer in the chapter opener to give an analogy of a function.

Section 5.1: Relations

The concept of relations is an introduction to the concept of functions, so teachers should spend some time to explain it. Teachers should use the arrow diagram, which consists of two sets connected by arrows, to explain this. Teachers are to explain the terms: domain, codomain and image, which students are required to know for this chapter.

Section 5.2: Functions

Teachers should highlight what makes a relation a function and emphasise that when a function is applied to any input x, it will produce exactly one output y. However, some students may have a misconception that each value of x will give a different value of y. Hence, teachers should point out that a function can produce the same value of y for different values of x, e.g. the function $f(x) = x^2$.

Once students have understood the relationship between the input x and output y, they are then able to represent a function using an equation, a table and a graph. Teachers may wish to note that students are to learn to write equations using the notation of a function in this section.

Section 5.3: Inverse functions

Students may find it confusing to understand that if $f(x) = y$, then $f^{-1}(y) = x$. Teachers may explain using the example on page 177 of the textbook, by representing Set A as the input x and Set B as the output y. After which, students should be guided to conclude that when mapping from Set A to Set B, the relationship is represented by $f(x) = y$; when mapping from the reverse direction, where Set B is the input and Set A is the output, the relationship is represented by $f^{-1}(y) = x$.

Since students need to express x as the subject of the equation, teachers are also recommended to revise the process of expressing one variable in terms of another and give some practice questions to the students. For example, in the equation $y = 3x + 2$, expressing x in terms of y would give $x = \frac{y-2}{3}$.

Section 5.4: Composite functions

Students may find it difficult to understand mapping a function to another function. As such, teachers may also rewrite the variables to help students understand. For example, on page 182 of the textbook, to map function f to function g, teachers may first write $y = f(x) = x^2$ and $g(y) = y + 1$. Then, $g(f(x)) = f(x) + 1 = x^2 + 1$.

Teachers may also wish to cite real-world contexts involving composite functions to help students appreciate and understand composite functions. An example is a savings account that pays interest monthly with the use of a composite function. Another example is in computer programming, where composite functions are used to determine the output of a series of instructions or operations. That is, the output or parameters from one function are passed to the next function.

Introductory Problem

(i) Rule: Multiply the input by 2, and add 1 to it

(ii) **(a)** $y = 2 \times 2 + 1$
$= \mathbf{5}$

(b) $y = -1 \times 2 + 1$
$= \mathbf{-1}$

(iii) By letting x be the subject of the equation $y = 2x + 1$,

$2x = y - 1$

$x = \frac{y-1}{3}$

(a) $x = \frac{11-1}{2}$
$= \mathbf{5}$

(b) $x = \frac{-5-1}{2}$
$= \mathbf{-3}$

(iv) **(a)** $y = 5^2$
$= \mathbf{25}$

(b) $y = (-5)^2$
$= \mathbf{25}$

(v) $y = \pm\sqrt{25}$
$= \mathbf{\pm 5}$

(vi) Since a function is a relationship between two variables x and y such that every specified input x produces exactly one output y, $y = x^2$ is a function but $y = \pm\sqrt{x}$ is not. For each value of x, $y = x^2$ always gives only 1 output value but $y = \pm\sqrt{x}$ gives 2 output values for values of x which are not 0.

5.2 Functions

Thinking Time (Page 170)

The relation shown in Fig. 5.3 is a one-to-one function for which every image in the codomain corresponds to exactly one element of the domain.

The relation in Fig. 5.4 is not a one-to-one function because A and B are images of more than one element in the domain.

The relation in Fig. 5.5 is not a one-to-one function because there are elements in the codomain that are not images of the elements in the domain.

Practise Now 1

(a) **Yes**. The relation is a function since every element in the domain 'Students' has a unique image in the codomain 'Scores'.

(b) **No**. The relation is not a function since the element d in the domain A has no image in the codomain B.

Practise Now 2

$f(x) = 10x + 4$

$g(x) = 4x - 6$

(a) $f(4) = 10(4) + 4$
$= \mathbf{44}$

(b) $f(-7) = 10(-7) + 4$
$= \mathbf{-66}$

(c) $f\left(-\frac{2}{3}\right) = 10\left(-\frac{2}{3}\right) + 4$
$= \mathbf{-2\frac{2}{3}}$

(d) $g(2) = 4(2) - 6$
$= \mathbf{2}$

(e) $2g(6) = 2[4(6) - 6]$
$= 36$

(f) $g\left(\frac{7}{8}\right) = 4\left(\frac{7}{8}\right) - 6$
$= \mathbf{-2\frac{1}{2}}$

(g) $f\left(\frac{1}{2}\right) = 10\left(\frac{1}{2}\right) + 4$
$= 5 + 4$
$= 9$

$g(1) = 4(1) - 6$
$= -2$

$f\left(\frac{1}{2}\right) + g(1) = 9 - 2$
$= \mathbf{7}$

(h) $f(x) = g(x)$

$10x + 4 = 4x - 6$

$10x - 4x = -6 - 4$

$6x = -10$

$x = \mathbf{-1\frac{2}{3}}$

(i) $f(x) = 34$

$10x + 4 = 34$

$10x = 30$

$x = \mathbf{3}$

Practise Now 3

$f(x) = 2x - 5$

$F(x) = 7x + 12$

(a) $f(b) = \mathbf{2b - 5}$

(b) $F(b - 1) = 7(b - 1) + 12$
$= 7b - 7 + 12$
$= \mathbf{7b + 5}$

(c) $f(2b) = 2(2b) - 5$
$= 4b - 5$

$F(2b - 5) = 7(2b - 5) + 12$
$= 14b - 35 + 12$
$= 14b - 23$

$f(2b) + F(2b - 5) = (4b - 5) + (14b - 23)$
$= \mathbf{18b - 28}$

Practise Now 4

(a) $f(3x) = 4(3x)^2 - 5(3x) + 2$
$= \mathbf{36x^2 - 15x + 2}$

(b) $f(2x + 3) = 4(2x + 3)^2 - 5(2x + 3) + 2$
$= 4(4x^2 + 12x + 9) - 10x - 15 + 2$
$= 16x^2 + 48x + 36 - 10x - 13$
$= \mathbf{16x^2 + 38x + 23}$

(c) $f(x^2 - 3) = 4(x^2 - 3)^2 - 5(x^2 - 3) + 2$
$= 4(x^4 - 6x^2 + 9) - 5x^2 + 15 + 2$
$= 4x^4 - 24x^2 + 36 - 5x^2 + 17$
$= \mathbf{4x^4 - 29x^2 + 53}$

Practise Now 5

$f(3) = a(3)^2 + 3b$
$= 9a + 3b$
$= 15$ — (1)

$f(-2) = a(-2)^2 + b(-2)$
$= 4a - 2b$
$= 8$
$2b = 4a - 8$
$b = 2a - 4$ — (2)

Substitute (2) into (1):
$9a + 3(2a - 4) = 15$
$9a + 6a - 12 = 15$
$15a = 27$
$a = 1\frac{4}{5}$

From (2), $b = 2\left(1\frac{4}{5}\right) - 4$
$= -\frac{2}{5}$

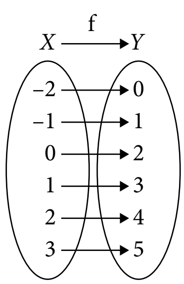
$\therefore\ a = \mathbf{1\frac{4}{5}},\ b = \mathbf{-\frac{2}{5}}$

$f(1) = 1\frac{4}{5}(1)^2 - \frac{2}{5}(1)$
$= \mathbf{1\frac{2}{5}}$

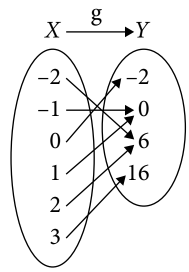
$f(-5) = 1\frac{4}{5}(-5)^2 - \frac{2}{5}(-5)$
$= \mathbf{47}$

Exercise 5A

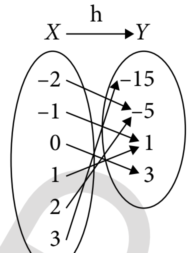
1. (a) **Yes**
   (b) **No**, because the element 2 has two images.
   (c) **Yes**
   (d) **Yes**
   (e) **No**, because the element 4 has no image.
   (f) **Yes**
2. Let set X be the domain and set Y be the codomain for each of the following functions.
   (a)



   (b)



   (c)



3. $f(2) = 6(2) - 4$
   $= \mathbf{8}$
   $f(-4) = 6(-4) - 4$
   $= \mathbf{-28}$
   $f\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right) - 4$
   $= \mathbf{-1}$
   $f\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{2}\right) - 4$
   $= \mathbf{-7}$
4. **(a)** $f(1) = 5 - 2(1)$
   $= \mathbf{3}$
   (b) $f(-2) = 5 - 2(-2)$
   $= 5 + 4$
   $= \mathbf{9}$
   (c) $f(0) = 5 - 2(0)$
   $= \mathbf{5}$
   (d) $f(3) = 5 - 2(3)$
   $= 5 - 6$
   $= -1$
   $f(-3) = 5 - 2(-3)$
   $= 5 + 6$
   $= 11$
   $f(3) + f(-3) = -1 + 11$
   $= \mathbf{10}$
5. **(a)** $g(2) = 7(2) + 4$
   $= \mathbf{18}$
   (b) $g(-3) = 7(-3) + 4$
   $= \mathbf{-17}$
   (c) $g\left(\frac{4}{7}\right) = 7\left(\frac{4}{7}\right) + 4$
   $= \mathbf{8}$
   (d) $g(0) = 7(0) + 4$
   $= 4$
   $g(-1) = 7(-1) + 4$
   $= -3$
   $g(0) + g(-1) = 4 + (-3)$
   $= 4 - 3$
   $= \mathbf{1}$

(e) $g\left(\frac{1}{7}\right) = 7\left(\frac{1}{7}\right) + 4$
$= 5$
$g\left(-\frac{1}{7}\right) = 7\left(-\frac{1}{7}\right) + 4$
$= 3$
$g\left(\frac{1}{7}\right) - g\left(-\frac{1}{7}\right) = 5 - 3$
$= \mathbf{2}$

6. (a) $g(a) = \boldsymbol{a^2 + 5}$
(b) $g(a + 1) = (a + 1)^2 + 5$
$= a^2 + 2a + 1 + 5$
$= \boldsymbol{a^2 + 2a + 6}$
(c) $g(a + 1) - g(a - 1) = (a^2 + 2a + 6) - [(a - 1)^2 + 5]$
$= a^2 + 2a + 6 - (a^2 - 2a + 1 + 5)$
$= \boldsymbol{4a}$

7. (a) $F(x - 1) = \frac{1}{2}(x - 1)(x - 1 + 1)$
$= \boldsymbol{\frac{1}{2}x(x - 1)}$
(b) $F(x + 1) = \frac{1}{2}(x + 1)(x + 1 + 1)$
$= \boldsymbol{\frac{1}{2}(x + 1)(x + 2)}$
(c) $F(x) - F(x - 1) = \frac{1}{2}x(x + 1) - \frac{1}{2}x(x - 1)$
$= \frac{1}{2}x[x + 1 - (x - 1)]$
$= \frac{1}{2}x(x + 1 - x + 1)$
$= \frac{1}{2}x(2)$
$= \boldsymbol{x}$
(d) $F(x^2) = \boldsymbol{\frac{1}{2}x^2(x^2 + 1)}$

8. $f(x) = \frac{x}{2} + 3$
$g(x) = \frac{3}{4}x - 2$
(a) (i) $f(2) = \frac{2}{2} + 3$
$= 4$
$g(2) = \frac{3}{4}(2) - 2$
$= \frac{3}{2} - 2$
$= -\frac{1}{2}$
$f(2) + g(2) = 4 - \frac{1}{2}$
$= \mathbf{3\frac{1}{2}}$

(ii) $f(-1) = \frac{-1}{2} + 3$
$= 2\frac{1}{2}$
$g(-1) = \frac{3}{4}(-1) - 2$
$= -2\frac{3}{4}$
$f(-1) - g(-1) = 2\frac{1}{2} - \left(-2\frac{3}{4}\right)$
$= \mathbf{5\frac{1}{4}}$

(iii) $2f(4) = 2\left(\frac{4}{3} + 3\right)$
$= 2(5)$
$= 10$
$3g(6) = 3\left[\frac{3}{4}(6) - 2\right]$
$= 3\left[\frac{5}{2}\right]$
$= 7\frac{1}{2}$
$2f(4) - 3g(6) = 10 - 7\frac{1}{2}$
$= \mathbf{2\frac{1}{2}}$

(iv) $5f(-2) = 5\left(\frac{-2}{2} + 3\right)$
$= 5(2)$
$= 10$
$7g(-4) = 7\left[\frac{3}{4}(-4) - 2\right]$
$= 7[-5]$
$= -35$
$5f(-2) - 7g(-4) = 10 - (-35)$
$= \mathbf{45}$

(b) $f(x) = g(x)$
$\frac{x}{2} + 3 = \frac{3}{4}x - 2$
$\frac{1}{4}x = 5$
$x = \mathbf{20}$
For $f(x) = 17$,
$\frac{x}{2} + 3 = 17$
$\frac{x}{2} = 14$
$x = \mathbf{28}$

9. (a) $f(x) = 16$
$5x - 9 = 16$
$5x = 25$
$x = \mathbf{5}$
(b) $g(x) = 14$
$2 - 6x = 14$
$-6x = 12$
$x = \mathbf{-2}$

(c) $g(x) = x$

$2 - 6x = x$

$2 = 7x$

$x = \mathbf{\frac{2}{7}}$

(d) $f(x) = 2x$

$5x - 9 = 2x$

$3x = 9$

$x = \mathbf{3}$

(e) $f(x) = g(x)$

$5x - 9 = 2 - 6x$

$11x = 11$

$x = \mathbf{1}$

(f) $2f(x) = 3g(x)$

$2(5x - 9) = 3(2 - 6x)$

$10x - 18 = 6 - 18x$

$28x = 24$

$x = \mathbf{\frac{6}{7}}$

10. **(a)** $h(2a) = (2a)^2 - 5(2a) + 4$

$= 4a^2 - 10a + 4$

$h(a) = a^2 - 5a + 4$

$h(2a) - h(a) = (4a^2 - 10a + 4) - (a^2 - 5a + 4)$

$= \mathbf{3a^2 - 5a}$

(b) $a^2 - 5a + 4 = 0$

$(a - 4)(a - 1) = 0$

$a - 4 = 0$ or $a - 1 = 0$

$a = \mathbf{4}$ $\quad a = \mathbf{1}$

(c) $h(a^2) = (a^2)^2 - 5(a^2) + 4$

$= a^4 - 5a^2 + 4$

$h(a) = a^2 - 5a + 4$

$h(a^2) + h(a) = (a^4 - 5a^2 + 4) + (a^2 - 5a + 4)$

$= \mathbf{a^4 - 4a^2 - 5a + 8}$

11. $g(x) = mx + c$

$g(1) = m + c = 5$ — (1)

$g(5) = 5m + c = -4$ — (2)

(2) – (1): $4m = -9$

$m = -2\frac{1}{4}$

Substitute $m = -2\frac{1}{4}$ into (1): $-2\frac{1}{4} + c = 5$

$c = 5 + 2\frac{1}{4}$

$= 7\frac{1}{4}$

$\therefore m = \mathbf{-2\frac{1}{4}}, c = \mathbf{7\frac{1}{4}}$

$g(3) = -2\frac{1}{4}(3) + 7\frac{1}{4}$

$= \mathbf{\frac{1}{2}}$

$g(-4) = -2\frac{1}{4}(-4) + 7\frac{1}{4}$

$= \mathbf{16\frac{1}{4}}$

12. $f(x) = 4x + 9$

$f(1) = 4(1) + 9$

$= \mathbf{13}$

$f(2) = 4(2) + 9$

$= \mathbf{17}$

$f(3) = 4(3) + 9$

$= \mathbf{21}$

(a) $f(1) + f(2) = 13 + 17$

$= 30$

$f(1 + 2) = f(3) = 21$

$\therefore f(1) + f(2) \neq f(1 + 2)$

(b) $f(3) - f(2) = 21 - 17$

$= 4$

$f(3 - 2) = f(1) = 13$

$\therefore f(3) - f(2) \neq f(3 - 2)$

(c) $f(1) \times f(2) = 13 \times 17$

$= 221$

$f(1 \times 2) = f(2) = 17$

$\therefore f(1) \times f(2) \neq f(1 \times 2)$

(d) $f(2) \div f(1) = 17 \div 13$

$= 1\frac{4}{13}$

$f(2 \times 1) = f(2) = 17$

$\therefore f(2) \div f(1) \neq f(2 \div 1)$

13. $f(x) = \frac{3}{4}x + \frac{1}{2}$

$g(x) = 1\frac{1}{2} - \frac{2}{3}x$

$f(2) = \frac{3}{4}(2) + \frac{1}{2}$

$= \mathbf{2}$

$f\left(\frac{1}{2}\right) = \frac{3}{4}\left(\frac{1}{2}\right) + \frac{1}{2}$

$= \mathbf{\frac{7}{8}}$

$g(3) = 1\frac{1}{2} - \frac{2}{3}(3)$

$= \mathbf{-\frac{1}{2}}$

$g(-6) = 1\frac{1}{2} - \frac{2}{3}(-6)$

$= \mathbf{5\frac{1}{2}}$

(a) $f(3) = \frac{3}{4}(3) + \frac{1}{2}$

$= 2\frac{3}{4}$

$f(2) + f(3) = 2 + 2\frac{3}{4}$

$= 4\frac{3}{4}$

$f(2 + 3) = f(5)$

$= \frac{3}{4}(5) + \frac{1}{2}$

$= 4\frac{1}{4}$

$\therefore f(2) + f(3) \neq f(2 + 3)$

(b) $g(4) = 1\frac{1}{2} - \frac{2}{3}(4)$

$= -1\frac{1}{6}$

$g(2) = 1\frac{1}{2} - \frac{2}{3}(2)$

$= \frac{1}{6}$

$g(4) - g(2) = -1\frac{1}{6} - \frac{1}{6}$

$= -1\frac{1}{3}$

$\therefore g(4) - g(2) \neq g(4 - 2)$

(c) $f(x) = g(x)$

$\frac{3}{4}x + \frac{1}{2} = 1\frac{1}{2} - \frac{2}{3}x$

$\frac{3}{4}x + \frac{2}{3}x = 1\frac{1}{2} - \frac{1}{2}$

$\frac{17}{12}x = 1$

$x = \mathbf{\frac{12}{17}}$

(d) $f(a) = \mathbf{\frac{3}{4}a + \frac{1}{2}}$

$f(2a) = \frac{3}{4}(2a) + \frac{1}{2}$

$= \mathbf{\frac{3}{2}a + \frac{1}{2}}$

$g(3a) = 1\frac{1}{2} - \frac{2}{3}(3a)$

$= \mathbf{1\frac{1}{2} - 2a}$

(e) $f(a + 1) + g(a) = 5$

$\frac{3}{4}(a + 1) + \frac{1}{2} + 1\frac{1}{2} - \frac{2}{3}(a) = 5$

$\frac{3}{4}a + \frac{3}{4} + \frac{1}{2} + 1\frac{1}{2} - \frac{2}{3}a = 5$

$\frac{1}{12}a + \frac{11}{4} = 5$

$\frac{1}{12}a = \frac{9}{4}$

$a = \mathbf{27}$

(f) $f(2a) = g(6a)$

$\frac{3}{2}a + \frac{1}{2} = 1\frac{1}{2} - \frac{2}{3}(6a)$

$\frac{3}{2}a + \frac{1}{2} = 1\frac{1}{2} - 4a$

$\frac{3}{2}a + 4a = 1\frac{1}{2} - \frac{1}{2}$

$\frac{11}{2}a = 1$

$a = \mathbf{\frac{2}{11}}$

14. $h(2) = 34$

$p(2)^2 + q(2) + 2 = 34$

$4p + 2q + 2 = 34$

$4p + 2q = 32$

$2p + q = 16$ — (1)

$h(-3) = 29$

$p(-3)^2 + q(-3) + 2 = 29$

$9p - 3q + 2 = 29$

$9p - 3q = 27$

$3p - q = 9$ — (2)

(1) + (2):

$2p + q + 3p - q = 16 + 9$

$5p = 25$

$p = \mathbf{5}$

Substitute $p = 5$ into (1):

$2(5) + q = 16$

$10 + q = 16$

$q = \mathbf{6}$

$\therefore h(x) = 5x^2 + 6x + 2$

$h(4) = 5(4)^2 + 6(4) + 2$

$= 80 + 24 + 2$

$= \mathbf{106}$

$h(-2) = 5(-2)^2 + 6(-2) + 2$

$= 20 - 12 + 2$

$= \mathbf{10}$

5.3 Inverse functions

Practise Now 6

$f(x) = 8x + 3$

Let $y = 8x + 3$

$f(x) = y$ and $f^{-1}(y) = x$

$x = \frac{1}{8}(y - 3)$

$\therefore f^{-1}(y) = \frac{1}{8}(y - 3)$

$f^{-1}(x) = \mathbf{\frac{1}{8}(x - 3)}$

Practise Now 7

$f(x) = 7x - 4$

Let $y = 7x - 4$

$f(x) = y$ and $f^{-1}(y) = x$

$x = \frac{1}{7}(y + 4)$

$\therefore f^{-1}(y) = \frac{1}{7}(y + 4)$

Hence $f^{-1}(x) = \mathbf{\frac{1}{7}(x + 4)}$

$f^{-1}(10) = \frac{1}{7}(10 + 4)$

$= \mathbf{2}$

$f^{-1}(-4) = \frac{1}{7}(-4 + 4)$

$= \mathbf{0}$

$f^{-1}\left(\frac{1}{7}\right) = \frac{1}{7}\left(\frac{1}{7} + 4\right)$

$= \mathbf{\frac{29}{49}}$

Practise Now 8

$f(x) = \frac{2x}{x-5}$

Let $y = \frac{2x}{x-5}$

$f(x) = y$ and $f^{-1}(y) = x$

$y(x-5) = 2x$

$xy - 5y = 2x$

$xy - 2x = 5y$

$x(y-2) = 5y$

$x = \frac{5y}{y-2}$

$\therefore f^{-1}(y) = \frac{5y}{y-2}$

Hence $f^{-1}(x) = \mathbf{\frac{5x}{x-2}}$

f^{-1} is not defined when $x = \mathbf{2}$.

$f^{-1}(6) = \frac{5(6)}{6-2}$

$= \mathbf{7\frac{1}{2}}$

$f^{-1}(-3) = \frac{5(-3)}{-3-2}$

$= \mathbf{3}$

$f^{-1}\left(\frac{1}{4}\right) = \frac{5\left(\frac{1}{4}\right)}{\frac{1}{4} - 2}$

$= \mathbf{-\frac{5}{7}}$

Practise Now 9

$f(3) = 15$

$p(3) + q = 15$

$3p + q = 15$ — (1)

$f^{-1}(3) = 6$

$f(6) = 3$

$p(6) + q = 3$

$6p + q = 3$ — (2)

(2) – (1):

$6p + q - (3p + q) = 3 - 15$

$6p + q - 3p - q = -12$

$3p = -12$

$p = \mathbf{-4}$

Substitute $p = -4$ into (1):

$3(-4) + q = 15$

$-12 + q = 15$

$q = \mathbf{27}$

Exercise 5B

1. $f(x) = \frac{1}{4}x - 3$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = \frac{1}{4}x - 3$

$x = 4y + 12$

$\therefore f^{-1}(y) = 4y + 12$

Hence $f^{-1}(x) = \mathbf{4x + 12}$

2. $f(x) = x - 7$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = x - 7$

$x = y + 7$

$\therefore f^{-1}(y) = y + 7$

Hence $f^{-1} : x \mapsto \mathbf{x + 7}$

$f^{-1}(3) = 3 + 7$

$= \mathbf{10}$

$f^{-1}(7) = 7 + 7$

$= \mathbf{14}$

$f^{-1}(-5) = -5 + 7$

$= \mathbf{2}$

$f^{-1}\left(\frac{1}{3}\right) = \frac{1}{3} + 7$

$= \mathbf{7\frac{1}{3}}$

3. $g(x) = 3x + 4$

Let $y = g(x)$

$g(x) = y$ and $g^{-1}(y) = x$

$y = 3x + 4$

$x = \frac{y+4}{3}$

$\therefore g^{-1}(y) = \frac{y+4}{3}$

Hence $g^{-1} : x \mapsto \mathbf{\frac{x-4}{3}}$

$g^{-1}(3) = \frac{1}{3}(3-4)$

$= \mathbf{-\frac{1}{3}}$

$g^{-1}(-4) = \frac{1}{3}(-4-4)$

$= \mathbf{-2\frac{2}{3}}$

$g^{-1}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{1}{2} - 4\right)$

$= \mathbf{-1\frac{1}{6}}$

$g^{-1}\left(-\frac{3}{4}\right) = \frac{1}{3}\left(-\frac{3}{4} - 4\right)$

$= \mathbf{-1\frac{7}{12}}$

4. $h(x) = 5x + 6$

Let $y = h(x)$

$h(x) = y$ and $h^{-1}(y) = x$

$y = 5x + 6$

$x = \frac{y-6}{5}$

$\therefore h^{-1}(y) = \frac{y-6}{5}$

Hence $h^{-1} : x \mapsto \mathbf{\frac{x-6}{5}}$

$h^{-1}(6) = \frac{6-6}{5}$

$= \mathbf{0}$

$h^{-1}(10) = \frac{10-6}{5}$

$= \mathbf{\frac{4}{5}}$

$h^{-1}\left(-\frac{2}{5}\right) = \frac{-\frac{2}{5} - 6}{5}$

$= -\frac{32}{25}$

$= \mathbf{-1\frac{7}{25}}$

$h^{-1}\left(12\frac{1}{2}\right) = \frac{12\frac{1}{2} - 6}{5}$

$= \frac{13}{10}$

$= \mathbf{1\frac{3}{10}}$

5. $f(x) = 8 - 3x$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = 8 - 3x$

$x = \frac{8-y}{3}$

$\therefore f^{-1}(y) = \frac{8-y}{3}$

Hence $f^{-1}(x) = \mathbf{\frac{8-x}{3}}$

$f^{-1}(9) = \frac{8-9}{3}$

$= \mathbf{-\frac{1}{3}}$

$f^{-1}(-12) = \frac{8-(-12)}{3}$

$= \mathbf{6\frac{2}{3}}$

$f^{-1}\left(3\frac{1}{3}\right) = \frac{8 - \left(3\frac{1}{2}\right)}{3}$

$= \mathbf{1\frac{5}{9}}$

$f^{-1}\left(-\frac{3}{16}\right) = \frac{8 - \left(-\frac{3}{16}\right)}{3}$

$= \mathbf{2\frac{35}{48}}$

6. For $g(x) = 10$, $6x - 8 = 10$

$6x = 18$

$x = \mathbf{3}$

For $g(x) = 40$, $6x - 8 = 40$

$6x = 48$

$x = \mathbf{8}$

For $g(x) = -4$, $6x - 8 = -4$

$6x = 4$

$x = \mathbf{\frac{2}{3}}$

For $g(x) = -6$, $6x - 8 = -6$

$6x = 2$

$x = \mathbf{\frac{1}{3}}$

7. $f(x) = 7 - \frac{3}{5}x$

Let $y_1 = f(x)$

$f(x) = y_1$ and $f^{-1}(y_1) = x$

$y_1 = 7 - \frac{3}{5}x$

$x = \frac{7 - y_1}{\frac{3}{5}}$

$= \frac{5(7-y_1)}{3}$

$\therefore f^{-1}(y_1) = \frac{5(7-y_1)}{3}$

Hence $f^{-1}(x) = \frac{5(7-x)}{3}$

$g(x) = \frac{1}{4}x - 6$

Let $y_2 = g(x)$

$g(x) = y_2$ and $g^{-1}(y_2) = x$

$y_2 = \frac{1}{4}x - 6$

$x = \frac{y_2 + 6}{\frac{1}{4}}$

$= 4(y_2 + 6)$

$\therefore g^{-1}(y_2) = 4(y_2 + 6)$

Hence $g^{-1}(x) = 4(x + 6)$

(a) $f^{-1}(3) = \frac{5(7-3)}{3}$

$= \mathbf{6\frac{2}{3}}$

(b) $f^{-1}(-17) = \frac{5[7-(-17)]}{3}$

$= \mathbf{40}$

(c) $g^{-1}(5) = 4(5 + 6)$

$= \mathbf{44}$

(d) $g^{-1}(-6) = 4(-6 + 6)$

$= \mathbf{0}$

(e) $f^{-1}(2) = \frac{5(7-2)}{3}$

$= 8\frac{1}{3}$

$g^{-1}(1) = 4(1+6)$

$= 28$

$f^{-1}(2) + g^{-1}(1) = 8\frac{1}{3} + 28$

$= \mathbf{36\frac{1}{3}}$

(f) $f^{-1}(4) = \frac{5(7-4)}{3}$

$= 5$

$g^{-1}(4) = 4(4+6)$

$= 40$

$f^{-1}(4) - g^{-1}(4) = 5 - 40$

$= \mathbf{-35}$

8. $f(-2) = 20$

$a(-2) - b = 20$

$-2a - b = 20$

$2a + b = -20$ — (1)

$f^{-1}(32) = 4$

$f(4) = 32$

$a(4) - b = 32$

$4a - b = 32$ — (2)

(1) + (2):

$2a + b + 4a - b = -20 + 32$

$6a = 12$

$a = \mathbf{2}$

Substitute $a = 2$ into (1):

$2(2) + b = -20$

$4 + b = -20$

$b = \mathbf{-24}$

9. $f(x) = \frac{5x}{2-4x}$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = \frac{5x}{2-4x}$

$y(2 - 4x) = 5x$

$2y - 4xy = 5x$

$5x + 4xy = 2y$

$x(5 + 4y) = 2y$

$x = \frac{2y}{5+4y}$

$\therefore f^{-1}(y) = \frac{2y}{5+4y}$

Hence $f^{-1}(x) = \mathbf{\frac{2x}{4x+5}}$

$f^{-1}(x)$ is not defined when $5 + 4x = 0$ i.e. $x = -\mathbf{\frac{5}{4}}$

$f^{-1}(4) = \frac{2(4)}{5+4(4)}$

$= \mathbf{\frac{8}{21}}$

$f^{-1}(-6) = \frac{2(-6)}{5+4(-6)}$

$= \frac{-12}{-19}$

$= \mathbf{\frac{12}{19}}$

10. $f(x) = \frac{3x-1}{x-2}$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = \frac{3x-1}{x-2}$

$y(x-2) = 3x - 1$

$xy - 2y - 3x = -1$

$x(y-3) = 2y - 1$

$x = \frac{2y-1}{y-3}$

$\therefore f^{-1}(y) = \frac{2y-1}{y-3}$

Hence $f^{-1}(x) = \mathbf{\frac{2x-1}{x-3}}$

$f^{-1}(x)$ is not defined when $x - 3 = 0$ i.e. $x = 3$.

$f^{-1}(5) = \frac{2(5)-1}{5-3}$

$= \frac{9}{2}$

$= \mathbf{4\frac{1}{2}}$

$f^{-1}(7) = \frac{2(7)-1}{7-3}$

$= \frac{13}{4}$

$= \mathbf{3\frac{1}{4}}$

11. $h(1) = 2$

$p(1)^2 + q(1) = 2$

$p + q = 2$ — (1)

$h^{-1}(36) = 3$

$h(3) = 36$

$p(3)^2 + q(3) = 36$

$9p + 3q = 36$

$3p + q = 12$ — (2)

(2) – (1):

$3p + q - (p + q) = 12 - 2$

$3p + q - p - q = 10$

$2p = 10$

$p = \mathbf{5}$

Substitute $p = 5$ into (1):

$5 + q = 2$

$q = \mathbf{-3}$

$\therefore h(x) = 5x^2 - 3x$

$h(-1) = 5(-1)^2 - 3(-1)$

$= 5 + 3$

$= \mathbf{8}$

$h(2) = 5(2)^2 - 3(2)$

$= 20 - 6$

$= \mathbf{14}$

12. $f(1) = 1$

$\frac{a(1) - b}{4} = 1$

$a - b = 4$ — (1)

$f^{-1}(5) = 2$

$f(2) = 5$

$\frac{a(2) - b}{4} = 5$

$2a - b = 20$ — (2)

(2) – (1):

$2a - b - (a - b) = 20 - 4$

$2a - b - a + b = 16$

$a = \mathbf{16}$

Substitute $a = 16$ into (1):

$16 - b = 4$

$b = 16 - 4$

$= \mathbf{12}$

$\therefore f(x) = \frac{16x - 12}{4}$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = \frac{16x - 12}{4}$

$4y = 16x - 12$

$16x = 4y + 12$

$x = \frac{4y + 12}{16}$

$= \frac{y + 3}{4}$

$\therefore f^{-1}(y) = \frac{y + 3}{4}$

Hence $f^{-1}(x) = \mathbf{\frac{x + 3}{4}}$

$f^{-1}(7) = \frac{7 + 3}{4}$

$= \mathbf{2\frac{1}{2}}$

$f^{-1}\left(-5\frac{1}{2}\right) = \frac{-5\frac{1}{2} + 3}{4}$

$= \frac{-2\frac{1}{2}}{4}$

$= \mathbf{-\frac{5}{8}}$

13. $f(1) = 3$

$a(1) + b = 3$

$a + b = 3$ — (1)

$f^{-1}(7) = 5$

$f(5) = 7$

$a(5) + b = 7$

$5a + b = 7$ — (2)

(2) – (1):

$5a + b - (a + b) = 7 - 3$

$5a + b - a - b = 4$

$4a = 4$

$a = \mathbf{1}$

Substitute $a = 1$ into (1):

$1 + b = 3$

$b = \mathbf{2}$

$\therefore f(x) = x + 2$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = x + 2$

$x = y - 2$

$\therefore f^{-1}(y) = y - 2$

Hence $f^{-1}(x) = \mathbf{x - 2}$

14. $f(1) = -5$

$p(1) + q = -5$

$p + q = -5$ — (1)

$f(-2) = -10$

$p(-2) + q = -10$

$-2p + q = -10$ — (2)

(1) – (2):

$p + q - (-2p + q) = -5 - (-10)$

$p + q + 2p - q = -5 + 10$

$3p = 5$

$p = \mathbf{1\frac{2}{3}}$

Substitute $p = 1\frac{2}{3}$ into (1):

$1\frac{2}{3} + q = -5$

$q = \mathbf{-6\frac{2}{3}}$

$\therefore f(x) = \frac{5}{3}x - \frac{20}{3}$

Let $y = f(x)$

$f(x) = y$ and $f^{-1}(y) = x$

$y = \frac{5}{3}x - \frac{20}{3}$

$3y = 5x - 20$

$5x = 3y + 20$

$x = \frac{3}{5}y + 4$

$\therefore f^{-1}(y) = \frac{3}{5}y + 4$

Hence $f^{-1}(x) = \mathbf{\frac{3}{5}x + 4}$

15. $g^{-1}(-3) = 0$

$g(0) = -3$

$m(0) + c = -3$

$c = -3$

$g^{-1}(1) = 2$

$g(2) = 1$

$m(2) - 3 = 1$

$2m - 3 = 1$

$2m = 4$

$m = 2$

$\therefore g(x) = 2x - 3$

Let $y = g(x)$

$g(x) = y$ and $g^{-1}(y) = x$

$y = 2x - 3$

$2x = y + 3$

$x = \frac{y+3}{2}$

$\therefore g^{-1}(y) = \frac{y+3}{2}$

Hence $g^{-1}(x) = \frac{x+3}{2}$

$g(5) = 2(5) - 3$

$= \mathbf{7}$

$g^{-1}(4) = \frac{4+3}{2}$

$= \mathbf{3\frac{1}{2}}$

5.4 Composite functions

Practise Now 10

$gf(x) = g(f(x))$

$= g\left(\frac{2}{x-1}\right)$

$= 4\left(\frac{2}{x-1} + 2\right)^2$

$= 4\left[\frac{4}{(x-1)^2} + \frac{8}{x-1} + 4\right]$

$= \frac{16}{(x-1)^2} + \frac{32}{x-1} + 16$

$= \frac{16 + 32(x-1) + 16(x-1)^2}{(x-1)^2}$

$= \frac{16 + 32x - 32 + 16(x^2 - 2x + 1)}{(x-1)^2}$

$= \frac{16 + 32x - 32 + 16x^2 - 32x + 16}{(x-1)^2}$

$= \mathbf{\frac{16x^2}{(x-1)^2}}$

$fg(x) = f(g(x))$

$= f(4(x+2)^2)$

$= \frac{2}{4(x+2)^2 - 1}$

$= \frac{2}{4(x^2 + 4x + 4) - 1}$

$= \frac{2}{4x^2 + 16x + 16 - 1}$

$= \frac{2}{4x^2 + 16x + 15}$

$= \mathbf{\frac{2}{(2x+3)(2x+5)}}$

$gf(-2) = \frac{16(-2)^2}{(-2-1)^2}$

$= \mathbf{7\frac{1}{9}}$

$fg(3) = \frac{2}{4(3)^2 + 16(3) + 15}$

$= \mathbf{\frac{2}{99}}$

Practise Now 11

1. $fg(x) = gf(x) + 1$

$f(g(x)) = g(f(x)) + 1$

$f(2x + b) = g\left(\frac{x}{2} + a\right) + b + 1$

$\frac{2x+b}{2} + a = 2\left(\frac{x}{2} + a\right) + b + 1$

$\frac{2x+b}{2} + a = x + 2a + b + 1$

$2x + b + 2a = 2x + 4a + 2b + 2$

$b + 2a = 4a + 2b + 2$

$-b - 2 = 2a$

$\therefore a = \mathbf{-\frac{b}{2} - 1}$

2. **(a)** $fg(x) = 2$

$f(g(x)) = 2$

$f\left(\frac{1}{x+1}\right) = 2$

$\frac{1}{x+1} + 1 = 2$

$\frac{1}{x+1} = 1$

$x + 1 = 1$

$\therefore x = \mathbf{0}$

(b) $gf(x) = \frac{1}{3}$

$g(f(x)) = \frac{1}{3}$

$g(x+1) = \frac{1}{3}$

$\frac{1}{x+1+1} = \frac{1}{3}$

$x + 2 = 3$

$\therefore x = \mathbf{1}$

Exercise 5C

1. **(a)** $f(x) = 9 - x$, $g(x) = 3x + 4$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f(3x + 4) \\ &= 9 - (3x + 4) \\ &= 9 - 3x - 4 \\ &= \mathbf{-3x + 5} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(9 - x) \\ &= 3(9 - x) + 4 \\ &= 27 - 3x + 4 \\ &= \mathbf{-3x + 31} \end{aligned}$$

(b) $f(x) = 2x - 3$, $g(x) = x^2 + 5$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f(x^2 + 5) \\ &= 2(x^2 + 5) - 3 \\ &= 2x^2 + 10 - 3 \\ &= \mathbf{2x^2 + 7} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(2x - 3) \\ &= (2x - 3)^2 + 5 \\ &= 4x^2 - 12x + 9 + 5 \\ &= \mathbf{4x^2 - 12x + 14} \end{aligned}$$

(c) $f(x) = x - 1$, $g(x) = \dfrac{4}{x}$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f\left(\frac{4}{x}\right) \\ &= \mathbf{\frac{4}{x} - 1} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(x - 1) \\ &= \mathbf{\frac{4}{x - 1}} \end{aligned}$$

(d) $f(x) = 2x - 1$, $g(x) = \dfrac{4}{x} - 3$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f\left(\frac{4}{x} - 3\right) \\ &= 2\left(\frac{4}{x} - 3\right) - 1 \\ &= \frac{8}{x} - 6 - 1 \\ &= \mathbf{\frac{8}{x} - 7} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(2x - 1) \\ &= \mathbf{\frac{4}{2x - 1} - 3} \end{aligned}$$

(e) $f(x) = x + 2$, $g(x) = \dfrac{3}{x + 1}$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f\left(\frac{3}{x + 1}\right) \\ &= \mathbf{\frac{3}{x + 1} + 2} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(x + 2) \\ &= \frac{3}{x + 2 + 1} \\ &= \mathbf{\frac{3}{x + 3}} \end{aligned}$$

(f) $f(x) = 3x$, $g(x) = 4x + 5$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f(4x + 5) \\ &= 3(4x + 5) \\ &= \mathbf{12x + 15} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(3x) \\ &= 4(3x) + 5 \\ &= \mathbf{12x + 5} \end{aligned}$$

(g) $f(x) = x - 2$, $g(x) = \dfrac{2}{x} + 3$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f\left(\frac{2}{x} + 3\right) \\ &= \frac{2}{x} + 3 - 2 \\ &= \mathbf{\frac{2}{x} + 1} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(x - 2) \\ &= \mathbf{\frac{2}{x - 2} + 3} \end{aligned}$$

(h) $f(x) = x^2$, $g(x) = \dfrac{1 + 2x}{x - 1}$

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f\left(\frac{1 + 2x}{x - 1}\right) \\ &= \mathbf{\left(\frac{1 + 2x}{x - 1}\right)^2} \end{aligned}$$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(x^2) \\ &= \mathbf{\frac{1 + 2x^2}{x^2 - 1}} \end{aligned}$$

(i) $f(x) = 5 - 2x, g(x) = \frac{x+2}{x-1}$

$fg(x) = f(g(x))$

$= f\left(\frac{x+2}{x-1}\right)$

$= 5 - 2\left(\frac{x+2}{x-1}\right)$

$= \frac{5(x-1) - 2(x+2)}{x-1}$

$= \frac{5x - 5 - 2x - 4}{x-1}$

$= \frac{3x-9}{x-1}$

$= \mathbf{\frac{3(x-3)}{x-1}}$

$gf(x) = g(f(x))$

$= g(5 - 2x)$

$= \frac{5-2x+2}{5-2x-1}$

$= \frac{7-2x}{4-2x}$

$= \mathbf{\frac{7-2x}{2(2-x)}}$

2. $f(x) = x + 1, g(x) = 3x + 2$

(i) $fg(x) = f(g(x))$

$= f(3x + 2)$

$= 3x + 2 + 1$

$= \mathbf{3x + 3}$

$gf(x) = g(f(x))$

$= g(x + 1)$

$= 3(x + 1) + 2$

$= 3x + 3 + 2$

$= \mathbf{3x + 5}$

(ii) $fg(3) = 3(3) + 3$

$= \mathbf{12}$

$gf(3) = 3(3) + 5$

$= \mathbf{14}$

$fg(-1) = 3(-1) + 3$

$= \mathbf{0}$

$gf(-1) = 3(-1) + 5$

$= \mathbf{2}$

3. $f(x) = 2x^2 + 3, g(x) = 2x + 1$

(i) $fg(x) = f(g(x))$

$= f(2x + 1)$

$= 2(2x + 1)^2 + 3$

$= 2(4x^2 + 4x + 1) + 3$

$= 8x^2 + 8x + 2 + 3$

$= \mathbf{8x^2 + 8x + 5}$

$gf(x) = g(f(x))$

$= g(2x^2 + 3)$

$= 2(2x^2 + 3) + 1$

$= 4x^2 + 6 + 1$

$= \mathbf{4x^2 + 7}$

(ii) $fg(-1) = 8(-1)^2 + 8(-1) + 5$

$= \mathbf{5}$

$gf(-1) = 4(-1)^2 + 7$

$= \mathbf{11}$

$fg(3) = 8(3)^2 + 8(3) + 5$

$= \mathbf{101}$

$gf(3) = 4(3)^2 + 7$

$= \mathbf{43}$

4. $f(x) = ax + b, g(x) = x + 7$

(i) $fg(x) = f(g(x))$

$= f(x + 7)$

$= a(x + 7) + b$

$fg(1) = 7$

$a(1 + 7) + b = 7$

$8a + b = 7 \quad — (1)$

$fg(2) = 15$

$a(2 + 7) + b = 15$

$9a + b = 15 \quad — (2)$

(2) – (1):

$9a + b - (8a + b) = 15 - 7$

$9a + b - 8a - b = 8$

$a = 8$

Substitute $a = 8$ into (1):

$8(8) + b = 7$

$64 + b = 7$

$b = -57$

$\therefore f(x) = 8x - 57$

$gf(x) = g(8x - 57)$

$= 8x - 57 + 7$

$= 8x - 50$

$gf(5) = 8(5) - 50$

$= \mathbf{-10}$

(ii) $gf(x) = 14$

$8x - 50 = 14$

$8x = 64$

$x = \mathbf{8}$

5. $f(x) = kx - 3, g(x) = 2x + 5$

(i) $fg(x) = f(g(x))$

$= f(2x + 5)$

$= k(2x + 5) - 3$

$= \mathbf{2kx + 5k - 3}$

(ii) $fg(x) = gf(x)$

$2kx + 5k - 3 = g(f(x))$

$2kx + 5k - 3 = g(kx - 3)$

$2kx + 5k - 3 = 2(kx - 3) + 5$

$2kx + 5k - 3 = 2kx - 6 + 5$

$2kx + 5k - 3 = 2kx - 1$

$5k - 3 = -1$

$5k = 2$

$k = \mathbf{\frac{2}{5}}$

6. $f(x) = \frac{2}{x}$, $g(x) = 3x - 4$

(i) $fg(x) = f(g(x))$
$= f(3x - 4)$
$= \mathbf{\frac{2}{3x - 4}}$

$gf(x) = g(f(x))$
$= g\left(\frac{2}{x}\right)$
$= 3\left(\frac{2}{x}\right) - 4$
$= \mathbf{\frac{6}{x} - 4}$

(ii) $fg(x) = gf(x)$
$\frac{2}{3x - 4} = \frac{6}{x} - 4$
$2x = 6(3x - 4) - 4x(3x - 4)$
$2x = 18x - 24 - 12x^2 + 16x$
$2x = 34x - 24 - 12x^2$
$12x^2 - 32x + 24 = 0$
$3x^2 - 8x + 6 = 0$
$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(6)}}{2(3)}$
$= \frac{8 \pm \sqrt{-8}}{6}$

$\therefore$ there are no real values of x for which $fg(x) = gf(x)$ (shown).

7. $f(x) = 2x + 3$, $g(x) = ax + b$

(i) $fg(x) = f(g(x))$
$= f(ax + b)$
$= 2(ax + b) + 3$
$= \mathbf{2ax + 2b + 3}$

$gf(x) = g(f(x))$
$= g(2x + 3)$
$= a(2x + 3) + b$
$= \mathbf{2ax + 3a + b}$

(ii) $gf(x) = x$
$2ax + 3a + b = x$
$2ax + (3a + b) = x$
$2a = 1$ — (1)
$3a + b = 0$ — (2)
From (1):
$a = \mathbf{\frac{1}{2}}$
Substitute $a = \frac{1}{2}$ into (2):
$3\left(\frac{1}{2}\right) + b = 0$
$\frac{3}{2} + b = 0$
$b = \mathbf{-1\frac{1}{2}}$

(iii) $fg(x) = 3gf(x)$
$2\left(\frac{1}{2}\right)x + 2\left(-1\frac{1}{2}\right) + 3 = 3\left[2\left(\frac{1}{2}\right)x + 3\left(\frac{1}{2}\right) - 1\frac{1}{2}\right]$
$x - 3 + 3 = 3x$
$x = 3x$
$2x = 0$
$x = \mathbf{0}$

8. $f(x) = 2x - 1$, $g(x) = x^2 + 5$

(i) $fg(x) = f(g(x))$
$= f(x^2 + 5)$
$= 2(x^2 + 5) - 1$
$= 2x^2 + 10 - 1$
$= \mathbf{2x^2 + 9}$

$gf(x) = g(f(x))$
$= g(2x - 1)$
$= (2x - 1)^2 + 5$
$= 4x^2 - 4x + 1 + 5$
$= \mathbf{4x^2 - 4x + 6}$

(ii) $fg(x) = gf(x)$
$2x^2 + 9 = 4x^2 - 4x + 6$
$2x^2 - 4x - 3 = 0$
$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$
$= \frac{4 \pm \sqrt{40}}{4}$
$=$ **2.58** or **−0.581** (to 3 s.f.)

Chapter 6 Further Trigonometry

TEACHING NOTES

Suggested Approach

Teachers may want to introduce this topic by asking students how measurements are obtained for towers and buildings in real life. Teachers can mention that such measurements can be easily obtained using trigonometry before introducing Sine Rule and Cosine Rule to them. This chapter exposes students to problems involving triangles which can be solved using trigonometric ratios of acute and obtuse angles.

Section 6.1: Sine and Cosine of obtuse angles

Teachers may wish to recap with students the trigonometric ratios of an acute angle before introducing the trigonometric ratios of the sine and cosine of obtuse angles. Before students move on to solve simple trigonometric equations, teachers can guide them along in discovering the relationship between trigonometric ratios of acute and obtuse angles (see Investigation: Relationship between trigonometric ratios of acute and obtuse angles).

Section 6.2: Area of triangle

In primary school, students are taught to find the area of a triangle by the formula, $\frac{1}{2} \times \text{base} \times \text{height}$. Teachers can build upon this and ask students to derive a formula with what they have learnt so far, such that the area of a triangle can still be calculated if the height of the triangle is not given. Some guidance will be needed while students work among themselves to derive the formula.

Section 6.3: Sine Rule

Teachers can start off with an activity to introduce the Sine Rule to students (see Investigation: Sine Rule). When using a calculator, it is important to remind students to check and see that the MODE is set to DEG. In addition, students are to give their answers correct to 3 significant figures and angles in degrees correct to 1 decimal place, unless otherwise stated. Therefore, students should have developed the habit of working with 4 or 5 significant figures, and angles in degrees to 2 decimal places and give their final answers correct to the required accuracy.

Teachers should highlight the ambiguous cases when using the Sine Rule to students (see Investigation: Ambiguous case and Worked Example 8). After which, students should be able to identify ambiguous cases. In cases where diagrams are provided, unless stated, students may exercise visual observation to determine if an angle is acute or obtuse, even though it may fall under an ambiguous case.

Section 6.4: Cosine Rule

Similar to the Sine Rule, teachers can start off with an activity to introduce the Cosine Rule to students (see Investigation: Cosine Rule). Teachers can show students how the Cosine Rule is derived when the included angle is an acute angle, and then challenge them to prove the Cosine Rule where the included angle is an obtuse angle (see Thinking Time on page 212 of the textbook).

Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 3).

6.1 Sine and cosine of obtuse angles

Investigation (Relationship between trigonometric ratios of acute and obtuse angles)

	A	$180° - A$	$\sin A$	$\sin (180° - A)$	$\cos A$	$\cos (180° - A)$
(a)	30°	150°	**0.5**	**0.5**	**0.866 (to 3 s.f.)**	**–0.866 (to 3 s.f.)**
(b)	76°	104°	**0.970 (to 3 s.f.)**	**0.970 (to 3 s.f.)**	**0.242 (to 3 s.f.)**	**–0.242 (to 3 s.f.)**
(c)	111°	69°	**0.934 (to 3 s.f.)**	**0.934 (to 3 s.f.)**	**–0.358 (to 3 s.f.)**	**0.358 (to 3 s.f.)**
(d)	167°	13°	**0.225 (to 3 s.f.)**	**0.225 (to 3 s.f.)**	**–0.974 (to 3 s.f.)**	**0.974 (to 3 s.f.)**

1. $\sin A = \mathbf{\sin (180° - A)}$
2. $\cos A = \mathbf{-\cos (180° - A)}$
3. The **sine** of obtuse angles is positive and the **cosine** of obtuse angles is negative.

Practise Now 1

1. **(a)** $\sin 96° = \sin (180° - 96°)$
$= \sin 84°$
$= \mathbf{0.995}$

(b) $\cos 51° = -\cos (180° - 51°)$
$= -\cos 129°$
$= -(-0.629)$
$= \mathbf{0.629}$

2. $\sin 8° - \cos 140° = \sin (180° - 8°) - [-\cos (180° - 140°)]$
$= \sin 172° - (-\cos 40°)$
$= \sin 172° + \cos 40°$
$= 0.139 + 0.766$
$= \mathbf{0.905}$

Practise Now 2

1. **(a)** $\sin \angle ACD = \sin (180° - \angle ACD)$
$= \sin \angle ACB$
$= \dfrac{\text{opp}}{\text{hyp}}$
$= \dfrac{AB}{AC}$
$= \mathbf{\dfrac{3}{5}}$

(b) $\cos \angle ACD = -\cos (180° - \angle ACD)$
$= -\cos \angle ACB$
$= -\dfrac{\text{adj}}{\text{hyp}}$
$= -\dfrac{BC}{AC}$
$= \mathbf{-\dfrac{4}{5}}$

(c) $\tan \angle BAC = \dfrac{\text{opp}}{\text{adj}}$
$= \dfrac{BC}{AB}$
$= \mathbf{\dfrac{4}{3}}$

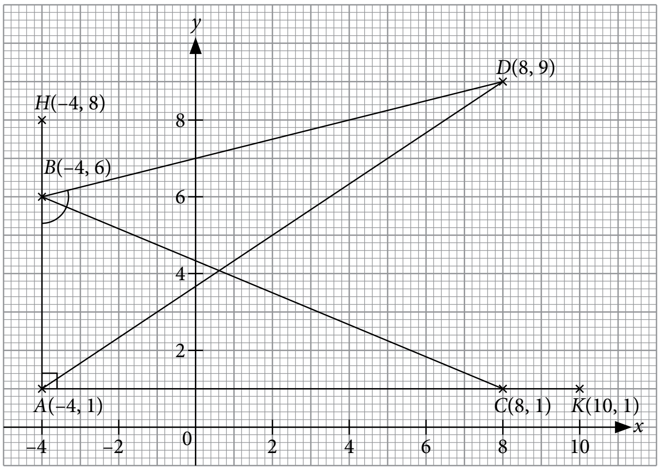
2. **(i)** $BC = \sqrt{[8 - (-4)]^2 + (1 - 6)^2}$
$= \sqrt{12^2 + (-5)^2}$
$= \sqrt{169}$
$= \mathbf{13\ units}$

(ii) (a) $\sin \angle HBC = \sin (180° - \angle HBC)$
$= \sin \angle ABC$
$= \dfrac{\text{opp}}{\text{hyp}}$
$= \dfrac{AC}{BC}$
$= \mathbf{\dfrac{12}{13}}$

(b) $\cos \angle BCK = -\cos (180° - \angle BCK)$
$= -\cos \angle ACB$
$= -\dfrac{\text{adj}}{\text{hyp}}$
$= -\dfrac{AC}{BC}$
$= \mathbf{-\dfrac{12}{13}}$

(c) $\tan \angle ABC = \dfrac{\text{opp}}{\text{adj}}$
$= \dfrac{AC}{AB}$
$= \mathbf{\dfrac{12}{5}}$

(iii) Since $\cos \angle ABD$ is negative when $\angle ABD$ is obtuse, a possible set of coordinates for D are **(8, 9)**.



Practise Now 3

(a) Since $\sin x$ is positive, x can either be an acute angle or an obtuse angle.

$\sin x = 0.415$

$x = \sin^{-1} 0.415 = 24.5°$ (to 1 d.p.)

or $180° - 24.5° = 155.5°$ (to 1 d.p.)

$\therefore$ x = **24.5° or 155.5°**

(b) Since $\cos x$ is negative, x is an obtuse angle.

$\cos x = -0.234$

$x = \cos^{-1}(-0.234)$

= **103.5°** (to 1 d.p.)

(c) Since $\cos x$ is positive, x is an acute angle.

$\cos x = 0.104$

$x = \cos^{-1} 0.104$

= **84.0°** (to 1 d.p.)

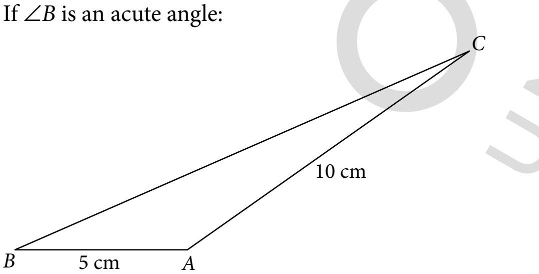
Introductory Problem Revisited

In $\triangle ABC$, $\sin B = \frac{2}{5}$, $AB = 5$ cm and $AC = 10$ cm.

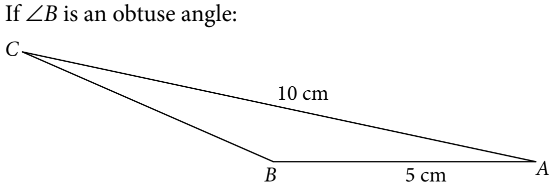
Since $\sin B$ is a positive ratio, B can be an acute angle or an obtuse angle.

We can sketch 2 different types of triangles for $\triangle ABC$ as shown.

If $\angle B$ is an acute angle:



If $\angle B$ is an obtuse angle:



Exercise 6A

1. **(a)** $\sin 110° = \sin(180° - 110°)$

$= \mathbf{\sin 70°}$

(b) $\sin 176° = \sin(180° - 176°)$

$= \mathbf{\sin 4°}$

(c) $\sin 98° = \sin(180° - 98°)$

$= \mathbf{\sin 82°}$

(d) $\cos 99° = -\cos(180° - 99°)$

$= \mathbf{-\cos 81°}$

(e) $\cos 107° = -\cos(180° - 107°)$

$= \mathbf{-\cos 73°}$

(f) $\cos 175° = -\cos(180° - 175°)$

$= \mathbf{-\cos 5°}$

2. **(a)** $\sin 148° = \sin(180° - 148°)$

$= \sin 32°$

$= \mathbf{0.530}$

(b) $\cos 35° = -\cos(180° - 35°)$

$= -\cos 145°$

$= -(-0.819)$

$= \mathbf{0.819}$

3. **(a)** $2\cos 45° + 3\sin 135° = 2\cos 45° + 3\sin(180° - 135°)$

$= 2\cos 45° + 3\sin 45°$

$= 2(0.707) + 3(0.707)$

$= 1.414 + 2.121$

$= \mathbf{3.535}$

(b) $3\cos 135° + 4\sin 135° = 3[-\cos(180° - 135°)] + 4\sin(180° - 135°)$

$= -3\cos 45° + 4\sin 45°$

$= -3(0.707) + 4(0.707)$

$= -2.121 + 2.828$

$= \mathbf{0.707}$

(c) $\cos 135° - 2\sin 45° = -\cos(180° - 135°) - 2\sin 45°$

$= -\cos 45° - 2\sin 45°$

$= -0.707 - 2(0.707)$

$= -0.707 - 1.414$

$= \mathbf{-2.121}$

4. **(a)** $\sin \angle ABD = \sin(180° - \angle ABD)$

$= \sin \angle CBD$

$= \frac{CD}{BD}$

$= \frac{8}{10}$

$= \mathbf{\frac{4}{5}}$

(b) $\cos \angle DBA = -\cos(180° - \angle DBA)$

$= -\cos \angle CBD$

$= -\frac{BC}{BD}$

$= -\frac{6}{10}$

$= \mathbf{-\frac{3}{5}}$

(c) $\tan \angle CBD = \frac{CD}{BC}$

$= \frac{8}{6}$

$= \mathbf{\frac{4}{3}}$

5. **(i)** Using Pythagoras' Theorem,

$PR^2 = PQ^2 + QR^2$

$41^2 = x^2 + 40^2$

$x^2 = 41^2 - 40^2$

$= 81$

$x = \sqrt{81}$ (since $x > 0$)

$= \mathbf{9}$

(ii) **(a)** $\sin \angle PRS = \sin (180° - \angle PRS)$

$= \sin \angle PRQ$

$= \frac{PQ}{PR}$

$= \mathbf{\frac{9}{41}}$

(b) $\cos \angle PRS = -\cos (180° - \angle PRS)$

$= -\cos \angle PRQ$

$= -\frac{QR}{PR}$

$= -\mathbf{\frac{40}{41}}$

(c) $\tan \angle PRQ = \frac{PQ}{QR}$

$= \mathbf{\frac{9}{40}}$

6. **(a)** Let x be the acute angle.

Since x is an acute angle,

$\sin x = 0.52$

$x = \sin^{-1} 0.52 = 31.3°$ (to 1 d.p.)

$\therefore x = \mathbf{31.3°}$

(b) Let x be the acute angle.

Since x is an acute angle,

$\sin x = 0.75$

$x = \sin^{-1} 0.75 = 48.6°$ (to 1 d.p.)

$\therefore x = \mathbf{48.6°}$

(c) Let x be the acute angle.

Since x is an acute angle,

$\sin x = 0.875$

$x = \sin^{-1} 0.875 = 61.0°$ (to 1 d.p.)

$\therefore x = \mathbf{61.0°}$

(d) Let x be the acute angle.

Since x is an acute angle,

$\sin x = 0.3456$

$x = \sin^{-1} 0.3456 = 20.2°$ (to 1 d.p.)

$\therefore x = \mathbf{20.2°}$

7. **(a)** Let x be the obtuse angle.

Since x is an obtuse angle,

$\sin x = 0.52$

$x = 180° - \sin^{-1} 0.52 = 148.7°$ (to 1 d.p.)

$\therefore x = \mathbf{148.7°}$

(b) Let x be the obtuse angle.

Since x is an obtuse angle,

$\sin x = 0.75$

$x = 180° - \sin^{-1} 0.75 = 131.4°$ (to 1 d.p.)

$\therefore x = \mathbf{131.4°}$

(c) Let x be the obtuse angle.

Since x is an obtuse angle,

$\sin x = 0.875$

$x = 180° - \sin^{-1} 0.875 = 119.0°$ (to 1 d.p.)

$\therefore x = \mathbf{119.0°}$

(d) Let x be the obtuse angle.

Since x is an obtuse angle,

$\sin x = 0.3456$

$x = 180° - \sin^{-1} 0.3456 = 159.8°$ (to 1 d.p.)

$\therefore x = \mathbf{159.8°}$

8. **(a)** Let x be the acute angle.

Since x is an acute angle,

$\cos x = 0.67$

$x = \cos^{-1} 0.67 = 47.9°$ (to 1 d.p.)

$\therefore x = \mathbf{47.9°}$

(b) Let x be the acute angle.

Since x is an acute angle,

$\cos x = 0.756$

$x = \cos^{-1} 0.756 = 40.9°$ (to 1 d.p.)

$\therefore x = \mathbf{40.9°}$

(c) Let x be the acute angle.

Since x is an acute angle,

$\cos x = 0.5$

$x = \cos^{-1} 0.5$

$= 60°$

$\therefore x = \mathbf{60°}$

(d) Let x be the acute angle.

Since x is an acute angle,

$\cos x = 0.985$

$x = \cos^{-1} 0.985$

$= 9.9°$ (to 1 d.p.)

$\therefore x = \mathbf{9.9°}$

9. **(a)** Since $\sin x$ is positive, x can either be an acute angle or an obtuse angle.

$\sin x = 0.753$

$x = \sin^{-1} 0.753 = 48.9°$ (to 1 d.p.)

or $180° - 48.9° = 131.1°$ (to 1 d.p.)

$\therefore x = \mathbf{48.9°}$ or $\mathbf{131.1°}$

(b) Since $\sin x$ is positive, x can either be an acute angle or an obtuse angle.

$\sin x = 0.952$

$x = \sin^{-1} 0.952 = 72.2°$ (to 1 d.p.)

or $180° - 72.2° = 107.8°$ (to 1 d.p.)

$\therefore x = \mathbf{72.2°}$ or $\mathbf{107.8°}$

(c) Since $\sin x$ is positive, x can either be an acute angle or an obtuse angle.

$\sin x = 0.4714$

$x = \sin^{-1} 0.4714 = 28.1°$ (to 1 d.p.)

or $180° - 28.1° = 151.9°$ (to 1 d.p.)

$\therefore x = \mathbf{28.1°}$ or $\mathbf{151.9°}$

(d) Since $\cos x$ is negative, x is an obtuse angle.

$\cos x = -0.238$

$x = \cos^{-1} (-0.238)$

$= \mathbf{103.8°}$ (to 1 d.p.)

(e) Since $\cos x$ is negative, x is an obtuse angle.

$\cos x = -0.783$

$x = \cos^{-1} (-0.783)$

$= \mathbf{141.5°}$ (to 1 d.p.)

(f) Since $\cos x$ is positive, x is an acute angle.

$\cos x = 0.524$

$x = \cos^{-1} 0.524$

$= \mathbf{58.4°}$ (to 1 d.p.)

10. (a) Using Pythagoras' Theorem,

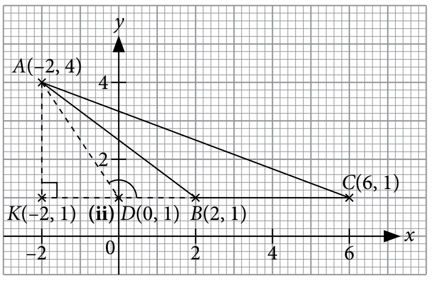
$PR^2 = PQ^2 + QR^2$
$= 8^2 + 15^2$
$= 64 + 225$
$= 289$
$PR = \sqrt{289}$ (since $PR > 0$)
$= 17$

$\sin \angle PRS = \sin (180° - \angle PRS)$
$= \sin \angle PRQ$
$= \frac{PQ}{PR}$
$= \mathbf{\frac{8}{17}}$

(b) $\cos \angle SRP = -\cos (180° - \angle SRP)$
$= -\cos \angle PRQ$
$= -\frac{QR}{PR}$
$= \mathbf{-\frac{15}{17}}$

(c) $\tan \angle PRQ = \frac{PQ}{QR}$
$= \mathbf{\frac{8}{15}}$

11.



K is the point (–2, 1).
$AK = 3$ units, $BK = 4$ units, $CK = 8$ units
Using Pythagoras' Theorem,

$AB^2 = AK^2 + BK^2$
$= 3^2 + 4^2$
$= 25$
$AB = \sqrt{25}$
$= 5$ units

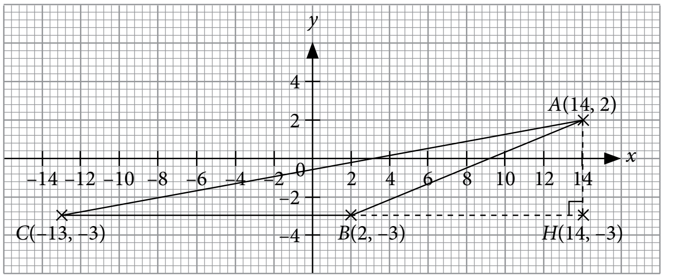
(i) (a) $\sin \angle ABC = \sin (180° - \angle ABC)$
$= \sin \angle ABK$
$= \frac{AK}{AB}$
$= \mathbf{\frac{3}{5}}$

(b) $\cos \angle ABC = -\cos (180° - \angle ABC)$
$= -\cos \angle ABK$
$= -\frac{BK}{AB}$
$= \mathbf{-\frac{4}{5}}$

(c) $\tan \angle ACB = \frac{AK}{CK}$
$= \mathbf{\frac{3}{8}}$

(ii) Since $\cos \angle ADB$ is negative when $\angle ADB$ is obtuse, a possible set of coordinates are ***D*(0, 1)**.

12.



H is the point (14, –3).
$AH = 5$ units, $BH = 12$ units, $CH = 27$ units
Using Pythagoras' Theorem,

$AB^2 = AH^2 + BH^2$
$= 5^2 + 12^2$
$= 169$
$AB = \sqrt{169}$
$= 13$ units

(a) $\sin \angle ABC = \sin (180° - \angle ABC)$
$= \sin \angle ABH$
$= \frac{AH}{AB}$
$= \mathbf{\frac{5}{13}}$

(b) $\cos \angle ABC = -\cos (180° - \angle ABC)$
$= -\cos \angle ABH$
$= -\frac{BH}{AB}$
$= \mathbf{-\frac{12}{13}}$

(c) $\tan \angle ACB = \frac{AH}{CH}$
$= \mathbf{\frac{5}{27}}$

13. (a) Since $\sin (x + °10)$ is positive, $(x + 10°)$ can either be an acute angle or an obtuse angle.

$\sin (x + 10°) = 0.47$
$x + 10° = \sin^{-1} 0.47 = 28.0°$ (to 1 d.p.)
or $180° - 28.0° = 152.0°$ (to 1 d.p.)
$\therefore x = \mathbf{18.0°}$ or $\mathbf{142.0°}$

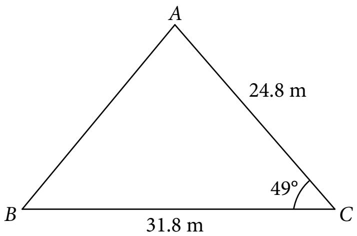
(b) Since $\cos (x - °10)$ is negative, $(x - 10°)$ is an obtuse angle.

$\cos (x - 10°) = -0.56$
$x - 10° = \cos^{-1} (-0.56)$
$= 124.1°$ (to 1 d.p.)
$\therefore x = \mathbf{134.1°}$

6.2 Area of triangle

Practise Now 4

1.



We have $a = 31.8$, $b = 24.8$ and $C = 49°$.

$$\text{Area of } \triangle ABC = \frac{1}{2}ab \sin C$$
$$= \frac{1}{2} \times 31.8 \times 24.8 \times \sin 49°$$
$$= \mathbf{298\ m^2} \text{ (to 3 s.f.)}$$

2. We have $l = 7.5$, $n = 5.6$, $L = °78$ and $N = °47$.

$\angle M = 180° - 78° - 47°$ ($\angle$ sum of $\triangle$)
$= 55°$

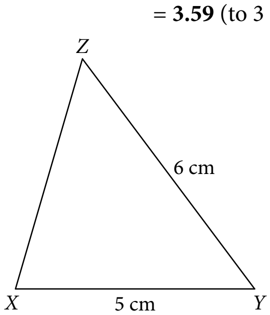
$$\text{Area of } \triangle LMN = \frac{1}{2}ln \sin M$$
$$= \frac{1}{2} \times 7.5 \times 5.6 \times \sin 55°$$
$$= \mathbf{17.2\ cm^2} \text{ (to 3 s.f.)}$$

Practise Now 5

1. We have $q = x$, $r = 2x$ and $P = 104°$.

$$\text{Area of } \triangle PQR = \frac{1}{2}qr \sin P$$
$$12.5 = \frac{1}{2} \times x \times 2x \times \sin 104°$$
$$= x^2 \sin 104°$$
$$x^2 = \frac{12.5}{\sin 104°}$$
$$x = \sqrt{\frac{12.5}{\sin 104°}} \text{ (since } x > 0)$$
$$= \mathbf{3.59} \text{ (to 3 s.f.)}$$

2.



We have $x = 6$ and $z = 5$.

$$\text{Area of } \triangle XYZ = \frac{1}{2}xz \sin \angle XYZ$$
$$12 = \frac{1}{2} \times 6 \times 5 \times \sin \angle XYZ$$
$$= 15 \sin \angle XYZ$$
$$\sin \angle XYZ = \frac{12}{15}$$
$$= \frac{4}{5}$$

Since $\angle XYZ$ is an acute angle,

$$\sin \angle XYZ = \frac{4}{5}$$
$$\angle XYZ = \sin^{-1} \frac{4}{5} = 53.1° \text{ (to 1 d.p.)}$$
$$\therefore \angle XYZ = \mathbf{53.1°}$$

Exercise 6B

1. (a) We have $b = 9$, $c = 8$ and $A = 72°$.

$$\text{Area of } \triangle ABC = \frac{1}{2}bc \sin A$$
$$= \frac{1}{2} \times 9 \times 8 \times \sin 72°$$
$$= \mathbf{34.2\ cm^2} \text{ (to 3 s.f.)}$$

(b) We have $d = 9$, $f = 7$ and $E = 111°$.

$$\text{Area of } \triangle DEF = \frac{1}{2}df \sin E$$
$$= \frac{1}{2} \times 9 \times 7 \times \sin 111°$$
$$= \mathbf{29.4\ cm^2} \text{ (to 3 s.f.)}$$

(c) We have $g = 10$, $i = 9.5$, $G = 62°$ and $I = 57°$.

$\angle H = 180° - 62° - 57°$ ($\angle$ sum of $\triangle$)
$= 61°$

$$\text{Area of } \triangle GHI = \frac{1}{2}gi \sin H$$
$$= \frac{1}{2} \times 10 \times 9.5 \times \sin 61°$$
$$= \mathbf{41.5\ m^2} \text{ (to 3 s.f.)}$$

(d) We have $j = 13.35$, $l = 6.5$, $J = 105°$ and $L = 28°$.

$\angle K = 180° - 105° - 28°$ ($\angle$ sum of $\triangle$)
$= 47°$

$$\text{Area of } \triangle JKL = \frac{1}{2}jl \sin K$$
$$= \frac{1}{2} \times 13.35 \times 6.5 \times \sin 47°$$
$$= \mathbf{31.7\ m^2} \text{ (to 3 s.f.)}$$

(e) Area of $MNOP = 2 \times$ Area of $\triangle MNP$

$$= 2 \times \left(\frac{1}{2} \times MP \times MN \times \sin \angle NMP\right)$$
$$= 2 \times \left(\frac{1}{2} \times 5.3 \times 5.8 \times \sin 117°\right)$$
$$= \mathbf{27.4\ cm^2} \text{ (to 3 s.f.)}$$

(f) Area of $QRST = 2 \times$ Area of $\triangle QRT$

$$= 2 \times \left(\frac{1}{2} \times QT \times QR \times \sin \angle RQT\right)$$
$$= 2 \times \left(\frac{1}{2} \times 8.5 \times 8.5 \times \sin 78°\right)$$
$$= \mathbf{70.7\ m^2} \text{ (to 3 s.f.)}$$

2. $$\text{Area of } \triangle ABC = \frac{1}{2} \times AB \times AC \times \sin \angle BAC$$
$$= \frac{1}{2} \times 22 \times 15 \times \sin 45°$$
$$= \mathbf{117\ cm^2} \text{ (to 3 s.f.)}$$

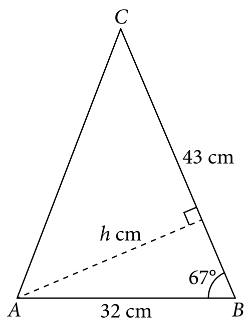
3. $$\text{Area of } \triangle PQR = \frac{1}{2}qr \sin P$$
$$= \frac{1}{2} \times 152 \times 125 \times \sin 72°$$
$$= \mathbf{9040\ cm^2} \text{ (to 3 s.f.)}$$

4. $\angle YXZ = 180° - 48° - 32°$ ($\angle$ sum of $\triangle$)
$= 100°$

Area of $\triangle XYZ = \frac{1}{2} \times XZ \times XY \times \sin \angle YXZ$
$= \frac{1}{2} \times 2.8 \times 2 \times \sin 100°$
$= \mathbf{2.76\ m^2}$ (to 3 s.f.)

5. **(i)** Area of $\triangle ABC = \frac{1}{2} \times AB \times BC \times \sin \angle ABC$
$= \frac{1}{2} \times 32 \times 43 \times \sin 67°$
$= 633.31\ cm^2$ (to 5 s.f.)
$= \mathbf{633\ cm^2}$ (to 3 s.f.)

(ii) Let the perpendicular distance from A to BC be h cm.



Area of $\triangle ABC = 633.31$
$\frac{1}{2} \times BC \times h = 633.31$
$\frac{1}{2} \times 43 \times h = 633.31$
$21.5h = 633.31$
$h = 29.5$ (to 3 s.f.)
$\therefore$ the perpendicular distance from A to BC is **29.5 cm**.

6. Total area $= \frac{1}{2} \times 112 \times 202 \times \sin 30° + \frac{1}{2} \times 202 \times 197 \times \sin 60.5°$
$= 5656 + 17\ 317.467\ 28$
$= \mathbf{23\ 000\ m^2}$ (to 3 s.f.)

7. **(i)** $\sin \angle ACD = \frac{3.7}{8.0}$
$\angle ACD = \sin^{-1} \frac{3.7}{8.0}$
$= \mathbf{27.5°}$ (to 1 d.p.)

(ii) $\cos \angle BAC = \frac{8.0}{AB}$
$\cos 40.4° = \frac{8.0}{AB}$
$AB = \frac{8.0}{\cos 40.4°}$
$= \mathbf{10.5\ cm}$ (to 3 s.f.)

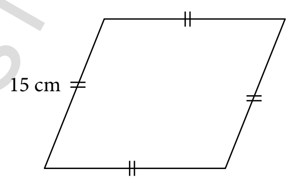
(iii) Area of $\triangle AED = \frac{1}{2} \times AE \times AD \times \sin \angle EAD$
$= \frac{1}{2} \times 4.1 \times 3.7 \times \sin 55.1°$
$= \mathbf{6.22\ cm^2}$ (to 3 s.f.)

8. Area of $\triangle ABC = 97\ cm^2$
$\frac{1}{2} \times AC \times AB \times \sin \angle BAC = 97$
$\frac{1}{2} \times 4x \times 5x \times \sin 68° = 97$
$10x^2 \sin 68° = 97$
$x^2 = \frac{97}{10 \sin 68°}$
$x = \sqrt{\frac{97}{10 \sin 68°}}$ (since $x > 0$)
$= \mathbf{3.23}$ (to 3 s.f.)

9. Let the length of QR be x cm.
$3QR = 4PR$
$3x = 4PR$
$PR = \frac{3}{4}x$ cm
Area of $\triangle PQR = 158\ cm^2$
$\frac{1}{2} \times QR \times PR \times \sin \angle PRQ = 158$
$\frac{1}{2} \times x \times \frac{3}{4}x \times \sin 55° = 158$
$\frac{3}{8}x^2 \sin 55° = 158$
$x^2 = \frac{1264}{3 \sin 55°}$
$x = \sqrt{\frac{1264}{3 \sin 55°}}$ (since $x > 0$)
$= 22.7$ (to 3 s.f.)
$\therefore$ the length of QR is **22.7 cm**.

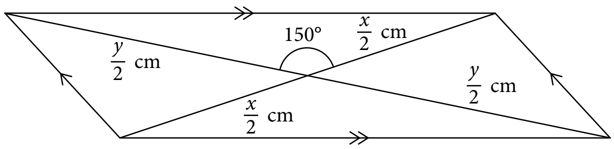
10. Area of $\triangle XYZ = 59.5\ cm^2$
$\frac{1}{2} \times XY \times YZ \times \sin \angle XYZ = 59.5$
$\frac{1}{2} \times 13 \times 16.2 \times \sin \angle XYZ = 59.5$
$105.3 \sin \angle XYZ = 59.5$
$\sin \angle XYZ = \frac{59.5}{105.3}$
Since $\sin \angle XYZ$ is positive, $\angle XYZ$ can either be an acute angle or an obtuse angle.
$\sin \angle XYZ = \frac{59.5}{105.3}$
$\angle XYZ = \sin^{-1} \frac{59.5}{105.3} = 34.4°$ (to 1 d.p.)
or $180° - 34.4° = 145.6°$ (to 1 d.p.)
$\therefore$ $\angle XYZ =$ **34.4° and 145.6°**

11. Let the angle of one side of the rhombus be $a°$.



Area of rhombus $= 40\ cm^2$
$2 \times \left(\frac{1}{2} \times 15 \times 15 \times \sin a°\right) = 40$
$225 \sin a° = 40$
$\sin a° = \frac{40}{225}$
Since $\sin a°$ is positive, $a°$ can either be an acute angle or an obtuse angle.
$\sin a° = \frac{40}{225}$
$a° = \sin^{-1} \frac{40}{225} = 10.2°$ (to 1 d.p.)
or $180° - 10.2° = 169.8°$ (to 1 d.p.)
$\therefore$ angles of rhombus = **10.2° and 169.8°**

12.



$$\text{Area of parallelogram} = 100 \text{ cm}^2$$

$$2 \times \left(\frac{1}{2} \times \frac{x}{2} \times \frac{y}{2} \times \sin 150° + \frac{1}{2} \times \frac{x}{2} \times \frac{y}{2} \times \sin 30° \right) = 100$$

$$2 \times 0.125xy = 100$$

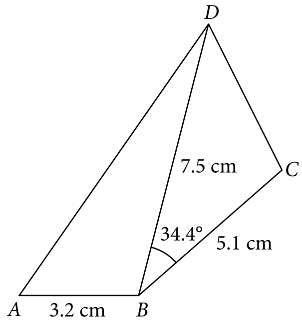
$$0.25xy = 100$$

$$xy = 400$$

$$y = \frac{400}{x}$$

A possible value of x is **16** and the corresponding value of y is **25**.

13. **(i)**



$$\text{Area of } \triangle BCD = \frac{1}{2} \times BC \times BD \times \sin \angle CBD$$

$$= \frac{1}{2} \times 5.1 \times 7.5 \times \sin 34.4°$$

$= \mathbf{10.8\ cm^2}$ (to 3 s.f.)

(ii) Area of $\triangle ABD = 11.62 \text{ cm}^2$

$$\frac{1}{2} \times AB \times BD \times \sin \angle ABD = 11.62$$

$$\frac{1}{2} \times 3.2 \times 7.5 \times \sin \angle ABD = 11.62$$

$$12 \sin \angle ABD = 11.62$$

$$\sin \angle ABD = \frac{11.62}{12}$$

$$\angle ABD = \sin^{-1} \frac{11.62}{12}$$

$= 75.5°$ (to 1 d.p.)

or $180° - 75.5° = 104.5°$ (to 1 d.p.)

Since $\angle ABD$ is obtuse,

$\therefore \angle ABD = \mathbf{104.5°}$

6.3 Sine Rule

Investigation (Sine Rule)

1. **(b)** The length of the triangle opposite vertex B is labelled $\boldsymbol{b}$.
   (c) The length of the triangle opposite vertex C is labelled $\boldsymbol{c}$.
2. *Teachers may guide students to fill in the necessary information in the table.*
3. The 3 quantities are equal.
4. $\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$
5. **Yes**. We can manipulate $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ to obtain $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
6. The lengths of the sides of a triangle are **proportional** to the sine of the angles opposite the sides.

Practise Now 6

(i) $\angle A = 180° - 58.3° - 39.4°$ ($\angle$ sum of $\triangle$)

$= \mathbf{82.3°}$

(ii) Using Sine Rule,

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 39.4°} = \frac{12.5}{\sin 82.3°}$$

$$c = \frac{12.5 \sin 39.4°}{\sin 82.3°}$$

$= 8.0063$ (to 5 s.f.)

$= 8.01$ cm (to 3 s.f.)

$\therefore AB = \mathbf{8.01\ cm}$

(iii) Using Sine Rule,

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 58.3°} = \frac{12.5}{\sin 82.3°}$$

$$b = \frac{12.5 \sin 58.3°}{\sin 82.3°}$$

$= 10.732$ (to 5 s.f.)

$= 10.7$ cm (to 3 s.f.)

$\therefore AC = \mathbf{10.7\ cm}$

Practise Now 7

1. **(i)** Using Sine Rule,

$$\frac{\sin R}{r} = \frac{\sin Q}{q}$$

$$\frac{\sin R}{10.2} = \frac{\sin 42°}{12}$$

$$\sin R = \frac{10.2 \sin 42°}{12}$$

$= 0.568\,76$ (to 5 s.f.)

$\angle R = \sin^{-1} 0.568\,76$

$= 34.664°$ (to 3 d.p.)

or $180° - 34.664° = 145.336°$

Since $r < q$, then $\angle R < \angle Q$, hence $\angle R$ cannot be $145.336°$.

$\therefore \angle R = \mathbf{34.7°}$ (to 1 d.p.)

(ii) $\angle P = 180° - 34.664° - 42°$ ($\angle$ sum of $\triangle$)

$= 103.336°$

$= \mathbf{103.3°}$ (to 1 d.p.)

(iii) Using Sine Rule,

$$\frac{p}{\sin P} = \frac{q}{\sin Q}$$

$$\frac{p}{\sin 103.336°} = \frac{12}{\sin 42°}$$

$$p = \frac{12 \sin 103.336°}{\sin 42°}$$

$= 17.5$ cm (to 3 s.f.)

$\therefore QR = \mathbf{17.5\ cm}$

2. **(i)** Using Sine Rule,

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin B}{12.4} = \frac{\sin 96.8°}{15.6}$$

$$\sin B = \frac{12.4 \sin 96.8°}{15.6}$$

$= 0.789\ 28$ (to 5 s.f.)

$\angle B = \sin^{-1} 0.789\ 28$

$= 52.118°$ (to 3 d.p.)

or $180° - 52.118° = 127.882°$

Since $b < a$, then $\angle B < \angle A$, hence $\angle B$ cannot be 127.882°.

$\therefore \angle ABC =$ **52.1°** (to 1 d.p.)

(ii) $\angle BCA = 180° - 96.8° - 52.118°$ ($\angle$ sum of $\triangle$)

$= 31.082°$

$=$ **31.1°** (to 1 d.p.)

(iii) Using Sine Rule,

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 31.082°} = \frac{15.6}{\sin 96.8°}$$

$$c = \frac{15.6 \sin 31.082°}{\sin 96.8°}$$

$= 8.11$ cm (to 3 s.f.)

$\therefore AB =$ **8.11 cm**

Investigation (Ambiguous case)

Case 1: $a > c$

Teachers may guide students to fill in the necessary information for a and c.

When $a > c$, **1** triangle(s) can be constructed.

Therefore, there is **1** value of $\angle ACB$.

Case 2: $a < c$

Teachers may guide students to fill in the necessary information for a and c.

When $a < c$, **2** triangle(s) can be constructed.

Therefore, there are **2** possible values of $\angle ACB$.

Practise Now 8

1. Using Sine Rule,

$$\frac{\sin \angle QPR}{8.2} = \frac{\sin 26°}{5}$$

$$\sin \angle QPR = \frac{8.2 \sin 26°}{5}$$

$= 0.718\ 93$ (to 5 s.f.)

Since $\angle QPR$ is obtuse,

$\angle QPR = 180° - \sin^{-1} 0.718\ 93$

$= 134.034°$ (to 3 d.p.)

$= 134.0°$ (to 1 d.p.)

$\angle PRQ = 180° - 26° - 134.034°$ ($\angle$ sum of $\triangle$)

$= 19.966°$

Using Sine Rule,

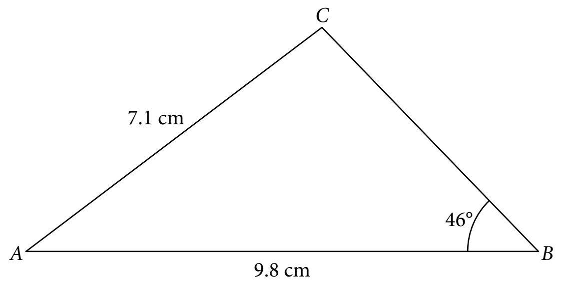
$$\frac{PQ}{\sin 19.966°} = \frac{5}{\sin 26°}$$

$$PQ = \frac{5 \sin 19.966°}{\sin 26°}$$

$= 3.89$ cm (to 3 s.f.)

$\therefore \angle QPR =$ **134.0°** and $PQ =$ **3.89 cm**

2.



Using Sine Rule,

$$\frac{\sin \angle ACB}{9.8} = \frac{\sin 46°}{7.1}$$

$$\sin \angle ACB = \frac{9.8 \sin 46°}{7.1}$$

$= 0.992\ 89$ (to 5 s.f.)

Since $\angle ACB$ is obtuse,

$\angle ACB = 180° - \sin^{-1} 0.992\ 89$

$= 96.836°$ (to 3 d.p.)

$= 96.8°$ (to 1 d.p.)

$\angle BAC = 180° - 46° - 96.836°$ ($\angle$ sum of $\triangle$)

$= 37.164°$

Using Sine Rule,

$$\frac{BC}{\sin 37.164°} = \frac{7.1}{\sin 46°}$$

$$BC = \frac{7.1 \sin 37.164°}{\sin 46°}$$

$= 5.96$ cm (to 3 s.f.)

$\therefore \angle ACB =$ **96.8°** and $BC =$ **5.96 cm**

Exercise 6C

1. **(a)** $\angle C = 180° - 42° - 76°$ ($\angle$ sum of $\triangle$)

$=$ **62°**

Using Sine Rule,

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 76°} = \frac{7.4}{\sin 42°}$$

$$b = \frac{7.4 \sin 76°}{\sin 42°}$$

$=$ **10.7 cm** (to 3 s.f.)

Using Sine Rule,

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 62°} = \frac{7.4}{\sin 42°}$$

$$c = \frac{7.4 \sin 62°}{\sin 42°}$$

$=$ **9.76 cm** (to 3 s.f.)

(b) $\angle F = 180° - 38.7° - 62°$ ($\angle$ sum of $\triangle$)

$= \mathbf{79.3°}$

Using Sine Rule,

$$\frac{d}{\sin D} = \frac{e}{\sin E}$$

$$\frac{d}{\sin 38.7°} = \frac{6.25}{\sin 62°}$$

$$d = \frac{6.25 \sin 38.7°}{\sin 62°}$$

$= \mathbf{4.43\ m}$ (to 3 s.f.)

Using Sine Rule,

$$\frac{f}{\sin F} = \frac{e}{\sin E}$$

$$\frac{f}{\sin 79.3°} = \frac{6.25}{\sin 62°}$$

$$f = \frac{6.25 \sin 79.3°}{\sin 62°}$$

$= \mathbf{6.96\ m}$ (to 3 s.f.)

(c) $\angle H = 180° - 24° - 118°$ ($\angle$ sum of $\triangle$)

$= \mathbf{38°}$

Using Sine Rule,

$$\frac{g}{\sin G} = \frac{h}{\sin H}$$

$$\frac{g}{\sin 118°} = \frac{8}{\sin 38°}$$

$$g = \frac{8 \sin 118°}{\sin 38°}$$

$= \mathbf{11.5\ mm}$ (to 3 s.f.)

Using Sine Rule,

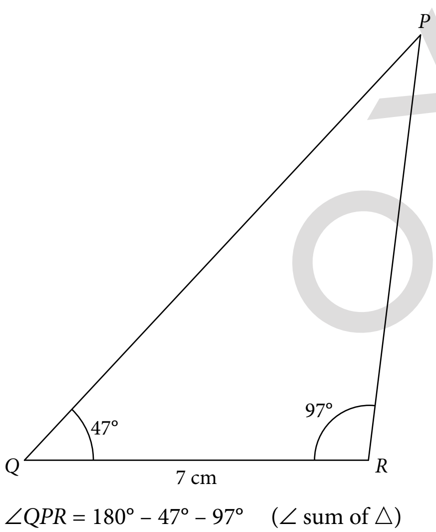
$$\frac{i}{\sin I} = \frac{h}{\sin H}$$

$$\frac{i}{\sin 24°} = \frac{8}{\sin 38°}$$

$$i = \frac{8 \sin 24°}{\sin 38°}$$

$= \mathbf{5.29\ mm}$ (to 3 s.f.)

2.



$\angle QPR = 180° - 47° - 97°$ ($\angle$ sum of $\triangle$)

$= 36°$

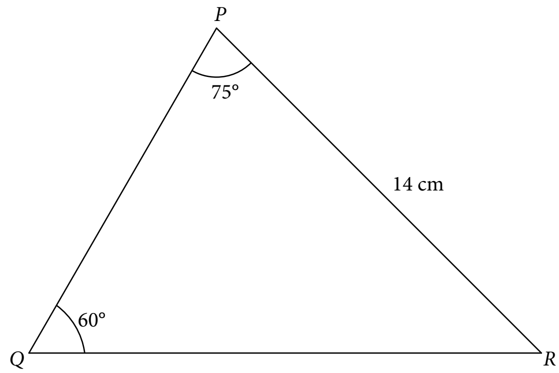
Using Sine Rule,

$$\frac{PQ}{\sin 97°} = \frac{7}{\sin 36°}$$

$$PQ = \frac{7 \sin 97°}{\sin 36°}$$

$= \mathbf{11.8\ cm}$ (to 3 s.f.)

3.



$\angle R = 180° - 60° - 75°$ ($\angle$ sum of $\triangle$)

$= 45°$

The longest side is the side opposite the largest angle, i.e. QR.

Using Sine Rule,

$$\frac{QR}{\sin 75°} = \frac{14}{\sin 60°}$$

$$QR = \frac{14 \sin 75°}{\sin 60°}$$

$= \mathbf{15.6\ cm}$ (to 3 s.f.)

4. **(a)** Using Sine Rule,

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin B}{6.93} = \frac{\sin 92.0°}{15.3}$$

$$\sin B = \frac{6.93 \sin 92.0°}{15.3}$$

$= 0.452\ 67$ (to 5 s.f.)

$\angle B = \sin^{-1} 0.452\ 67$

$= 26.915°$ (to 3 d.p.)

or $180° - 26.915° = 153.085°$

Since $b < a$, then $\angle B < \angle A$, hence $\angle B$ cannot be $153.085°$.

$\therefore$ $\mathbf{\angle B = 26.9°}$ (to 1 d.p.)

$\angle C = 180° - 26.915° - 92.0°$ ($\angle$ sum of $\triangle$)

$= 61.085°$

$= \mathbf{61.1°}$ (to 1 d.p.)

Using Sine Rule,

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 61.085°} = \frac{15.3}{\sin 92.0°}$$

$$c = \frac{15.3 \sin 61.085°}{\sin 92.0°}$$

$= \mathbf{13.4\ cm}$ (to 3 s.f.)

(b) Using Sine Rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{14.5} = \frac{\sin 98.0°}{17.4}$$

$$\sin A = \frac{14.5 \sin 98.0°}{17.4}$$

$$= 0.825\ 22 \text{ (to 5 s.f.)}$$

$$\angle A = \sin^{-1} 0.825\ 22$$

$$= 55.611° \text{ (to 3 d.p.)}$$

$$\text{or } 180° - 55.611° = 124.389°$$

Since $a < b$, then $\angle A < \angle B$, hence $\angle B$ cannot be 124.389°.

$\therefore$ $\angle A$ = **55.6°** (to 1 d.p.)

$\angle C = 180° - 55.611° - 98.0°$ ($\angle$ sum of $\triangle$)

$= 26.389°$

$=$ **26.4°** (to 1 d.p.)

Using Sine Rule,

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 26.389°} = \frac{17.4}{\sin 98.0°}$$

$$c = \frac{17.4 \sin 26.389°}{\sin 98.0°}$$

$=$ **7.81 m** (to 3 s.f.)

(c) Using Sine Rule,

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin B}{8.7} = \frac{\sin 35.0°}{9.5}$$

$$\sin B = \frac{8.7 \sin 35.0°}{9.5}$$

$$= 0.525\ 28 \text{ (to 5 s.f.)}$$

$$\angle B = \sin^{-1} 0.525\ 28$$

$$= 31.687° \text{ (to 3 d.p.)}$$

$$\text{or } 180° - 31.687° = 148.313°$$

Since $b < c$, then $\angle B < \angle C$, hence $\angle B$ cannot be 148.313°.

$\therefore$ $\angle B$ = **31.7°** (to 1 d.p.)

$\angle A = 180° - 31.687° - 35.0°$ ($\angle$ sum of $\triangle$)

$= 113.313°$

$=$ **113.3°** (to 1 d.p.)

Using Sine Rule,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 113.313°} = \frac{9.5}{\sin 35.0°}$$

$$a = \frac{9.5 \sin 113.313°}{\sin 35.0°}$$

$=$ **15.2 cm** (to 3 s.f.)

5. **(i)** Using Sine Rule,

$$\frac{\sin R}{13.4} = \frac{\sin 101°}{20.8}$$

$$\sin R = \frac{13.4 \sin 101°}{20.8}$$

$$= 0.632\ 39 \text{ (to 5 s.f.)}$$

$$\angle R = \sin^{-1} 0.632\ 39$$

$$= 39.227° \text{ (to 3 d.p.)}$$

$$\text{or } 180° - 39.227° = 140.773°$$

Since $r < p$, then $\angle R < \angle P$, hence $\angle R$ cannot be 140.773°.

$\therefore$ $\angle R$ = **39.2°** (to 1 d.p.)

(ii) $\angle Q = 180° - 101° - 39.227°$ ($\angle$ sum of $\triangle$)

$= 39.773°$

$=$ **39.8°** (to 1 d.p.)

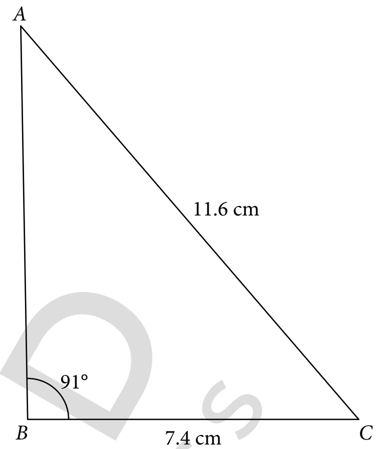
(iii) Using Sine Rule,

$$\frac{PR}{\sin 39.773°} = \frac{20.8}{\sin 101°}$$

$$PR = \frac{20.8 \sin 39.773°}{\sin 101°}$$

$=$ **13.6 cm** (to 3 s.f.)

6. **(i)**



Using Sine Rule,

$$\frac{\sin \angle BAC}{7.4} = \frac{\sin 91°}{11.6}$$

$$\sin \angle BAC = \frac{7.4 \sin 91°}{11.6}$$

$$= 0.637\ 83 \text{ (to 5 s.f.)}$$

$$\angle BAC = \sin^{-1} 0.637\ 83$$

$$= 39.630° \text{ (to 3 d.p.)}$$

$$\text{or } 180° - 39.630° = 140.370°$$

Since $BC < AC$, then $\angle BAC < \angle ABC$, hence $\angle BAC$ cannot be 140.370°.

$\therefore$ $\angle BAC$ = **39.6°** (to 1 d.p.)

(ii) $\angle ACB = 180° - 91° - 39.630°$ ($\angle$ sum of $\triangle$)

$= 49.37°$

$=$ **49.4°** (to 1 d.p.)

(iii) Using Sine Rule,

$$\frac{AB}{\sin 49.37°} = \frac{11.6}{\sin 91°}$$

$$AB = \frac{11.6 \sin 49.37°}{\sin 91°}$$

$=$ **8.80 cm** (to 3 s.f.)

7. **(i)** $\sin 25° = \frac{5.3}{AB}$

$$AB = \frac{5.3}{\sin 25°}$$

$=$ **12.5 m** (to 3 s.f.)

(ii) $\cos \angle DBN = \frac{5.3}{7.1}$

$$\angle DBN = \cos^{-1}\left(\frac{5.3}{7.1}\right)$$

$=$ **41.7°** (to 1 d.p.)

(iii) $\angle BCD = 180° - 103° - 46°$ ($\angle$ sum of $\triangle$)

$= 31°$

Using Sine Rule,

$$\frac{CD}{\sin 103°} = \frac{7.1}{\sin 31°}$$

$$CD = \frac{7.1 \sin 103°}{\sin 31°}$$

$=$ **13.4 m** (to 3 s.f.)

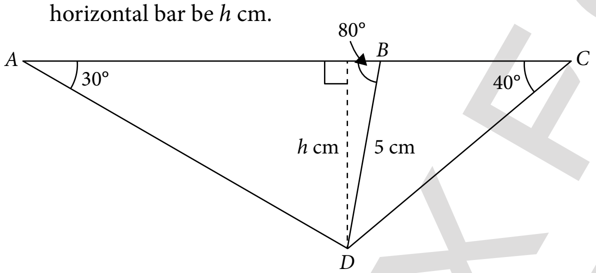
8. (i) $\angle BAC = 180° - 62° - 68°$ ($\angle$ sum of $\triangle$)
$= 50°$
Using Sine Rule,
$$\frac{AC}{\sin 62°} = \frac{6}{\sin 50°}$$
$$AC = \frac{6 \sin 62°}{\sin 50°}$$
$= 6.9156$ m (to 5 s.f.)
$=$ **6.92 m** (to 3 s.f.)

(ii) $\angle BCD = 180° - 68°$ (adj. $\angle$s on a str. line)
$= 112°$
Area of region
$$= \frac{1}{2} \times AC \times BC \times \sin \angle ACB + \frac{1}{2} \times BC \times CD \times \sin \angle BCD$$
$$= \frac{1}{2} \times 6.9156 \times 6 \times \sin 68° + \frac{1}{2} \times 6 \times 7.5 \times \sin 112°$$
$=$ **40.1 m²** (to 3 s.f.)

9. (i) $\angle ADB = 180° - 30° - 80°$ ($\angle$ sum of $\triangle$)
$= 70°$
Using Sine Rule,
$$\frac{AB}{\sin 70°} = \frac{5}{\sin 30°}$$
$$AB = \frac{5 \sin 70°}{\sin 30°}$$
$=$ **9.40 cm** (to 3 s.f.)

(ii) $\angle BDC = 80° - 40°$ (ext. $\angle$ of $\triangle$)
$= 40°$
Since $\angle BDC = \angle BCD =$ **°40,** $\triangle BCD$ is an isosceles triangle.
$\therefore BC = BD =$ **5 cm**

(iii) Let the vertical distance between the mass and the horizontal bar be h cm.



$$\sin 80° = \frac{h}{5}$$
$h = 5 \sin 80°$
$= 4.92$ cm (to 3 s.f.)
$\therefore$ the vertical distance between the mass and the horizontal bar is **4.92 cm**.

10. (i) Since RST is a straight line, QS is parallel to PT if $\angle QSR = \angle PTS$.
$\angle QSR = 180° - 74° - 90°$ (adj. $\angle$s on a str. line)
$= 16°$
$$\tan \angle PTS = \frac{PS}{ST}$$
$$= \frac{4.3}{5.7}$$
$$\angle PTS = \tan^{-1}\left(\frac{4.3}{5.7}\right)$$
$= 37.030°$ (to 3 d.p.)
Since $\angle QSR \neq \angle PTS$, QS is **not** parallel to PT.

(ii) $\angle PSR = 180° - 90°$ (adj. $\angle$s on a str. line)
$= 90°$
$\therefore \triangle PRS$ is a right-angled $\triangle$.
$$\cos \angle RPS = \frac{PS}{PR}$$
$$\cos 63° = \frac{4.3}{PR}$$
$$PR = \frac{4.3}{\cos 63°}$$
$=$ **9.47 cm** (to 3 s.f.)

(iii) $\angle PQS = 180° - 63° - 54°$ ($\angle$ sum of $\triangle$)
$= 63°$
Using Sine Rule,
Since $\angle SQP = \angle SPQ$, $\triangle SQP$ is an isosceles triangle.
$SQ = SP$ (base $\angle$s of isos. $\triangle$)
$=$ **4.3 cm**

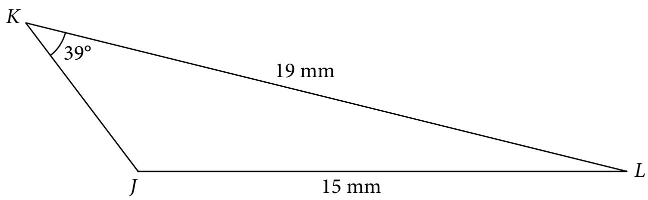
11. $$\cos 73° = \frac{QR}{5.7}$$
$QR = 5.7 \cos 73°$
$= 1.6665$ km (to 5 s.f.)
$\angle QPS = 180° - 48° - 55°$ ($\angle$ sum of $\triangle$)
$= 77°$
Using Sine Rule,
$$\frac{PQ}{\sin 55°} = \frac{5.7}{\sin 77°}$$
$$PQ = \frac{5.7 \sin 55°}{\sin 77°}$$
$= 4.7920$ km (to 5 s.f.)
Area of nature reserve
$=$ Area of $\triangle QRS$ + Area of $\triangle PQS$
$$= \frac{1}{2} \times 1.6665 \times 5.7 \times \sin 73° + \frac{1}{2} \times 4.7920 \times 5.7 \times \sin 48°$$
$=$ **14.7 km²** (to 3 s.f.)

12. (i) $$\sin 27.6° = \frac{QR}{5.7}$$
$QR = 5.7 \sin 27.6°$
$=$ **2.64 cm** (to 3 s.f.)

(ii) $$\cos \angle SPR = \frac{3.2}{5.7}$$
$$\angle SPR = \cos^{-1}\left(\frac{3.2}{5.7}\right)$$
$=$ **55.8°** (to 1 d.p.)

(iii) Using Sine Rule,
$$\frac{\sin \angle PST}{2.7} = \frac{\sin 64.2°}{3.2}$$
$$\sin \angle PST = \frac{2.7 \sin 64.2°}{3.2}$$
$= 0.759\ 64$ (to 5 s.f.)
$\angle PST = \sin^{-1} 0.759\ 64$
$= 49.432°$ (to 3 d.p.)
or $180° - 49.432° = 130.568°$
Since $PT < PS$, then $\angle PST < \angle PTS$, hence $\angle PST$ cannot be 130.568°.
$\therefore \angle PST =$ **49.4°** (to 1 d.p.)

13.



Using Sine Rule,

$$\frac{\sin \angle KJL}{19} = \frac{\sin 39°}{15}$$

$$\sin \angle KJL = \frac{19 \sin 39°}{15}$$

$$= 0.797\ 14 \text{ (to 5 s.f.)}$$

Since $\angle KJL$ is obtuse,

$$\angle KJL = 180° - \sin^{-1} 0.797\ 14$$

$$= 127.142° \text{ (to 3 d.p.)}$$

$$= 127.1° \text{ (to 1 d.p.)}$$

$$\angle JLK = 180° - 39° - 127.142° \text{ (}\angle \text{ sum of } \triangle\text{)}$$

$$= 13.858°$$

Using Sine Rule,

$$\frac{JK}{\sin 13.858°} = \frac{15}{\sin 39°}$$

$$JK = \frac{15 \sin 13.858°}{\sin 39°}$$

$$= 5.71 \text{ mm (to 3 s.f.)}$$

$\therefore$ **$\angle KJL = 127.1°$** and **$JK = 5.71$ mm**

14. Using Sine Rule,

$$\frac{\sin \angle MNO}{80} = \frac{\sin 43°}{67}$$

$$\sin \angle MNO = \frac{80 \sin 43°}{67}$$

$$= 0.814\ 33 \text{ (to 5 s.f.)}$$

Since $\angle MNO$ is obtuse,

$$\angle MNO = 180° - \sin^{-1} 0.814\ 33$$

$$= 125.479° \text{ (to 3 d.p.)}$$

$$= 125.5° \text{ (to 1 d.p.)}$$

$$\angle NMO = 180° - 43° - 125.479° \text{ (}\angle \text{ sum of } \triangle\text{)}$$

$$= 11.521°$$

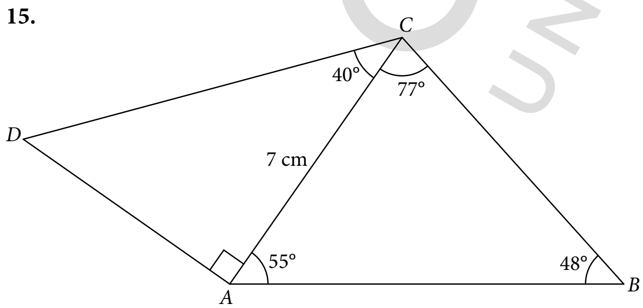
Using Sine Rule,

$$\frac{NO}{\sin 11.521°} = \frac{67}{\sin 43°}$$

$$NO = \frac{67 \sin 11.521°}{\sin 43°}$$

= **19.6 m** (to 3 s.f.)

15.



(i) Using Sine Rule,

$$\frac{AB}{\sin 77°} = \frac{7}{\sin 48°}$$

$$AB = \frac{7 \sin 77°}{\sin 48°}$$

= **9.18 cm** (to 3 s.f.)

(ii) $\tan 40° = \frac{AD}{7}$

$$AD = 7 \tan 40°$$

$$= 5.8737 \text{ cm (to 5 s.f.)}$$

Map		**Actual**
8 cm	represent	1 km
1 cm	represents	$\frac{1}{8}$ km = 0.125 km
5.8737 cm	represent	(5.8737 × 0.125) km
		= 0.734 21 km (to 5 s.f.)
		= 0.734 km (to 3 s.f.)

$\therefore$ the length represented by AD is **0.734 km**.

(iii)

Map		**Actual**
8 cm	represent	1 km
1 cm	represents	$\frac{1}{8}$ km = 0.125 km
7 cm	represent	(7 × 0.125) km
		= 0.875 km

$$\text{Area of } \triangle ADC = \frac{1}{2} \times 0.734\ 21 \times 0.875$$

= **0.321 km²** (to 3 s.f.)

16. (i) $\sin B = \frac{4}{3} \sin A$

$$= \frac{4}{3} \sin 35°$$

$$= 0.764\ 77 \text{ (to 5 s.f.)}$$

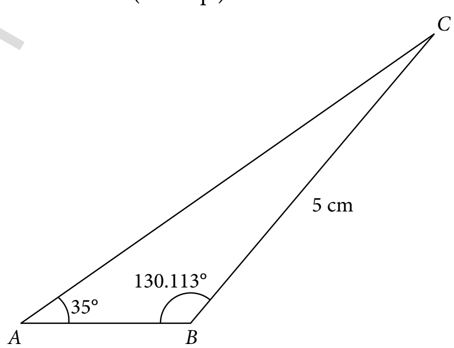
Since $\angle B$ is obtuse,

$$\angle B = 180° - \sin^{-1} 0.764\ 77$$

$$= 130.113° \text{ (to 3 d.p.)}$$

= **130.1°** (to 1 d.p.)

(ii)



$$\frac{AC}{\sin B} = \frac{5}{\sin A}$$

$$AC = \frac{5 \sin B}{\sin A}$$

$$= 5 \times \frac{4}{3}$$

= **$6\frac{2}{3}$ cm or 6.67 cm** (to 3 s.f.)

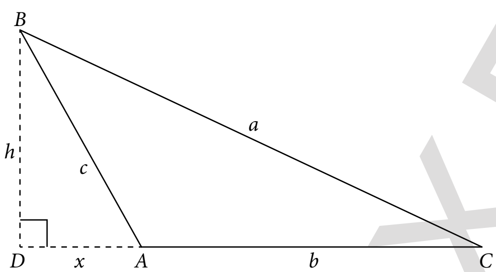
6.4 Cosine Rule

Investigation (Cosine Rule)

1. **(b)** The length of the side of the triangle opposite vertex B is labelled $\boldsymbol{b}$.
   (c) The length of the side of the triangle opposite vertex C is labelled $\boldsymbol{c}$.
2. *Teachers may guide students to fill in the necessary information in the table.*
3. The values in the 5th column are equal to the values in the 6th column.
   The values in the 7th column are equal to the values in the 8th column.
   The values in the 9th column are equal to the values in the 10th column.
4. $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$
   $\boldsymbol{b^2 = a^2 + c^2 - 2ac \cos B}$
   $\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C}$
5. $\boldsymbol{\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}}$
   $\boldsymbol{\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}}$
   $\boldsymbol{\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}}$

Thinking Time (Page 212)

1. When $\angle A$ is an obtuse angle,



In $\triangle BCD$,
$a^2 = h^2 + (b + x)^2$ (Pythagoras' Theorem)
$= h^2 + b^2 + 2bx + x^2$
$= b^2 + (h^2 + x^2) + 2bx$ — (1)
In $\triangle ABD$,
$c^2 = h^2 + x^2$ (Pythagoras' Theorem) — (2)
and $\cos A = -\cos (180° - \angle ABD) = -\dfrac{x}{c}$,
i.e. $x = -c \cos A$ — (3)
Substituting (2) and (3) into (1):
$a^2 = b^2 + c^2 + 2b(-c \cos A)$
$= b^2 + c^2 - 2bc \cos A$ (proven)

2. **(a)** If $\angle A = 90°$,
   $a^2 = b^2 + c^2 - 2bc \cos 90°$
   $a^2 = b^2 + c^2 - 2bc(0)$
   $a^2 = b^2 + c^2$
   (b) **Yes**, this is because when $\angle A = 90°$, we have a right-angled $\triangle$, and the formula $a^2 = b^2 + c^2$ (Pythagoras' Theorem), is a special case of the Cosine Rule.
3. **Pythagoras'** Theorem is a special case of the Cosine Rule.

Practise Now 9

(i) Using Cosine Rule,
$q^2 = p^2 + r^2 - 2pr \cos Q$
$PR^2 = 15.9^2 + 10.8^2 - 2 \times 15.9 \times 10.8 \times \cos 71°$
$= 257.64$ (to 5 s.f.)
$\therefore PR = \sqrt{257.64}$
$= \mathbf{16.1\ cm}$ (to 3 s.f.)

(ii) Using Sine Rule,
$\dfrac{\sin \angle QPR}{15.9} = \dfrac{\sin 71°}{\sqrt{257.64}}$
$\sin \angle QPR = \dfrac{15.9 \sin 71°}{\sqrt{257.64}}$
$= 0.936\ 61$ (to 5 s.f.)
$\angle QPR = \sin^{-1} 0.936\ 61$
$= 69.490°$ (to 3 d.p.)
or $180° - 69.490° = 110.510°$
Since $QR < PR$, then $\angle QPR < \angle PQR$, hence $\angle QPR$ cannot be 110.510°.
$\therefore \angle QPR = \mathbf{69.5°}$ (to 1 d.p.)

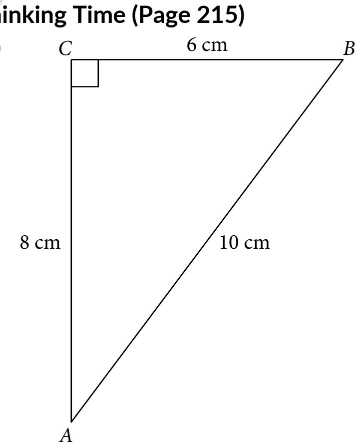
(iii) $\angle PRQ = 180° - 71° - 69.490°$ ($\angle$ sum of $\triangle$)
$= \mathbf{39.5°}$ (to 1 d.p.)

Practise Now 10

The largest angle is the angle opposite the longest side, i.e. $\angle QPR$.
Using Cosine Rule,
$\cos P = \dfrac{q^2 + r^2 - p^2}{2qr}$
$= \dfrac{11^2 + 13^2 - 18^2}{2 \times 11 \times 13}$
$= -\dfrac{17}{143}$
$\angle P = \cos^{-1}\left(-\dfrac{17}{143}\right)$
$= 96.8°$ (to 1 d.p.)
$\therefore$ the largest angle is **96.8°**.

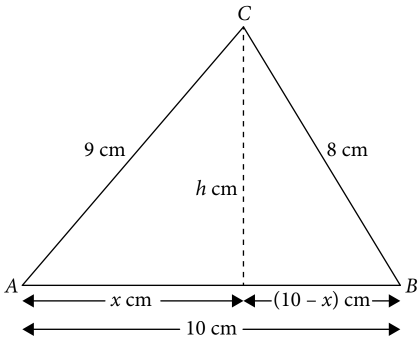
Thinking Time (Page 215)

(a)



Since $6^2 + 8^2 = 10^2$, $\angle C = 90°$.
Area of $\triangle ABC = \dfrac{1}{2} \times 6 \times 8 = 24\ \text{cm}^2$
$s = \dfrac{6 + 8 + 10}{2} = 12$ cm
Area of $\triangle ABC = \sqrt{12(12-6)(12-8)(12-10)} = 24\ \text{cm}^2$
$\therefore$ the formula is correct.

(b)



Using Pythagoras' Theorem,

$x^2 + h^2 = 9^2$

$h^2 = 81 - x^2$ — (1)

$h^2 + (10 - x)^2 = 8^2$

$h^2 = 64 - (10 - x)^2$ — (2)

Substitute (1) into (2):

$81 - x^2 = 64 - (10 - x)^2$

$= 64 - (100 - 20x + x^2)$

$= -x^2 + 20x - 36$

$20x = 117$

$x = 5.85$

Substitute $x = 5.85$ into (1):

$h^2 = 81 - 5.85^2$

$= 46.7775$

$h = \sqrt{46.7775}$ (since $h > 0$)

$= 6.8394$ (to 5 s.f.)

Area of $\triangle ABC = \frac{1}{2} \times 10 \times 6.8394$

$= 34.2$ cm^2 (to 3 s.f.)

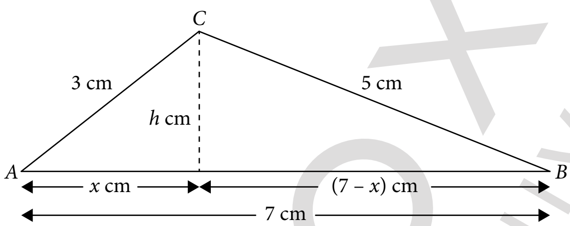
$s = \frac{8 + 9 + 10}{2} = 13.5$ cm

Area of $\triangle ABC = \sqrt{13.5(13.5 - 8)(13.5 - 9)(13.5 - 10)}$

$= 34.2$ cm^2 (to 3 s.f.)

$\therefore$ the formula is correct.

(c)



Using Pythagoras' Theorem,

$x^2 + h^2 = 3^2$

$h^2 = 9 - x^2$ — (1)

$h^2 + (7 - x)^2 = 5^2$

$h^2 = 25 - (7 - x)^2$ — (2)

Substitute (1) into (2):

$9 - x^2 = 25 - (7 - x)^2$

$= 25 - (49 - 14x + x^2)$

$= -x^2 + 14x - 24$

$14x = 33$

$x = 2\frac{5}{14}$

Substitute $x = 2\frac{5}{14}$ into (1):

$h^2 = 9 - \left(2\frac{5}{14}\right)^2$

$= 3\frac{87}{196}$

$h = \sqrt{3\frac{87}{196}}$ (since $h > 0$)

$= 1.8558$ (to 5 s.f.)

Area of $\triangle ABC = \frac{1}{2} \times 7 \times 1.8558$

$= 6.50$ cm^2 (to 3 s.f.)

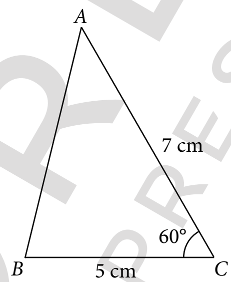
$s = \frac{7 + 5 + 3}{2} = 7.5$ cm

Area of $\triangle ABC = \sqrt{7.5(7.5 - 7)(7.5 - 5)(7.5 - 3)} = 6.50$ cm^2 (to 3 s.f.)

$\therefore$ the formula is correct.

Exercise 6D

1.



Using Cosine Rule,

$c^2 = a^2 + b^2 - 2ab \cos C$

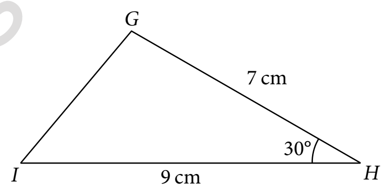
$= 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 60°$

$= 39$

$\therefore c = \sqrt{39}$

$=$ **6.24 cm** (to 3 s.f.)

2.



Using Cosine Rule,

$h^2 = g^2 + i^2 - 2gi \cos H$

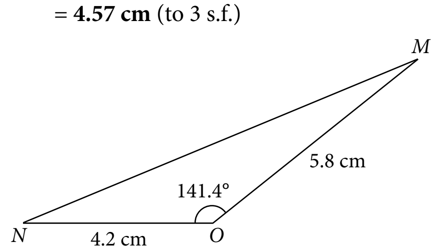
$= 9^2 + 7^2 - 2 \times 9 \times 7 \times \cos 30°$

$= 20.881$ (to 5 s.f.)

$\therefore h = \sqrt{20.881}$

$=$ **4.57 cm** (to 3 s.f.)

3.



Using Cosine Rule,

$o^2 = m^2 + n^2 - 2mn \cos O$

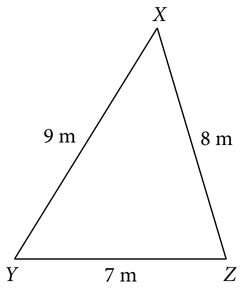
$= 4.2^2 + 5.8^2 - 2 \times 4.2 \times 5.8 \times \cos 141.4°$

$= 89.356$ (to 5 s.f.)

$\therefore o = \sqrt{89.356}$

$=$ **9.45 cm** (to 3 s.f.)

4.



Using Cosine Rule,

$$\cos X = \frac{y^2 + z^2 - x^2}{2yz}$$
$$= \frac{8^2 + 9^2 - 7^2}{2 \times 8 \times 9}$$
$$= \frac{2}{3}$$
$$\angle X = \cos^{-1} \frac{2}{3}$$
$$= \mathbf{48.2°} \text{ (to 1 d.p.)}$$

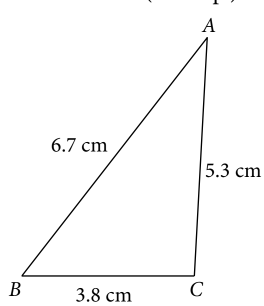
Using Cosine Rule,

$$\cos Y = \frac{x^2 + z^2 - y^2}{2xz}$$
$$= \frac{7^2 + 9^2 - 8^2}{2 \times 7 \times 9}$$
$$= \frac{11}{21}$$
$$\angle Y = \cos^{-1} \frac{11}{21}$$
$$= \mathbf{58.4°} \text{ (to 1 d.p.)}$$

Using Cosine Rule,

$$\cos Z = \frac{x^2 + y^2 - z^2}{2xy}$$
$$= \frac{7^2 + 8^2 - 9^2}{2 \times 7 \times 8}$$
$$= \frac{2}{7}$$
$$\angle Z = \cos^{-1} \frac{2}{7}$$
$$= \mathbf{73.4°} \text{ (to 1 d.p.)}$$

5.



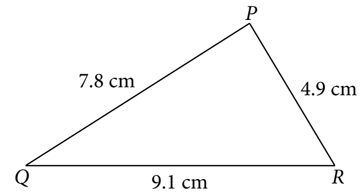
The smallest angle is the angle opposite the shortest side, i.e. $\angle A$.

Using Cosine Rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{5.3^2 + 6.7^2 - 3.8^2}{2 \times 5.3 \times 6.7}$$
$$= 0.824\,27 \text{ (to 5 s.f.)}$$
$$\angle A = \cos^{-1} 0.824\,27$$
$$= 34.5° \text{ (to 1 d.p.)}$$

$\therefore$ the smallest angle is **34.5°**.

6.



The largest angle is the angle opposite the longest side, i.e. $\angle P$.

Using Cosine Rule,

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$
$$= \frac{4.9^2 + 7.8^2 - 9.1^2}{2 \times 4.9 \times 7.8}$$
$$= 0.026\,688 \text{ (to 5 s.f.)}$$
$$\angle P = \cos^{-1} 0.026\,688$$
$$= 88.5° \text{ (to 1 d.p.)}$$

$\therefore$ the largest angle is **88.5°**.

7. **(i)** $\angle CBD = 180° - 125°$ (adj. $\angle$s on a str. line)

$= 55°$

Since $\angle CBD = \angle BCD = 55°$, $\triangle BCD$ is an isosceles triangle.

$\therefore\ CD = BD =$ **9 m**

(ii) Using Cosine Rule,

$$AD^2 = AB^2 + BD^2 - 2 \times AB \times BD \times \cos \angle ABD$$
$$= 8^2 + 9^2 - 2 \times 8 \times 9 \times \cos 125°$$
$$= 227.60 \text{ (to 5 s.f.)}$$
$$\therefore\ AD = \sqrt{227.60}$$
$$= \mathbf{15.1\ m} \text{ (to 3 s.f.)}$$

8. **(i)** $\angle APB = 180° - 60°$ (adj. $\angle$s on a str. line)

$= 120°$

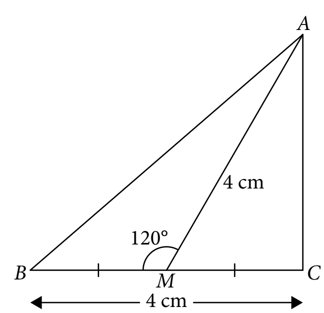
Using Sine Rule,

$$\frac{AB}{\sin 120°} = \frac{5}{\sin 45°}$$
$$AB = \frac{5 \sin 120°}{\sin 45°}$$
$$= \mathbf{6.12\ m} \text{ (to 3 s.f.)}$$

(ii) Using Cosine Rule,

$$AC^2 = AP^2 + CP^2 - 2 \times AP \times CP \times \cos \angle APC$$
$$= 5^2 + 8^2 - 2 \times 5 \times 8 \times \cos 60°$$
$$= 49$$
$$\therefore\ AC = \sqrt{49}$$
$$= \mathbf{7\ m}$$

9.



(i) $CM = BM = \frac{4}{2} = 2$ cm

$\angle AMC = 180° - 120°$ (adj. $\angle$s on a str. line)

$= 60°$

Using Cosine Rule,

$$AC^2 = AM^2 + CM^2 - 2 \times AM \times CM \times \cos \angle AMC$$
$$= 4^2 + 2^2 - 2 \times 4 \times 2 \times \cos 60°$$
$$= 12$$
$$\therefore\ AC = \sqrt{12}$$
$$= \mathbf{3.46\ cm} \text{ (to 3 s.f.)}$$

(ii) Using Cosine Rule,

$AB^2 = AM^2 + BM^2 - 2 \times AM \times BM \times \cos \angle AMB$
$= 4^2 + 2^2 - 2 \times 4 \times 2 \times \cos 120°$
$= 28$

$\therefore AB = \sqrt{28}$
$=$ **5.29 cm** (to 3 s.f.)

(iii) $\cos \angle ACB = \cos \angle ACM$

$= \dfrac{2^2 + 12 - 4^2}{2 \times 2 \times \sqrt{12}}$
$= 0$

$\angle ACB = \cos^{-1} 0$
$=$ **90°**

10. (i) $\angle ADB = 180° - 90°$ (adj. $\angle$s on a str. line)
$= 90°$

$\therefore \triangle ABD$ is a right-angled $\triangle$.

$\tan \angle BAD = \dfrac{5}{12}$

$\angle BAD = \tan^{-1} \dfrac{5}{12}$
$= 22.620°$ (to 3 d.p.)
$=$ **22.6°** (to 1 d.p.)

(ii) $\angle APD = 180° - 50° - 22.620°$
$= 107.380°$

Using Sine Rule,

$\dfrac{PD}{\sin 22.620°} = \dfrac{12}{\sin 107.380°}$

$PD = \dfrac{12 \sin 22.620°}{\sin 107.380°}$
$=$ **4.84 m** (to 3 s.f.)

(iii) $\tan \angle CBD = \dfrac{12}{5}$

$\angle CBD = \tan^{-1} \dfrac{12}{5}$
$= 67.380°$ (to 3 d.p.)

Using Cosine Rule,

$DQ^2 = BD^2 + BQ^2 - 2 \times BD \times BQ \times \cos \angle DBQ$
$= 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 67.380°$
$= 47.077$ (to 5 s.f.)

$\therefore DQ = \sqrt{47.077}$
$=$ **6.86 m** (to 3 s.f.)

11. (i) Using Cosine Rule,

$a^2 = 6^2 + 5^2 - 2 \times 6 \times 5 \times \cos 92°$
$= 63.094$

$\therefore a = \sqrt{63.094}$
$= 7.9432$ (to 5 s.f.)
$=$ **7.94** (to 3 s.f.)

(ii) Using Cosine Rule,

$\cos \theta = \dfrac{5^2 + 7^2 - 7.9432^2}{2 \times 5 \times 7}$
$= 0.155\ 79$ (to 5 s.f.)

$\theta = \cos^{-1} 0.155\ 79$
$=$ **81.0°** (to 1 d.p.)

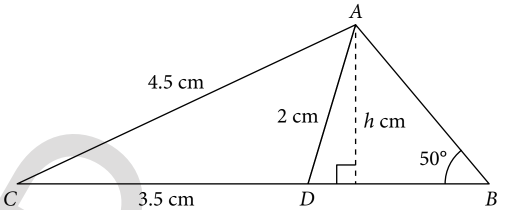
12. (i) Using Cosine Rule,

$\cos \angle ADC = \dfrac{2^2 + 3.5^2 - 4.5^2}{2 \times 2 \times 3.5}$
$= -\dfrac{2}{7}$

$\angle ADC = \cos^{-1}\left(-\dfrac{2}{7}\right)$
$= 106.602°$ (to 3 d.p.)

$\angle ADB = 180° - 106.602°$
$= 73.398°$
$=$ **73.4°** (to 1 d.p.)

(ii)



Let the shortest distance from A to CB be h cm.

$\sin \angle ADB = \dfrac{h}{2}$

$\sin 73.398° = \dfrac{h}{2}$

$h = 2 \sin 73.398°$
$=$ **1.92 cm** (to 3 s.f.)

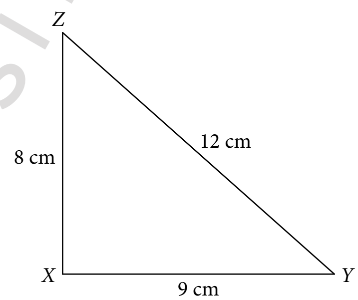
(iii) $\angle BAD = 180° - 50° - 73.398°$
$= 56.602°$

Using Sine Rule,

$\dfrac{BD}{\sin 56.602°} = \dfrac{2}{\sin 50°}$

$BD = \dfrac{2 \sin 56.602°}{\sin 50°}$
$=$ **2.18 cm** (to 3 s.f.)

13.



(i)

Map		**Actual**
2 cm	represent	5 km
1 cm	represents	$\dfrac{5}{2}$ km = 2.5 km
8 cm	represent	(8 × 2.5) km = 20 km

$\therefore$ the length represented by XZ is **20 km**.

(ii) Using Cosine Rule,

$\cos \angle YXZ = \dfrac{8^2 + 9^2 - 12^2}{2 \times 8 \times 9}$
$= \dfrac{1}{144}$

$\angle YXZ = \cos^{-1} \dfrac{1}{144}$
$= 89.602°$ (to 3 d.p.)
$=$ **89.6°** (to 1 d.p.)

(iii)

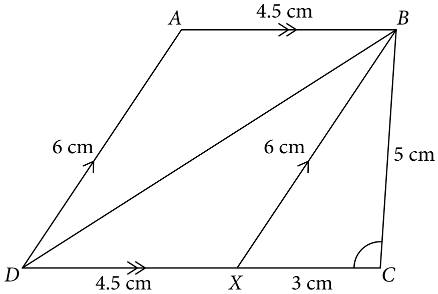
Map		**Actual**
2 cm	represent	5 km
1 cm	represents	$\frac{5}{2}$ km = 2.5 km
9 cm	represent	(9 × 2.5) km
		= 22.5 km

∴ the length represented by XY is 22.5 km.

Area of the farm $= \frac{1}{2} \times 20 \times 22.5 \times \sin 89.602°$

$= \mathbf{225\ km^2}$ (to 3 s.f.)

14.



$ABXD$ is a parallelogram.

Using Cosine Rule,

$\cos \angle BCX = \frac{5^2 + 3^2 - 6^2}{2 \times 5 \times 3}$

$= -\frac{1}{15}$

$\angle BCX = \cos^{-1}\left(-\frac{1}{15}\right)$

$= \mathbf{93.8°}$ (to 1 d.p.)

Using Cosine Rule,

$BD^2 = CD^2 + BC^2 - 2 \times CD \times BC \times \cos \angle BCD$

$= 7.5^2 + 5^2 - 2 \times 7.5 \times 5 \times \left(-\frac{1}{15}\right)$

$= 86.25$

$\therefore BD = \sqrt{86.25}$

$= \mathbf{9.29\ cm}$ (to 3 s.f.)

15. (i) $\frac{AD}{AB} = \frac{3+2}{3} = \frac{5}{3}$

$\frac{AE}{AC} = \frac{6+5}{6} = \frac{11}{6}$

$\frac{AD}{AB} \neq \frac{AE}{AC}$

∴ **no**, $\triangle ADE$ is not an enlargement of $\triangle ABC$.

(ii) Using Cosine Rule,

$\cos \theta = \frac{3^2 + 6^2 - 3.5^2}{2 \times 3 \times 6}$

$= \mathbf{\frac{131}{144}}$

(iii) Using Cosine Rule,

$x^2 = (3+2)^2 + (6+5)^2 - 2 \times (3+2) \times (6+5) \times \frac{131}{144}$

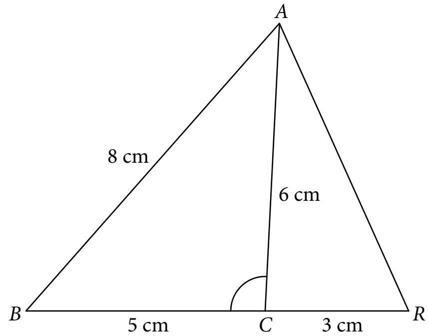
$= 5^2 + 11^2 - 2 \times 5 \times 11 \times \frac{131}{144}$

$= \frac{3307}{72}$

$x = \sqrt{\frac{3307}{72}}$

$= \mathbf{6.78}$ (to 3 s.f.)

16.



(i) Using Cosine Rule,

$\cos \angle BCA = \frac{5^2 + 6^2 - 8^2}{2 \times 5 \times 6}$

$= \mathbf{-\frac{1}{20}}$

(ii) Using Cosine Rule,

$AR^2 = 6^2 + 3^2 - 2 \times 6 \times 3 \times \cos \angle ACR$

$= 45 - 36 \times [-\cos(180° - \angle ACR)]$

$= 45 - 36 \times (-\cos \angle BCA)$

$= 45 + 36 \cos \angle BCA$

$= 45 + 36 \times \left(-\frac{1}{20}\right)$

$= 43.2$

$AR = \sqrt{43.2}$

$= \mathbf{6.57\ cm}$ (to 3 s.f.)

17. Using Cosine Rule,

$\cos A = \frac{(5+3)^2 + (6+7)^2 - 14^2}{2 \times (5+3) \times (6+7)}$

$= \frac{8^2 + 13^2 - 14^2}{2 \times 8 \times 13}$

$= \frac{37}{208}$

Using Cosine Rule,

$PQ^2 = 5^2 + 6^2 - 2 \times 5 \times 6 \times \frac{37}{208}$

$= \frac{2617}{52}$

$PQ = \sqrt{\frac{2617}{52}}$

$= \mathbf{7.09\ cm}$ (to 3 s.f.)

Chapter 7 Applications of Trigonometry

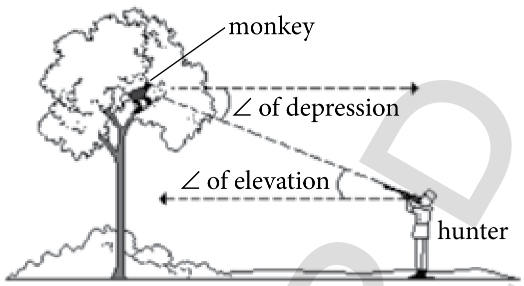
TEACHING NOTES

Suggested Approach

In Book 2, students have learnt how to use the clinometer to measure angles and find the heights of buildings, flagpoles, etc. This chapter introduces students to angles of elevation and depression. Teachers can use various illustrations to help students with their learning and understanding of angles of elevation and depression, and how they may be used to solve simple problems.

Section 7.1: Angles of elevation and depression

Teachers may use the following illustration to show students the angle of elevation and the angle of depression.



Using the board or a visualiser, draw a picture of a monkey up in a tree and a hunter with a gun, looking for game. When the hunter spots the monkey in the tree, his gun, which was initially down, is now raised through an angle towards the monkey. Teachers may illustrate this as the angle of elevation.

When the monkey notices that a hunter is pointing a gun at it, it looks at him at an angle and teachers may illustrate this as the angle of depression.

Section 7.2: Bearings

To introduce the concept of true bearing, teachers may revisit the general compass bearing that students might have learnt in Geography. Thus, true North is equivalent to 000° or 360° and East is equivalent to 090° while

North-West is represented by 315°, etc. From this explanation, students could better understand how the concept of bearing is used in different disciplines.

Students might also have difficulties identifying parallel lines and thus angles related to parallel lines. Teachers should highlight to students that all north-south lines are parallel, and the same is also true for all east-west lines. Therefore, students should first draw/identify such lines at all related points to help them solve problems involving bearings.

Section 7.3: Three-dimensional problems

Teachers can start off with an activity to introduce three-dimensional solids so that students can better visualise these solids when solving three-dimensional problems (see Investigation: Visualising right angles in 3D figures).

Teachers should highlight to students that the basic technique used in solving three-dimensional problems is to reduce it to a problem in a plane. Since students may encounter difficulties in this area, more practice and guidance will have to be given to them.

Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 6).

7.1 Angles of elevation and depression

Performance Task (Page 220)

1. Size of acute angle formed at the intersection between AB and the thread = $90° - \angle x$

Since the thread and the red horizontal dotted line that passes through A are perpendicular to each other,

$90° - \angle x + \angle y + 90° = 180°$

$\angle y = \angle x$

3. **(i)** Let the distance between the student and the object be k m.

(ii) Let the angle of elevation be $a°$.

(iii) Let the vertical height of the clinometer above the ground be h m and the vertical height between the top of the object and the clinometer be p m.

$\therefore \tan a° = \dfrac{p}{k}$

$p = k \tan a°$

Height of the object $= h + p$

$= h + k \tan a°$

Practise Now 1

1. Let x m be the distance of the point on the ground from the foot of the office tower.

$\tan 34° = \dfrac{43}{x}$

$x \tan 34° = 43$

$x = \dfrac{43}{\tan 34°}$

$= 63.8$ (to 3 s.f.)

$\therefore$ the distance of the point on the ground from the foot of the office tower is **63.8 m**.

2. $\tan 48° = \dfrac{50}{LA}$

$LA \tan 48° = 50$

$LA = \dfrac{50}{\tan 48°}$

$= 45.020$ m (to 5 s.f.)

$\tan 38° = \dfrac{50}{LB}$

$LB \tan 38° = 50$

$LB = \dfrac{50}{\tan 38°}$

$= 63.997$ m (to 5 s.f.)

Distance between boats A and $B = 63.997 - 45.020$

$=$ **19.0 m** (to 3 s.f.)

Practise Now 2

1. **(i)** $\angle ACB = 25°$ (alt. $\angle$s)

$\tan 25° = \dfrac{165}{BC}$

$BC = \dfrac{165}{\tan 25°}$

$= 353.84$ m (to 5 s.f.)

$=$ **354 m** (to 3 s.f.)

(ii) Let θ be the angle of depression of the sports car from the centre of the wheel.

$\tan \theta = \dfrac{165 - 75}{353.84}$

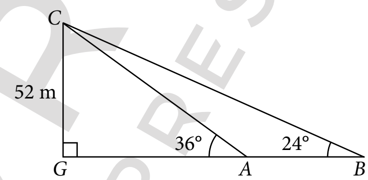
$= \dfrac{90}{353.84}$

$\theta = \tan^{-1} \dfrac{90}{353.84}$

$= 14.3°$ (to 1 d.p.)

$\therefore$ the angle of depression of the sports car from the centre of the wheel is **14.3°**.

2.



$\tan 36° = \dfrac{52}{GA}$

$GA = \dfrac{52}{\tan 36°}$

$= 71.572$ m (to 5 s.f.)

$\tan 24° = \dfrac{52}{71.572 + AB}$

$AB = \dfrac{52}{\tan 24°} - 71.572$

$= 45.2$ m (to 3 s.f.)

$\therefore$ the distance between the top ships is **45.2 m**.

Exercise 7A

1. Let h m be the height of the kite above Li Ting's hand.

$\sin 58° = \dfrac{h}{140}$

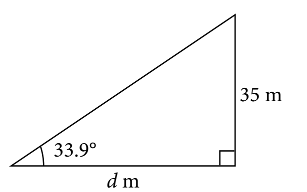
$h = 140 \sin 58°$

$=$ **119 m** (to 3 s.f.)

2. Let d m be the distance between the two buildings.

Height difference between the two buildings $= 120 - 85$

$= 35$ m



$\tan 33.9° = \dfrac{35}{d}$

$d = \dfrac{35}{\tan 33.9°}$

$= 52.1$ m (to 3 s.f.)

$\therefore$ the distance between the two buildings is **52.1 m**.

3. Let $x°$ be the angle of elevation of P from the point R.

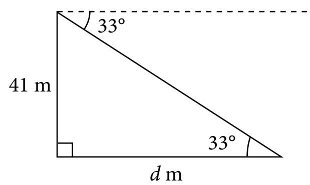
$\tan x° = \frac{44}{84}$

$x° = \tan^{-1} \frac{44}{84}$

$= 27.6°$ (to 1 d.p.)

$\therefore$ the angle of elevation of P from the point R is **27.6°**.

4. Let d m be the distance between the fire hydrant and the foot of the building.



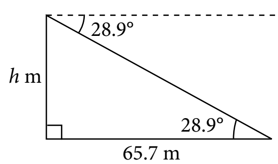
$\tan 33° = \frac{41}{d}$

$d = \frac{41}{\tan 33°}$

$= 63.1$ m (to 3 s.f.)

$\therefore$ the distance between the fire hydrant and the foot of the building is **63.1 m**.

5. Let h m be the height of the cliff.



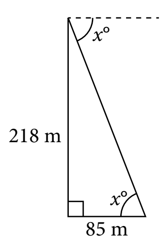
$\tan 28.9° = \frac{h}{65.7}$

$h = 65.7 \tan 28.9°$

$= 36.3$ m (to 3 s.f.)

$\therefore$ the height of the cliff is **36.3 m**.

6. Let $x°$ be the angle of depression of the bird from the top of the castle.



$\tan x° = \frac{218}{85}$

$x° = \tan^{-1} \frac{218}{85}$

$= 68.7°$ (to 1 d.p.)

$\therefore$ the angle of depression of the bird from the top of the castle is **68.7°**.

7. $\tan 37° = \frac{45}{BM}$

$BM = \frac{45}{\tan 37°}$

$= 59.717$ m (to 5 s.f.)

$\tan 42° = \frac{45}{AM}$

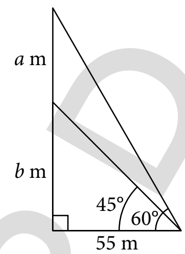
$AM = \frac{45}{\tan 42°}$

$= 49.978$ m (to 5 s.f.)

$AB = 59.717 - 49.978$

$=$ **9.74 m** (to 3 s.f.)

8. Let a m be the height of the castle and b m be the height of the mountain.



$\tan 45° = \frac{b}{55}$

$b = 55 \tan 45°$

$= 55$

$\tan 60° = \frac{a + 55}{55}$

$a + 55 = 55 \tan 60°$

$a = 55 \tan 60° - 55$

$= 40.3$ (to 3 s.f.)

$\therefore$ the height of the castle is **40.3 m**.

9. Let y m be the distance of Q from the bottom of the bridge and z m be the distance of P from the bottom of the bridge.

$\tan 23° = \frac{5.5}{y}$

$y = \frac{5.5}{\tan 23°}$

$= 12.957$ (to 5 s.f.)

$z = 12.957 - 5.1$

$= 7.857$

$\tan x° = \frac{5.5}{7.857}$

$x° = \tan^{-1} \frac{5.5}{7.857}$

$= 35.0°$ (to 1 d.p.)

$\therefore x =$ **35.0**

10. $\tan 38° = \frac{320}{AC}$

$AC = \frac{320}{\tan 38°}$

$= 409.58$ m (to 5 s.f.)

$\tan 58° = \frac{320}{BC}$

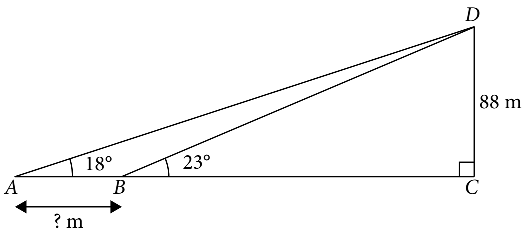
$BC = \frac{320}{\tan 58°}$

$= 199.96$ m (to 5 s.f.)

$x = 409.58 - 199.96$

$=$ **210** (to 3 s.f.)

11.



$\tan 23° = \dfrac{88}{BC}$

$BC = \dfrac{88}{\tan 23°}$

$= 207.32$ m (to 5 s.f.)

$\tan 18° = \dfrac{88}{AC}$

$AC = \dfrac{88}{\tan 18°}$

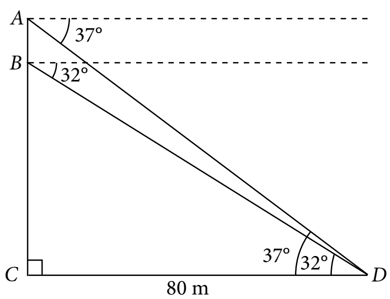
$= 270.84$ m (to 5 s.f.)

$AB = 270.84 - 207.32$

$= 63.5$ m (to 3 s.f.)

∴ the distance between the two boats is **63.5 m**.

12.



$\tan 32° = \dfrac{BC}{80}$

$BC = 80 \tan 32°$

$= 49.990$ m (to 5 s.f.)

$\tan 37° = \dfrac{AC}{80}$

$AC = 80 \tan 37°$

$= 60.284$ m (to 5 s.f.)

$AB = 60.284 - 49.990$

$= 10.3$ m (to 3 s.f.)

∴ the height of the satellite dish is **10.3 m**.

13. Let d m be the distance between the bottom of the building and point T on level ground.

$\tan 26° = \dfrac{h}{d}$

$d = \dfrac{h}{\tan 26°}$ — (1)

$\tan 35° = \dfrac{h + 12.2}{d}$

$d = \dfrac{h + 12.2}{\tan 35°}$ — (2)

Substitute (1) into (2):

$\dfrac{h}{\tan 26°} = \dfrac{h + 12.2}{\tan 35°}$

$h \tan 35° = h \tan 26° + 12.2 \tan 26°$

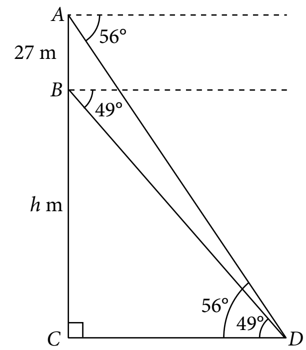
$h \tan 35° - h \tan 26° = 12.2 \tan 26°$

$h(\tan 35° - \tan 26°) = 12.2 \tan 26°$

$h = \dfrac{12.2 \tan 26°}{\tan 35° - \tan 26°}$

$=$ **28.0** (to 3 s.f.)

14. Let h m be the height of the cliff.



(i) $\tan 49° = \dfrac{h}{CD}$

$h = CD \tan 49°$ — (1)

$\tan 56° = \dfrac{h + 27}{CD}$

$h + 27 = CD \tan 56°$

$h = CD \tan 56° - 27$ — (2)

Substitute (2) into (1):

$CD \tan 56° - 27 = CD \tan 49°$

$CD \tan 56° - CD \tan 49° = 27$

$CD(\tan 56° - \tan 49°) = 27$

$CD = \dfrac{27}{\tan 56° - \tan 49°}$

$= 81.278$ m (to 5 s.f.)

$= 81.3$ m (to 3 s.f.)

∴ the distance between the base of the cliff and the guard house is **81.3 m**.

(ii) Substitute $CD = 81.278$ m into (1):

$h = 81.278 \tan 49°$

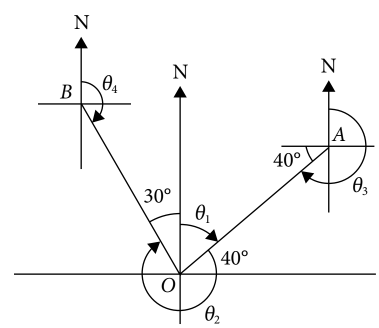
$= 93.5$ m (to 3 s.f.)

∴ the height of the cliff is **93.5 m**.

7.2 Bearings

Practise Now 3

1.



(a) The bearing of A from O is given by the acute angle θ_1, which is (90° – 40°).

∴ the bearing of A from O is **050°**.

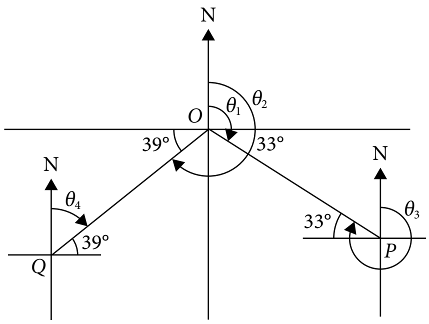
(b) The bearing of B from O is given by the reflex angle θ_2, which is (360° – 30°).

∴ the bearing of B from O is **330°**.

(c) The bearing of O from A is given by the reflex angle θ_3, which is $(270° - 40°)$.
$\therefore$ the bearing of O from A is **230°**.

(d) The bearing of O from B is given by the obtuse angle θ_4, which is $(180° - 30°)$.
$\therefore$ the bearing of O from B is **150°**.

2.



(a) The bearing of P from O is given by the obtuse angle θ_1, which is $(90° + 33°)$.
$\therefore$ the bearing of P from O is **123°**.

(b) The bearing of Q from O is given by the reflex angle θ_2, which is $(270° - 39°)$.
$\therefore$ the bearing of Q from O is **231°**.

(c) The bearing of O from P is given by the reflex angle θ_3, which is $(270° + 33°)$.
$\therefore$ the bearing of O from P is **303°**.

(d) The bearing of O from Q is given by the acute angle θ_4, which is $(90° - 39°)$.
$\therefore$ the bearing of O from Q is **051°**.

Practise Now 4

1. Since the bearing of R from Q is 118°,

$\angle PRQ = 118° - 44°$ (ext. $\angle$ of $\triangle$)
$= 74°$

$\angle PQR = 180° - 118°$ (adj. $\angle$s on a str. line)
$= 62°$

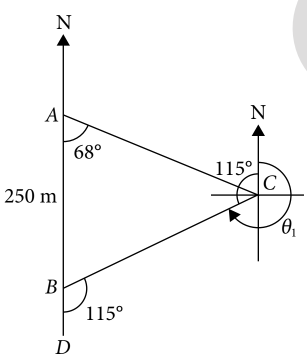
Using Sine Rule,

$$\frac{PQ}{\sin 74°} = \frac{150}{\sin 44°}$$

$$PQ = \frac{150 \sin 74°}{\sin 44°}$$

$= $ **208 m** (to 3 s.f.)

2.



(i) The bearing of B from C is given by the reflex angle θ_1, which is $(360° - 115°)$.
$\therefore$ the bearing of B from C is **245°**.

(ii) $\angle ABC = 180° - 115°$ (adj. $\angle$s on a str. line)
$= 65°$

$\angle ACB = 115° - 68°$ (ext. $\angle$ of $\triangle$)
$= 47°$

Using Sine Rule,

$$\frac{AC}{\sin 65°} = \frac{250}{\sin 47°}$$

$$AC = \frac{250 \sin 65°}{\sin 47°}$$

$= $ **310 m** (to 3 s.f.)

Using Sine Rule,

$$\frac{BC}{\sin 68°} = \frac{250}{\sin 47°}$$

$$BC = \frac{250 \sin 68°}{\sin 47°}$$

$= $ **317 m** (to 3 s.f.)

Practise Now 5

(i) $\angle ABC = 360° - 60° - 238°$ ($\angle$s at a point)
$= 62°$

Using Cosine Rule,

$AC^2 = 55^2 + 35^2 - 2 \times 55 \times 35 \times \cos 62°$
$= 2442.5$ (to 5 s.f.)

$AC = \sqrt{2442.5}$
$= $ **49.4 km** (to 3 s.f.)

(ii) Using Cosine Rule,

$$\cos \angle BAC = \frac{2442.5 + 35^2 - 55^2}{2 \times \sqrt{2442.5} \times 35}$$

$= 0.185\ 72$ (to 5 s.f.)

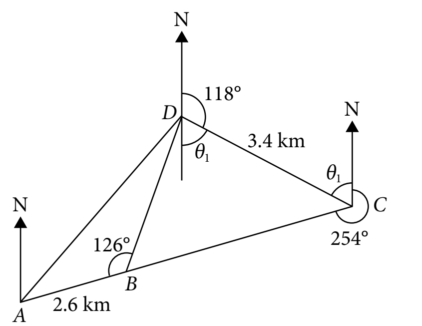
$\angle BAC = \cos^{-1} 0.185\ 72$
$= 79.297°$ (to 3 d.p.)

$120° + 79.297° = 199.3°$ (to 1 d.p.)

$\therefore$ the bearing of C from A is **199.3°**.

Practise Now 6

(a) (i)



$\theta_1 = 180° - 118°$ (int. $\angle$s)
$= 62°$

$\angle BCD = 360° - 254° - 62°$ ($\angle$s at a point)
$= 44°$

$\angle CDB = 126° - 44°$ (ext. $\angle$ of $\triangle$)
$= 82°$

$118° + 82° = 200°$

$\therefore$ the bearing of B from D is **200°**.

(ii) $\angle CBD = 180° - 126°$ (adj. ∠s on a str. line)
$= 54°$

Using Sine Rule,

$$\frac{BD}{\sin 44°} = \frac{3.4}{\sin 54°}$$

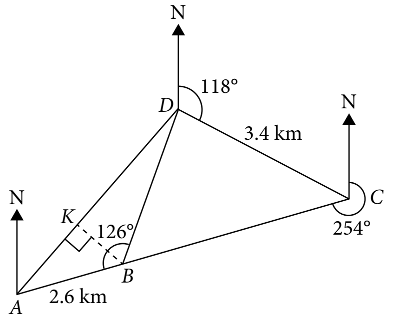
$$BD = \frac{3.4 \sin 44°}{\sin 54°}$$
$= 2.9194$ km (to 5 s.f.)
$=$ **2.92 km** (to 3 s.f.)

(iii) Using Cosine Rule,

$$AD^2 = 2.6^2 + 2.9194^2 - 2 \times 2.6 \times 2.9194 \times \cos 126°$$
$= 24.206$ (to 5 s.f.)

$$AD = \sqrt{24.206}$$
$=$ **4.92 km** (to 3 s.f.)

(b)



The shortest path corresponds to BK, where BK is perpendicular to AD.

$$\text{Area of } \triangle ABD = \frac{1}{2} \times AB \times BD \times \sin \angle ABD$$

$$\frac{1}{2} \times AD \times BK = \frac{1}{2} \times AB \times BD \times \sin \angle ABD$$

$$\frac{1}{2} \times \sqrt{24.206} \times BK = \frac{1}{2} \times 2.6 \times 2.9194 \times \sin 126°$$

$$BK = \frac{2.6 \times 2.9194 \times \sin 126°}{\sqrt{24.206}}$$
$= 1.25$ km (to 3 s.f.)

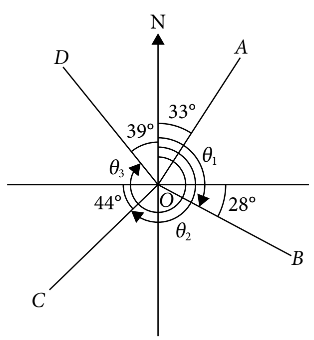
∴ the length of the path is **1.25 km**.

Introductory Problem Revisited

$$\cos 50° = \frac{TX}{BT}$$
$$= \frac{TX}{280}$$
$$TX = 280 \cos 50°$$
$=$ **180 m** (to 3 s.f.)

Exercise 7B

1. **(a)**



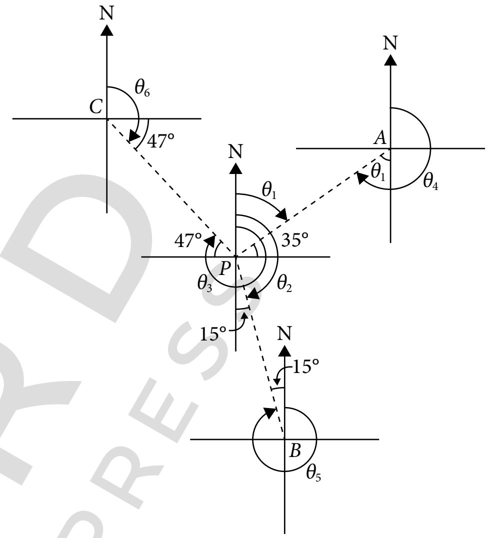
The bearing of A from O is **033°**.

(b) The bearing of B from O is given by the obtuse angle θ_1, which is (90° + 28°).
∴ the bearing of B from O is **118°**.

(c) The bearing of C from O is given by the reflex angle θ_2, which is (270° – 44°).
∴ the bearing of C from O is **226°**.

(d) The bearing of D from O is given by the reflex angle θ_3, which is (360° – 39°).
∴ the bearing of D from O is **321°**.

2. **(a)**



The bearing of A from P is given by the acute angle θ_1, which is (90° – 35°).
∴ the bearing of A from P is **055°**.

(b) The bearing of B from P is given by the obtuse angle θ_2, which is (180° – 15°).
∴ the bearing of B from P is **165°**.

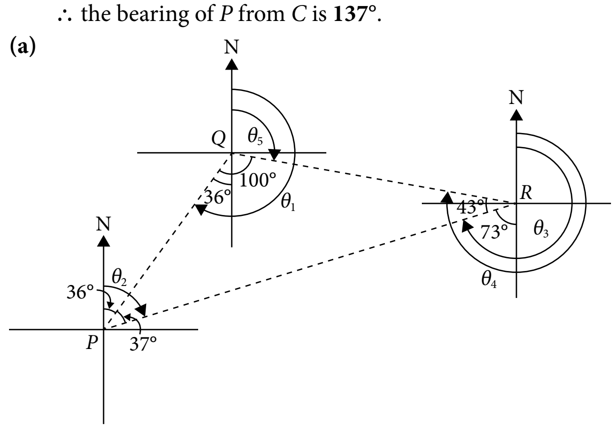
(c) The bearing of C from P is given by the reflex angle θ_3, which is (270° + 47°).
∴ the bearing of C from P is **317°**.

(d) The bearing of P from A is given by the reflex angle θ_4, which is (180° + 55°).
∴ the bearing of P from A is **235°**.

(e) The bearing of P from B is given by the reflex angle θ_5, which is (360° – 15°).
∴ the bearing of P from B is **345°**.

(f) The bearing of P from C is given by the obtuse angle θ_6, which is (90° + 47°).
∴ the bearing of P from C is **137°**.

3. **(a)**



The bearing of Q from P is **036°**.

(b) The bearing of P from Q is given by the reflex angle θ_1, which is $(180° + 36°)$.
$\therefore$ the bearing of P from Q is **216°**.

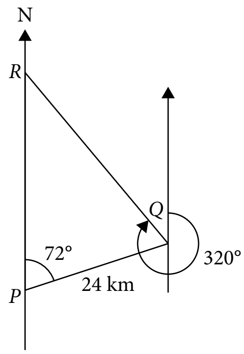
(c) The bearing of R from P is given by the acute angle θ_2, which is $(36° + 37°)$.
$\therefore$ the bearing of R from P is **073°**.

(d) The bearing of P from R is given by the reflex angle θ_3, which is $(180° + 73°)$.
$\therefore$ the bearing of P from R is **253°**.

(e) The bearing of Q from R is given by the reflex angle θ_4, which is $(253° + 43°)$.
$\therefore$ the bearing of Q from R is **296°**.

(f) The bearing of R from Q is given by the acute angle θ_5, which is $(216° − 100°)$.
$\therefore$ the bearing of R from Q is **116°**.

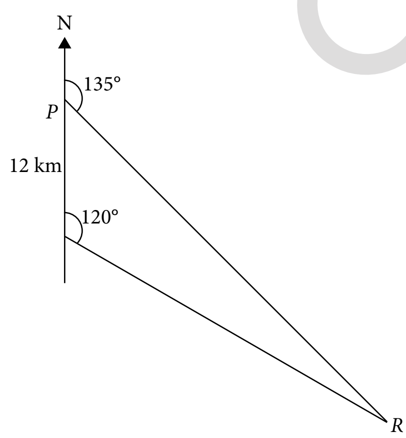
4.



(i) $\angle PQR = 320° − 180° − 72°$
$= 68°$
$\angle PRQ = 180° − 72° − 68°$ ($\angle$ sum of $\triangle$)
$= 40°$
Using Sine Rule,
$$\frac{PR}{\sin 68°} = \frac{24}{\sin 40°}$$
$$PR = \frac{24 \sin 68°}{\sin 40°}$$
$= \mathbf{34.6\ km}$ (to 3 s.f.)

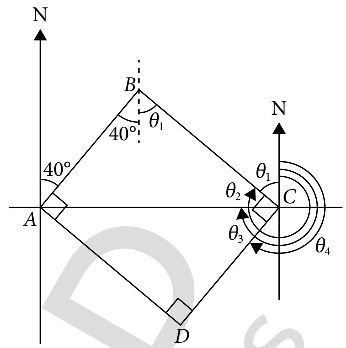
(ii) Using Sine Rule,
$$\frac{QR}{\sin 72°} = \frac{24}{\sin 40°}$$
$$QR = \frac{24 \sin 72°}{\sin 40°}$$
$= \mathbf{35.5\ km}$ (to 3 s.f.)

5.



$\angle PRQ = 135° − 120°$ (ext. $\angle$ of $\triangle$)
$= 15°$
Using Sine Rule,
$$\frac{PR}{\sin 120°} = \frac{12}{\sin 15°}$$
$$PR = \frac{12 \sin 120°}{\sin 15°}$$
$= \mathbf{40.2\ km}$ (to 3 s.f.)

6.

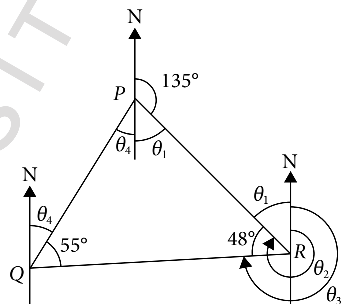


(i) $\theta_1 = 90° − 40°$
$= 50°$
The bearing of B from C is given by the reflex angle θ_2, which is $(360° − 50°)$.
$\therefore$ the bearing of B from C is **310°**.

(ii) The bearing of A from C is given by the reflex angle θ_3, which is 270°.
$\therefore$ the bearing of A from C is **270°**.

(iii) The bearing of D from C is given by the reflex angle θ_4, which is $(360° − 90° − 50°)$.
$\therefore$ the bearing of D from C is **220°**.

7.

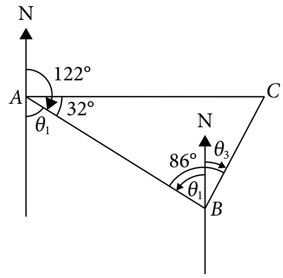


(i) $\theta_1 = 180° − 135°$ (adj. $\angle$s on a str. line)
$= 45°$
The bearing of P from R is given by the reflex angle θ_2, which is $(360° − 45°)$.
$\therefore$ the bearing of P from R is **315°**.

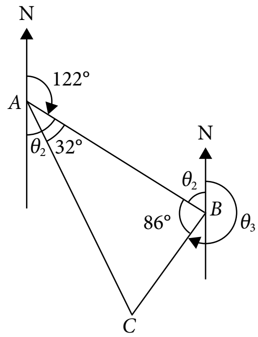
(ii) The bearing of Q from R is given by the reflex angle θ_3, which is $(315° − 48°)$.
$\therefore$ the bearing of Q from R is **267°**.

(iii) The bearing of P from Q is given by the acute angle θ_4.
$\angle QPR = 180° − 55° − 48°$ ($\angle$ sum of $\triangle$)
$= 77°$
$\theta_4 = 77° − 45°$
$= 32°$
$\therefore$ the bearing of P from Q is **032°**.

8.



Case 1



Case 2

Case 1:

$\theta_1 = 180° - 122°$ (adj. $\angle$s on a str. line)

$= 58°$

The bearing of C from B is given by the acute angle θ_3, which is $(86° - 58°)$.

$\therefore$ the bearing of C from B is 028°.

Case 2:

$\theta_2 = 180° - 122°$ (adj. $\angle$s on a str. line)

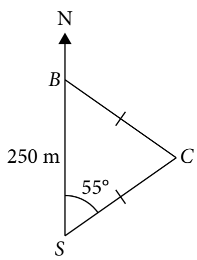
$= 58°$

The bearing of C from B is given by the reflex angle θ_3, which is $(360° - 86° - °58)$.

$\therefore$ the bearing of C from B is 216°.

$\therefore$ the possible bearings of C from B are **028° or 216°**.

9. **(a)**



Let B, C and S represent the bookshop, Sara and the supermarket respectively.

Since $BC = CS$, $\angle CBS = \angle BSC = 55°$.

$\angle BCS = 180° - 55° - 55°$ ($\angle$ sum of $\triangle$)

$= 70°$

Using Sine Rule,

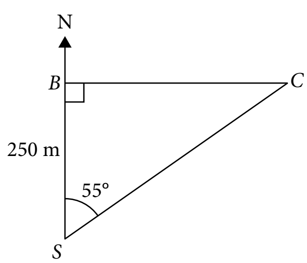
$$\frac{CS}{\sin 55°} = \frac{250}{\sin 70°}$$

$$CS = \frac{250 \sin 55°}{\sin 70°}$$

$= 218$ m (to 3 s.f.)

$\therefore$ Sara has to walk **218 m**.

(b)



Let B, C and S represent the bookshop, Sara and the supermarket respectively.

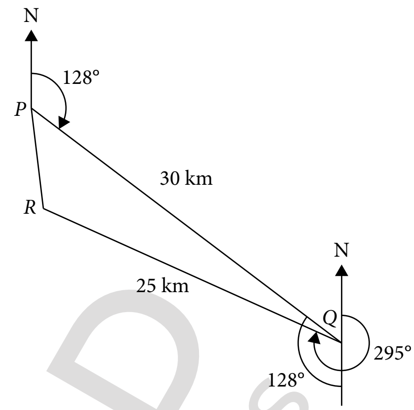
$$\cos 55° = \frac{250}{CS}$$

$$CS = \frac{250}{\cos 55°}$$

$= 436$ m (to 3 s.f.)

$\therefore$ Sara has to walk **436 m**.

10.



$\angle PQR = 128° + 180° - 295°$

$= 13°$

Using Cosine Rule,

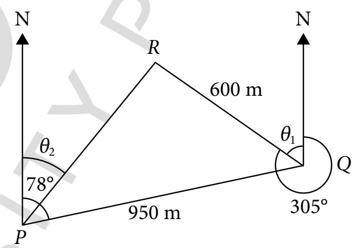
$PR^2 = 30^2 + 25^2 - 2 \times 30 \times 25 \times \cos 13°$

$= 63.445$ (to 5 s.f.)

$PR = \sqrt{63.445}$

$=$ **7.97 km** (to 3 s.f.)

11.



(i) $\theta_1 = 360° - 305°$ ($\angle$s at a point)

$= 55°$

$\angle PQR = 180° - 78° - 55°$ (int. $\angle$s)

$= 47°$

Using Cosine Rule,

$PR^2 = 950^2 + 600^2 - 2 \times 950 \times 600 \times \cos 47°$

$= 485\ 020$ (to 5 s.f.)

$PR = \sqrt{485\ 020}$

$=$ **696 m** (to 3 s.f.)

(ii) Using Cosine Rule,

$$\cos \angle QPR = \frac{PR^2 + PQ^2 - QR^2}{2 \times PR \times PQ}$$

$$= \frac{485\ 020 + 950^2 - 600^2}{2 \times \sqrt{485\ 020} \times 950}$$

$= 0.776\ 53$ (to 5 s.f.)

$\angle QPR = \cos^{-1} 0.776\ 53$

$= 39.056°$ (to 3 d.p.)

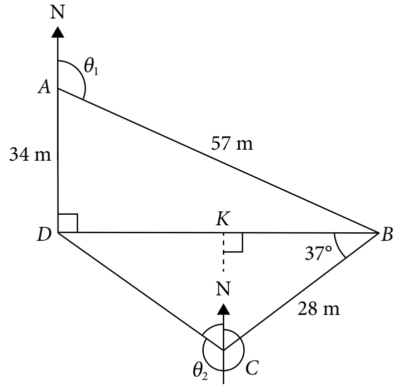
The bearing of R from P is given by the acute angle θ_2, which is $(78° - 39.056°)$.

$\theta_2 = 78° - 39.056°$

$= 38.9°$ (to 1 d.p.)

$\therefore$ the bearing of R from P is **038.9°**.

12.



(i) $\sin \angle ABD = \frac{34}{57}$

$\angle ABD = \sin^{-1} \frac{34}{57}$

$= 36.619°$ (to 3 d.p.)

The bearing of B from A is given by the acute angle θ_1, which is $(90° + 36.619°)$ (ext. $\angle$ of $\triangle$).

$\theta_1 = 90° + 36.619°$

$= 126.6°$ (to 1 d.p.)

$\therefore$ the bearing of B from A is **126.6°**.

(ii) The shortest path corresponds to CK, where CK is perpendicular to BD.

$\sin 37° = \frac{CK}{28}$

$CK = 28 \sin 37°$

$= 16.9$ m (to 3 s.f.)

$\therefore$ the length of the path is **16.9 m**.

(iii) Using Pythagoras' Theorem,

$BD^2 = 57^2 - 34^2$

$= 2093$

$BD = \sqrt{2093}$

$= 45.749$ m (to 5 s.f.)

Using Cosine Rule,

$CD^2 = 2093 + 28^2 - 2 \times 45.749 \times 28 \times \cos 37°$

$= 830.94$ (to 5 s.f.)

$CD = \sqrt{830.94}$

$= 28.826$ m (to 5 s.f.)

Using Cosine Rule,

$\cos \angle BCD = \frac{28^2 + 830.94 - 2093}{2 \times 28 \times 28.826}$

$= -0.296\ 15$ (to 5 s.f.)

$\angle BCD = \cos^{-1} (-0.296\ 15)$

$= 107.227°$ (to 3 d.p.)

$\angle BCK = 180° - 90° - 37°$ ($\angle$ sum of $\triangle$)

$= 53°$

$\angle DCK = 107.227° - 53°$

$= 54.227°$

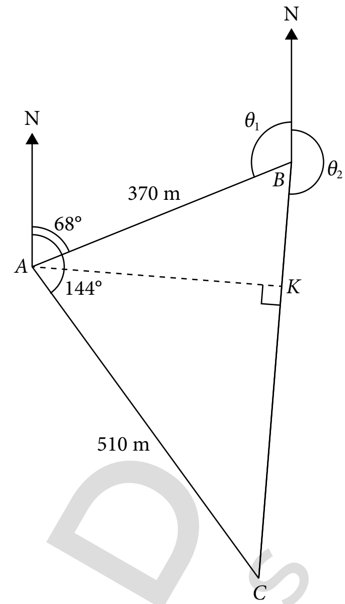
The bearing of D from C is given by the reflex angle θ_2, which is $(360° - 54.227°)$.

$\theta_2 = 360° - 54.227°$

$= 305.8°$ (to 1 d.p.)

$\therefore$ the bearing of D from C is **305.8°**.

13.



(i) $\angle BAC = 144° - 68°$

$= 76°$

Using Cosine Rule,

$BC^2 = 370^2 + 510^2 - 2 \times 370 \times 510 \times \cos 76°$

$= 305\ 700$ (to 5 s.f.)

$BC = \sqrt{305\ 700}$

$=$ **553 m** (to 3 s.f.)

(ii) Using Sine Rule,

$\frac{\sin \angle ACB}{370} = \frac{\sin 76°}{\sqrt{305\ 700}}$

$\sin \angle ACB = \frac{370 \sin 76°}{\sqrt{305\ 700}}$

$= 0.649\ 32$ (to 5 s.f.)

$\angle ACB = \sin^{-1} 0.649\ 32$

$= 40.490°$ (to 3 d.p.)

$=$ **40.5°** (to 1 d.p.)

(iii) $\theta_1 = 180° - 68°$ (int. $\angle$s)

$= 112°$

Using Sine Rule,

$\frac{\sin \angle ABC}{510} = \frac{\sin 76°}{\sqrt{305\ 700}}$

$\sin \angle ABC = \frac{510 \sin 76°}{\sqrt{305\ 700}}$

$= 0.895\ 01$ (to 5 s.f.)

$\angle ABC = \sin^{-1} 0.895\ 01$

$=$ **63.510°** (to 3 d.p.)

The bearing of C from B is given by the obtuse angle θ_2, which is $(360° - 112° - 63.510°)$.

$\theta_2 = 360° - 112° - 63.510°$

$= 184.5°$ (to 1 d.p.)

$\therefore$ the bearing of C from B is **184.5°**.

(iv) The shortest distance from A to BC corresponds to AK, where AK is perpendicular to BC.

$\sin 40.490° = \frac{AK}{510}$

$AK = 510 \sin 40.490°$

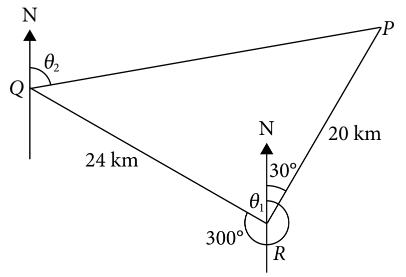
$= 331$ m (to 3 s.f.)

$\therefore$ the shortest distance from A to BC is **331 m**.

14. After 2 hours,

Distance travelled by $P = 10 \times 2 = 20$ km

Distance travelled by $Q = 12 \times 2 = 24$ km



$\theta_1 = 360° - 300°$ ($\angle$s at a point)

$= 60°$

$\angle PRQ = 60° + 30° = 90°$

$\therefore \triangle PRQ$ is a right-angled triangle.

Using Pythagoras' Theorem,

$PQ^2 = 24^2 + 20^2$

$= 976$

$PQ = \sqrt{976}$

$=$ **31.2 km** (to 3 s.f.)

$\tan \angle PQR = \frac{20}{24}$

$\angle PQR = \tan^{-1} \frac{20}{24}$

$= 39.806°$ (to 3 d.p.)

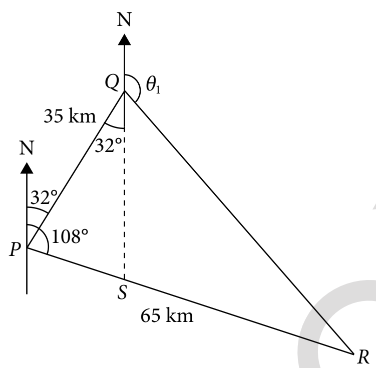
The bearing of P from Q is given by the acute angle θ_2, which is $(180° - 60° - 39.806°)$ (int. $\angle$s).

$\theta_2 = 180° - 60° - 39.806°$

$= 80.2°$ (to 1 d.p.)

$\therefore$ the bearing of P from Q is **080.2°**.

15.



(a) (i) $\angle QPR = 108° - 32°$

$= 76°$

Using Cosine Rule,

$QR^2 = 35^2 + 65^2 - 2 \times 35 \times 65 \times \cos 76°$

$= 4349.3$ (to 5 s.f.)

$QR = \sqrt{4349.3}$

$=$ **65.9 km** (to 3 s.f.)

(ii) Using Sine Rule,

$\frac{\sin \angle PQR}{65} = \frac{\sin 76°}{\sqrt{4349.3}}$

$\sin \angle PQR = \frac{65 \sin 76°}{\sqrt{4349.3}}$

$= 0.956\ 33$ (to 5 s.f.)

$\angle PQR = \sin^{-1} 0.956\ 33$

$= 73.005°$ (to 3 d.p.)

The bearing of R from Q is given by the acute angle θ_1, which is $[180° - (73.005° - 32°)]$.

$\theta_1 = 180° - (73.005° - 32°)$

$= 139.0°$ (to 1 d.p.)

$\therefore$ the bearing of R from Q is **139.0°**.

(b) $\angle PSQ = 180° - 76° - 32°$ ($\angle$ sum of $\triangle$)

$= 72°$

Using Sine Rule,

$\frac{PS}{\sin 32°} = \frac{35}{\sin 72°}$

$PS = \frac{35 \sin 32°}{\sin 72°}$

$= 19.502$ km (to 5 s.f.)

Time taken $= \frac{\text{Distance}}{\text{Speed}}$

$= \frac{19.502}{30}$

$= 0.650$ h (to 3 s.f.)

$= 39$ minutes

$\therefore$ the time when the ship reaches S is **1009 hours**.

7.3 Three-dimensional problems

Investigation (Visualising right angles in 3D figures)

2. **Yes**, $\angle NOB = 90°$. $\angle NOC = \angle NOD = 90°$. An object on a flat surface may make a right angle look smaller or larger than 90°.

3. **Method 1: Find a rectangle**

(a) **Yes**, the plane $EFGH$ is a rectangle because it is a face of the cuboid.

(b) Since $\angle EFG$ is a right angle of the rectangle $EFGH$, then $\angle EFG = 90°$ and $\triangle EFG$ is a right-angled triangle.

(c) $\triangle CGH$ lies on the plane $DCGH$. The plane $DCGH$ is a rectangle because it is a face of the cuboid. Since $\angle CGH$ is a right angle of the rectangle $DCGH$, then $\angle CGH = 90°$ and so $\triangle CGH$ is a right-angled triangle.

Method 2: Find a normal to a plane

(a) **Yes**, CG is a normal to the plane $EFGH$, because the plane $EFGH$ is a horizontal and the line CG is a vertical.

(b) **Yes**, the line GE is a line on the plane $EFGH$.

(c) Since CG is a normal to the plane $EFGH$, and GE is a line on the plane $EFGH$, then $\angle CGE = 90°$ because a normal to a plane is perpendicular to every line on the plane. Thus $\triangle CGE$ is a right-angled triangle.

(d) Since the plane $DCGH$ is vertical and the line EH is horizontal, then EH is a horizontal to the plane $DCGH$. Since the line HC lies on the plane $DCGH$, then the normal EH is perpendicular to the line HC, i.e. $\angle CHE = 90°$. Therefore, $\triangle CHE$ is a right-angled triangle.

4. **(a)** ***PS*, *QR*, *AD* and *BC*.**

(b) ***RS*, *PQ*, *CD* and *AB*.**

5. **(a)** **$\triangle NVZ$, $\triangle NVX$, $\triangle NVW$, $\triangle NVY$, $\triangle WYZ$, $\triangle XYZ$, $\triangle WXY$, $\triangle WXZ$, $\triangle NWZ$, $\triangle NYZ$, $\triangle NXY$ and $\triangle NWX$.**

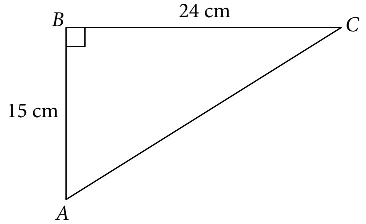
(b) **$\triangle NVZ$, $\triangle NVX$, $\triangle NVW$ and $\triangle NVY$** are identical.

$\triangle WYZ$, $\triangle XYZ$, $\triangle WXY$ and $\triangle WXZ$ are identical.

$\triangle NWZ$, $\triangle NYZ$, $\triangle NXY$ and $\triangle NWX$ are identical.

Practise Now 7

1. **(i)**



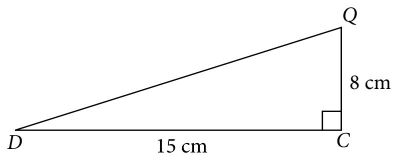
In $\triangle ABC$, $\angle ABC = 90°$.

$\tan \angle BAC = \frac{24}{15}$

$\angle BAC = \tan^{-1} \frac{24}{15}$

$= \mathbf{58.0°}$ (to 1 d.p.)

(ii)



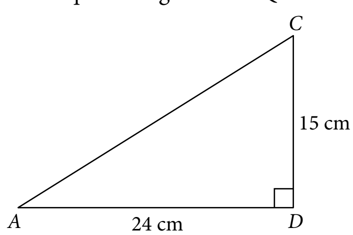
In $\triangle CDQ$, $\angle DCQ = 90°$.

$\tan \angle CDQ = \frac{8}{15}$

$\angle CDQ = \tan^{-1} \frac{8}{15}$

$= \mathbf{28.1°}$ (to 1 d.p.)

(iii) The required angle is $\angle CAQ$.



In $\triangle ACD$, $\angle ADC = 90°$.
Using Pythagoras' Theorem,

$AC^2 = AD^2 + CD^2$

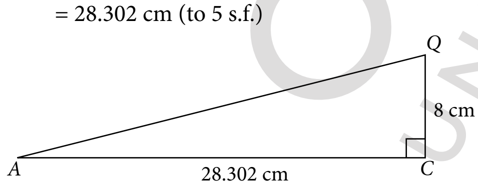
$= 24^2 + 15^2$

$= 576 + 225$

$= 801$

$AC = \sqrt{801}$

$= 28.302$ cm (to 5 s.f.)



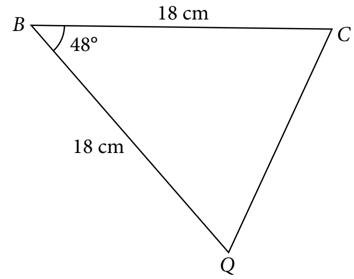
In $\triangle ACQ$, $\angle ACQ = 90°$.

$\tan \angle CAQ = \frac{8}{28.302}$

$\angle CAQ = \tan^{-1} \frac{8}{28.302}$

$= \mathbf{15.8°}$ (to 1 d.p.)

2. **(i)**



Using Cosine Rule,

$CQ^2 = 18^2 + 18^2 - 18 \times 18 \times 2 \times \cos °48$

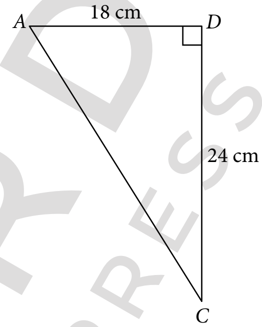
$= 214.40$ (to 5 s.f.)

$CQ = \sqrt{214.40}$

$= 14.642$ cm (to 5 s.f.)

$= \mathbf{14.6\ cm}$ (to 3 s.f.)

(ii)



In $\triangle ACD$, $\angle ADC = 90°$.
Using Pythagoras' Theorem,

$AC^2 = AD^2 + CD^2$

$= 18^2 + 24^2$

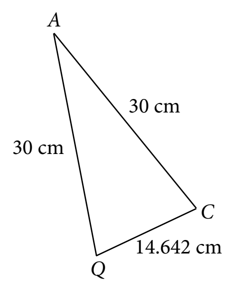
$= 324 + 576$

$= 900$

$AC = \sqrt{900}$

$= 30$ cm

$AQ = AC = 30$ cm



Using Cosine Rule,

$\cos \angle CAQ = \frac{30^2 + 30^2 - 214.40}{2 \times 30 \times 30}$

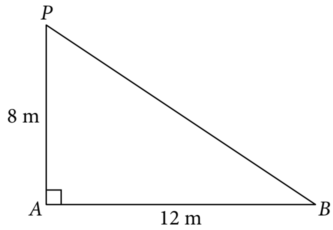
$= 0.880\ 89$ (to 5 s.f.)

$\angle CAQ = \cos^{-1} 0.880\ 89$

$= \mathbf{28.3°}$ (to 1 d.p.)

Practise Now 8

1. **(i)**



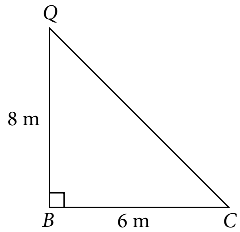
In $\triangle ABP$, $\angle BAP = 90°$.

$$\tan \angle ABP = \frac{8}{12}$$

$$\angle ABP = \tan^{-1} \frac{8}{12}$$

$= \mathbf{33.7°}$ (to 1 d.p.)

(ii)



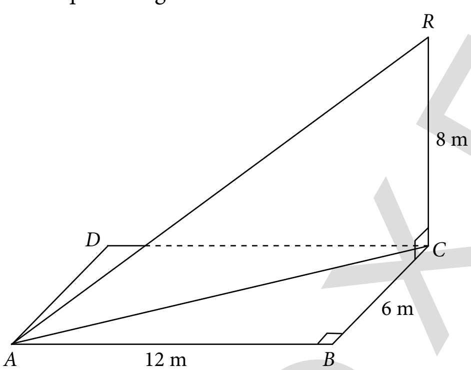
In $\triangle BCQ$, $\angle CBQ = 90°$.

$$\tan \angle BCQ = \frac{8}{6}$$

$$\angle BCQ = \tan^{-1} \frac{8}{6}$$

$= \mathbf{53.1°}$ (to 1 d.p.)

(iii) The required angle is $\angle CAR$.



In $\triangle ABC$, $\angle ABC = 90°$.

Using Pythagoras' Theorem,

$AC^2 = AB^2 + BC^2$

$= 12^2 + 6^2$

$= 144 + 36$

$= 180$

$AC = \sqrt{180}$

$= 13.416$ m (to 5 s.f.)

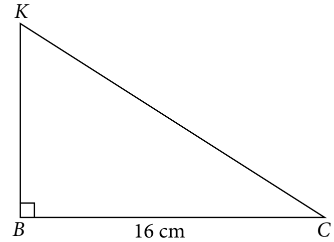
In $\triangle ACR$, $\angle ACR = 90°$.

$$\tan \angle CAR = \frac{8}{13.416}$$

$$\angle CAR = \tan^{-1} \frac{8}{13.416}$$

$= \mathbf{30.8°}$ (to 1 d.p.)

2. **(i)**



In $\triangle BCK$, $\angle CBK = 90°$.

$BK = 16 - 6$

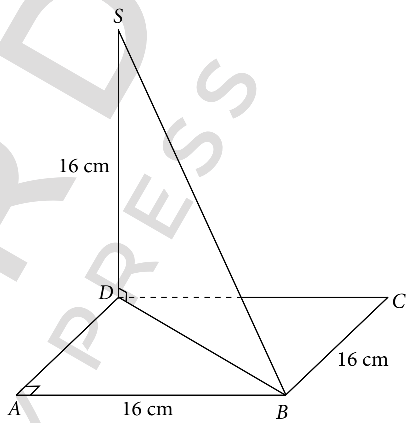
$= 10$ cm

$$\tan \angle BCK = \frac{10}{16}$$

$$\angle BCK = \tan^{-1} \frac{10}{16}$$

$= \mathbf{32.0°}$ (to 1 d.p.)

(ii)



In $\triangle ABD$, $\angle BAD = 90°$.

Using Pythagoras' Theorem,

$BD^2 = AB^2 + AD^2$

$= 16^2 + 16^2$

$= 256 + 256$

$= 512$

$BD = \sqrt{512}$

$= 22.627$ cm (to 5 s.f.)

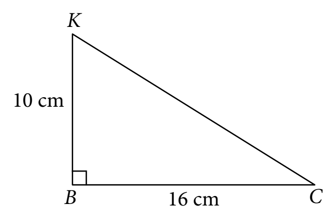
In $\triangle BDS$, $\angle BDS = 90°$.

$$\tan \angle SBD = \frac{16}{22.627}$$

$$\angle SBD = \tan^{-1} \frac{16}{22.627}$$

$= \mathbf{35.3°}$ (to 1 d.p.)

(iii) The required angle is $\angle CKD$.



In $\triangle BCK$, $\angle CBK = 90°$.

Using Pythagoras' Theorem,

$CK^2 = BC^2 + BK^2$

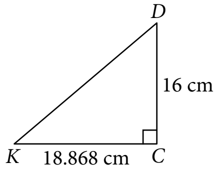
$= 16^2 + 10^2$

$= 256 + 100$

$= 356$

$CK = \sqrt{356}$

$= 18.868$ cm (to 5 s.f.)



In $\triangle CDK$, $\angle DCK = 90°$.

$$\tan \angle CKD = \frac{16}{18.868}$$

$$\angle CKD = \tan^{-1} \frac{16}{18.868}$$

$$= \mathbf{40.3°} \text{ (to 1 d.p.)}$$

Practise Now 9

(i) In $\triangle ABC$,

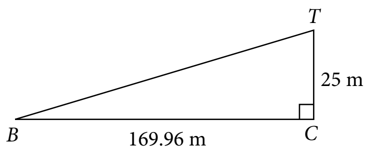
$$\angle ACB = 180° - 105° - 32° \ (\angle \text{ sum of } \triangle)$$
$$= 43°$$

Using Sine Rule,

$$\frac{BC}{\sin 105°} = \frac{120}{\sin 43°}$$

$$BC = \frac{120 \sin 105°}{\sin 43°}$$
$$= 169.96 \text{ m (to 5 s.f.)}$$
$$= \mathbf{170\ m} \text{ (to 3 s.f.)}$$

(ii)



In $\triangle BCT$, $\angle BCT = 90°$.

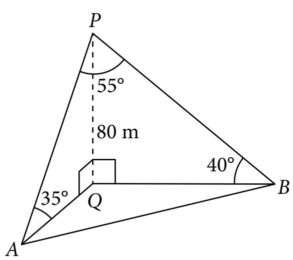
$$\tan \angle CBT = \frac{25}{169.96}$$

$$\angle CBT = \tan^{-1} \frac{25}{169.96}$$
$$= 8.4° \text{ (to 1 d.p.)}$$

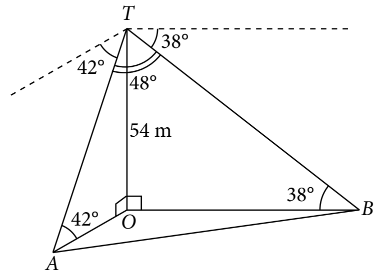
$\therefore$ the angle of elevation of T from B is **8.4°**.

Thinking Time (Page 242)

No, Albert's diagram is not correct because the base of P (ground level) and the points A and B do not form a straight line. This is evident from the fact that $55° + 35° + 40° = 130° \neq 180°$. The correct diagram should be:



Practise Now 10



In $\triangle AOT$, $\angle AOT = 90°$.

$$\sin 42° = \frac{54}{AT}$$

$$AT = \frac{54}{\sin 42°}$$
$$= 80.702 \text{ m (to 5 s.f.)}$$

In $\triangle BOT$, $\angle BOT = 90°$.

$$\sin 38° = \frac{54}{BT}$$

$$BT = \frac{54}{\sin 38°}$$
$$= 87.711 \text{ m (to 5 s.f.)}$$

Using Cosine Rule,

$$AB^2 = AT^2 + BT^2 - 2 \times AT \times BT \times \cos \angle ATB$$
$$= 80.702^2 + 87.711^2 - 2 \times 80.702 \times 87.711 \times \cos 48°$$
$$= 4733.2 \text{ (to 5 s.f.)}$$
$$AB = \sqrt{4733.2}$$
$$= \mathbf{68.8\ m} \text{ (to 3 s.f.)}$$

Exercise 7C

1. **(a)** $\angle$***RPV***
(b) $\angle$***RQV***
(c) $\angle$***QSU***
(d) $\angle$***PVS***
(e) $\angle$***TVW***
(f) $\angle$***TQW***

2. **(a)** In $\triangle QRV$, using Pythagoras' Theorem,

$$QV^2 = QR^2 + RV^2$$
$$= 5^2 + 5^2$$
$$= 50$$
$$QV = \sqrt{50}$$
$$= \mathbf{7.07\ cm} \text{ (to 3 s.f.)}$$

(b) The required angle is $\angle RQV$.
In $\triangle QRV$, $\angle QRV = 90°$.

$$\tan \angle RQV = \frac{5}{5}$$
$$= 1$$
$$\angle RQV = \tan^{-1} 1$$
$$= \mathbf{45°}$$

(c) In $\triangle PQV$, using Pythagoras' Theorem,

$$PV^2 = PQ^2 + QV^2$$
$$= 5^2 + 50$$
$$= 75$$
$$PV = \sqrt{75}$$
$$= \mathbf{8.66\ cm} \text{ (to 3 s.f.)}$$

(d) The required angle is $\angle RPV$.
In $\triangle PRV$, $\angle PRV = 90°$.

$$\sin \angle RPV = \frac{5}{\sqrt{75}}$$

$$\angle RPV = \sin^{-1} \frac{5}{\sqrt{75}}$$

$$= \mathbf{35.3°} \text{ (to 1 d.p.)}$$

3. (i) In $\triangle ABC$, using Pythagoras' Theorem,

$$AC^2 = BC^2 + AB^2$$
$$= 6^2 + 7^2$$
$$= 85$$
$$AC = \sqrt{85}$$
$$= \mathbf{9.22\ cm} \text{ (to 3 s.f.)}$$

(ii) In $\triangle ACP$, $\angle ACP = 90°$.
Using Pythagoras' Theorem,

$$PC^2 = AP^2 - AC^2$$
$$= 11^2 - 85$$
$$= 36$$
$$PC = \sqrt{36}$$
$$= \mathbf{6\ cm}$$

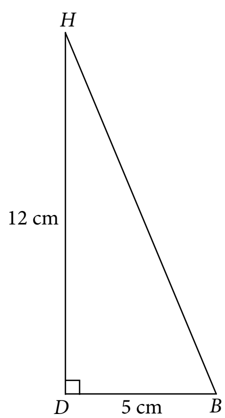
(iii) $\sin \angle PAC = \frac{6}{11}$

$$\angle PAC = \sin^{-1} \frac{6}{11}$$
$$= \mathbf{33.1°} \text{ (to 1 d.p.)}$$

(iv) In $\triangle BCP$,

$$\tan \angle PBC = \frac{6}{6}$$
$$= 1$$
$$\angle PBC = \tan^{-1} 1$$
$$= \mathbf{45°}$$

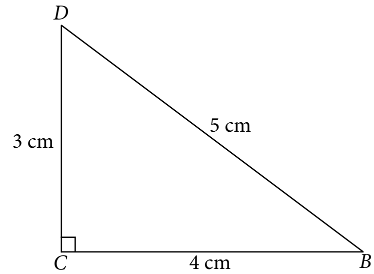
4. (i)



In $\triangle BDH$, $\angle BDH = 90°$.
Using Pythagoras' Theorem,

$$BH^2 = BD^2 + DH^2$$
$$= 5^2 + 12^2$$
$$= 25 + 144$$
$$= 169$$
$$BH = \sqrt{169}$$
$$= \mathbf{13\ cm}$$

(ii) The required angle is $\angle BDC$.

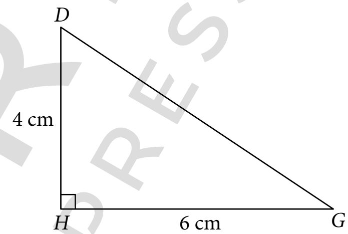


$$\tan \angle BDC = \frac{4}{3}$$
$$\angle BDC = \tan^{-1} \frac{4}{3}$$
$$= \mathbf{53.1°} \text{ (to 1 d.p.)}$$

(iii) $\tan \angle HBD = \frac{12}{5}$

$$\angle HBD = \tan^{-1} \frac{12}{5}$$
$$= \mathbf{67.4°} \text{ (to 1 d.p.)}$$

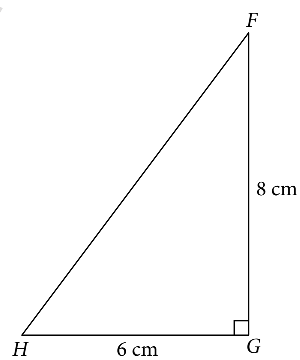
5. (i)



In $\triangle DGH$, $\angle DHG = 90°$.

$$\tan \angle DGH = \frac{4}{6}$$
$$\angle DGH = \tan^{-1} \frac{4}{6}$$
$$= \mathbf{33.7°} \text{ (to 1 d.p.)}$$

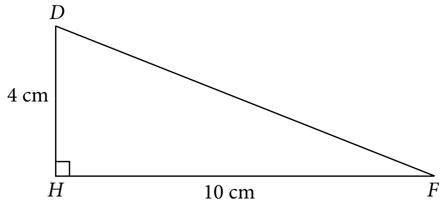
(ii)



In $\triangle FGH$, $\angle FGH = 90°$.
Using Pythagoras' Theorem,

$$HF^2 = GH^2 + FG^2$$
$$= 6^2 + 8^2$$
$$= 36 + 64$$
$$= 100$$
$$HF = \sqrt{100}$$
$$= \mathbf{10\ cm}$$

(iii) The required angle is $\angle DFH$.

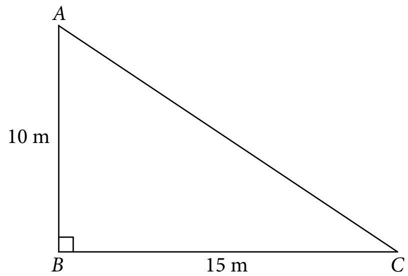


$\tan \angle DFH = \dfrac{4}{10}$

$\angle DFH = \tan^{-1} \dfrac{4}{10}$

$= \mathbf{21.8°}$ (to 1 d.p.)

6. (i)



In $\triangle ABC$, $\angle ABC = 90°$.

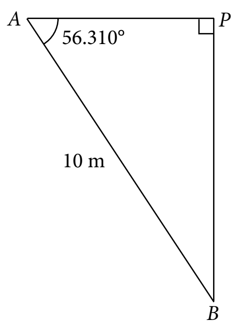
$\tan \angle BAC = \dfrac{15}{10}$

$\angle BAC = \tan^{-1} \dfrac{15}{10}$

$= 56.310°$ (to 3 d.p.)

$= \mathbf{56.3°}$ (to 1 d.p.)

(ii)



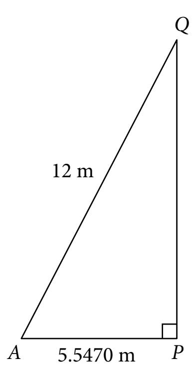
$\cos 56.310° = \dfrac{AP}{10}$

$AP = 10 \cos 56.310°$

$= 5.5470$ m (to 5 s.f.)

$=$ **5.55 m** (to 3 s.f.)

(iii)



In $\triangle AOP$, $\angle APO = 90°$.

$OP^2 = AO^2 - AP^2$

$= 12^2 - 5.5470^2$

$= 113.23$ (to 5 s.f.)

$OP = \sqrt{113.23}$

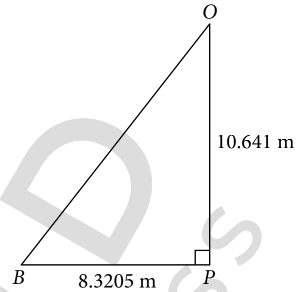
$= 10.641$ m (to 5 s.f.)

$=$ **10.6 m** (to 3 s.f.)

(iv) $\sin 56.310° = \dfrac{BP}{10}$

$BP = 10 \sin 56.310°$

$= 8.3205$ m (to 5 s.f.)



In $\triangle BOP$, $\angle BPO = 90°$.

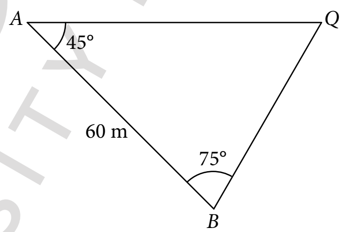
$\tan \angle OBP = \dfrac{10.641}{8.3205}$

$\angle OBP = \tan^{-1} \dfrac{10.641}{8.3205}$

$= 52.0°$ (to 1 d.p.)

$\therefore$ the angle of elevation of O from B is **52.0°**.

7.



$\angle AQB = 180° - 45° - 75°$ ($\angle$ sum of $\triangle$)

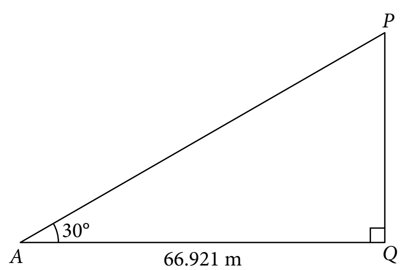
$= 60°$

Using Sine Rule,

$\dfrac{AQ}{\sin 75°} = \dfrac{60}{\sin 60°}$

$AQ = \dfrac{60 \sin 75°}{\sin 60°}$

$= 66.921$ m (to 5 s.f.)



In $\triangle APQ$, $\angle AQP = 90°$.

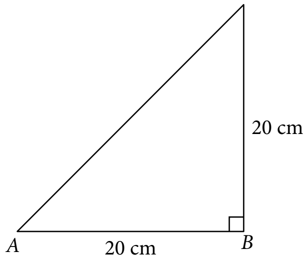
$\tan 30° = \dfrac{PQ}{66.921}$

$PQ = 66.921 \tan 30°$

$= 38.6$ m (to 3 s.f.)

$\therefore$ the height of the tower is **38.6 m**.

8. **(i)**

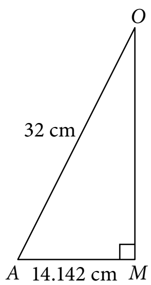


In $\triangle ABC$, $\angle ABC = 90°$.
Using Pythagoras' Theorem,

$AC^2 = AB^2 + BC^2$
$= 20^2 + 20^2$
$= 400 + 400$
$= 800$
$AC = \sqrt{800}$
$= 28.284$ cm (to 5 s.f.)

$AM = \dfrac{28.284}{2}$
$= 14.142$ cm
$=$ **14.1 cm** (to 3 s.f.)

(ii)



Using Pythagoras' Theorem,

$MO^2 = OA^2 - AM^2$
$= 32^2 - 14.142^2$
$= 824.00$ (to 5 s.f.)
$MO = \sqrt{824.00}$
$= 28.7$ cm (to 3 s.f.)

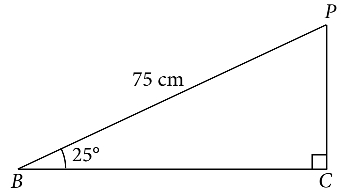
$\therefore$ the height of the pyramid is **28.7 cm**.

(iii) The required angle is $\angle OAM$.

$\cos \angle OAM = \dfrac{14.142}{32}$

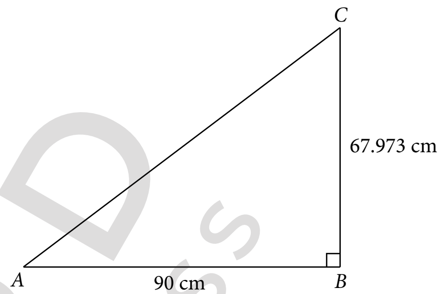
$\angle OAM = \cos^{-1} \dfrac{14.142}{32}$
$=$ **63.8°** (to 1 d.p.)

9. **(i)**



In $\triangle BCP$,

$\cos 25° = \dfrac{BC}{75}$
$BC = 75 \cos 25°$
$= 67.973$ cm (to 5 s.f.)

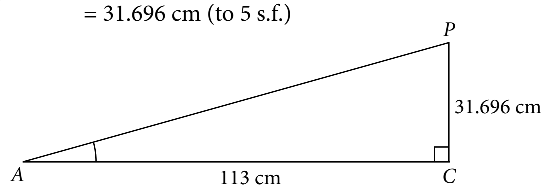


In $\triangle ABC$, $\angle ABC = 90°$.
Using Pythagoras' Theorem,

$AC^2 = AB^2 + BC^2$
$= 90^2 + 67.973^2$
$= 12\,720$ (to 5 s.f.)
$AC = \sqrt{12\,720}$
$= 112.78$ cm (to 5 s.f.)
$=$ **113 cm** (to 3 s.f.)

(ii) In $\triangle BCP$,

$\sin 25° = \dfrac{CP}{75}$
$CP = 75 \sin 25°$
$= 31.696$ cm (to 5 s.f.)



In $\triangle ACP$, $\angle ACP = 90°$.

$\tan \angle PAC = \dfrac{31.696}{113}$

$\angle PAC = \tan^{-1} \dfrac{31.696}{113}$
$=$ **15.7°** (to 1 d.p.)

(iii) The required angle is $\angle PBQ$.
In $\triangle BPQ$, $\angle BPQ = 90°$.

$\tan \angle PBQ = \dfrac{90}{75}$

$\angle PBQ = \tan^{-1} \dfrac{90}{75}$
$=$ **50.2°** (to 1 d.p.)

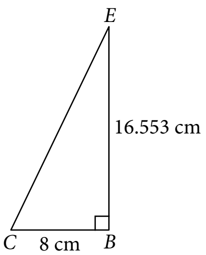
10. (i) The required angle is $\angle BEC$.

In $\triangle BEF$, $\angle BFE = 90°$.

Using Pythagoras' Theorem,

$BE^2 = EF^2 + BF^2$
$= 15^2 + 7^2$
$= 274$
$BE = \sqrt{274}$
$= 16.553$ cm (to 5 s.f.)

In $\triangle BEC$, $\angle CBE = 90°$.

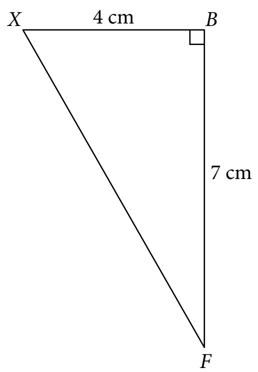


$\tan \angle BEC = \dfrac{8}{16.553}$

$\angle BEC = \tan^{-1} \dfrac{8}{16.553}$

$= \mathbf{25.8°}$ (to 1 d.p.)

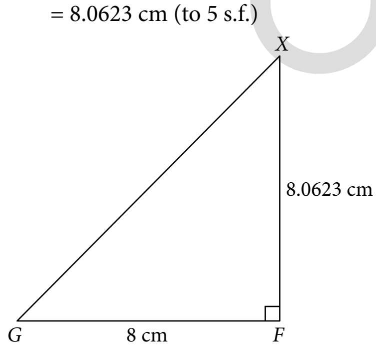
(ii)



In $\triangle BFX$, $\angle FBX = 90°$.

Using Pythagoras' Theorem,

$FX^2 = XB^2 + BF^2$
$= 4^2 + 7^2$
$= 16 + 49$
$= 65$
$FX = \sqrt{65}$
$= 8.0623$ cm (to 5 s.f.)



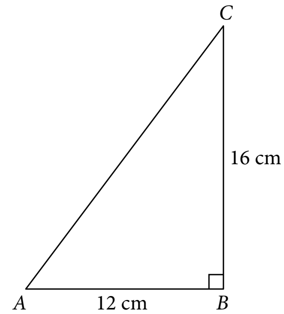
In $\triangle FGX$, $\angle GFX = 90°$.

$\tan \angle GXF = \dfrac{8}{8.0623}$

$\angle GXF = \tan^{-1} \dfrac{8}{8.0623}$

$= \mathbf{44.8°}$ (to 1 d.p.)

11. (i)

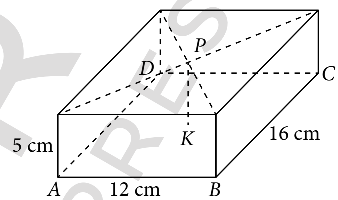


In $\triangle ABC$, $\angle ABC = 90°$.

Using Pythagoras' Theorem,

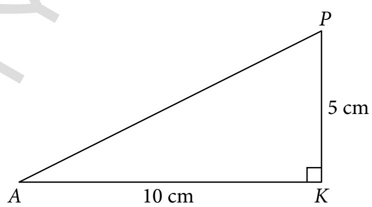
$AC^2 = AB^2 + BC^2$
$= 12^2 + 16^2$
$= 144 + 256$
$= 400$
$AC = \sqrt{400}$
$= 20$ cm

Let K be the point directly below P.



$AK = \dfrac{20}{2}$

$= 10$ cm

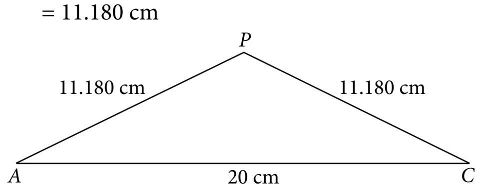


In $\triangle AKP$, $\angle AKP = 90°$.

Using Pythagoras' Theorem,

$AP^2 = AK^2 + KP^2$
$= 10^2 + 5^2$
$= 100 + 25$
$= 125$
$AP = \sqrt{125}$
$= 11.180$ cm (to 5 s.f.)

$CP = AP$
$= 11.180$ cm



Using Cosine Rule,

$\cos \angle PAC = \dfrac{11.180^2 + 20^2 - 11.180^2}{2 \times 11.180 \times 20}$

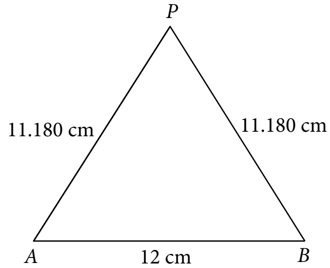
$= 0.894\ 45$ (to 5 s.f.)

$\angle PAC = \cos^{-1} 0.894\ 45$

$= \mathbf{26.6°}$ (to 1 d.p.)

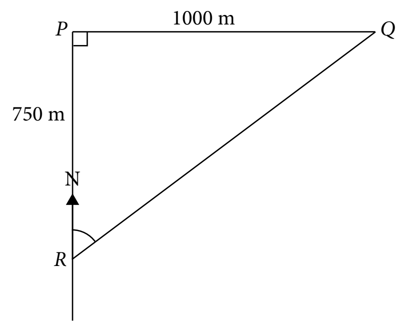
(ii) $BP = AP$

$= 11.180$ cm



$$\cos \angle PAB = \frac{11.180^2 + 12^2 - 11.180^2}{2 \times 11.180 \times 12}$$
$$= 0.536\,67 \text{ (to 5 s.f.)}$$
$$\angle PAB = \cos^{-1} 0.536\,67$$
$$= \mathbf{57.5°} \text{ (to 1 d.p.)}$$

12. (i)

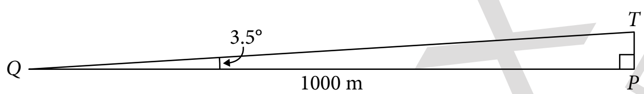


In $\triangle PQR$, $\angle QPR = 90°$.

$$\tan \angle PRQ = \frac{1000}{750}$$
$$= \frac{4}{3}$$
$$\angle PRQ = \tan^{-1} \frac{4}{3}$$
$$= 53.1° \text{ (to 1 d.p.)}$$

$\therefore$ the bearing of Q from R is **053.1°**.

(ii)

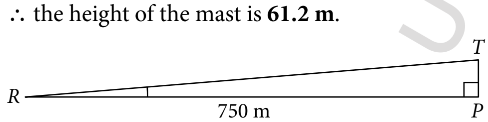


In $\triangle PQT$, $\angle QPT = 90°$.

$$\tan 3.5° = \frac{PT}{1000}$$
$$PT = 1000 \tan 3.5°$$
$$= 61.163 \text{ m (to 5 s.f.)}$$
$$= 61.2 \text{ m (to 3 s.f.)}$$

$\therefore$ the height of the mast is **61.2 m**.

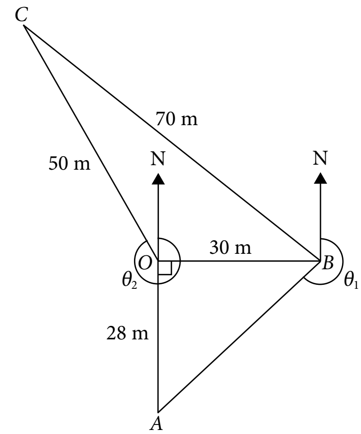
(iii)



$$\tan \angle PRT = \frac{61.163}{750}$$
$$\angle PRT = \tan^{-1} \frac{61.163}{750}$$
$$= 4.7° \text{ (to 1 d.p.)}$$

$\therefore$ the angle of elevation of T from R is **4.7°**.

13.



(a) (i) In $\triangle ABO$,

$$\tan \angle ABO = \frac{28}{30}$$
$$\angle ABO = \tan^{-1} \frac{28}{30}$$
$$= 43.025° \text{ (to 3 d.p.)}$$

The bearing of A from B is given by the reflex angle θ_1, which is $(270° - 43.025°)$.

$$\theta_1 = 270° - 43.025°$$
$$= 227.0° \text{ (to 1 d.p.)}$$

$\therefore$ the bearing of A from B is **227.0°**.

(ii) Using Cosine Rule,

$$\cos \angle COB = \frac{50^2 + 30^2 - 70^2}{2 \times 50 \times 30}$$
$$= -\frac{1}{2}$$
$$\angle COB = \cos^{-1}\left(-\frac{1}{2}\right)$$
$$= \mathbf{120°}$$

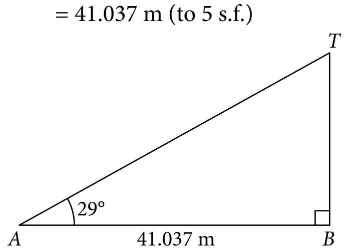
(iii) The bearing of C from O is given by the reflex angle θ_2, which is $[360° - (120° - 90°)]$.

$$\theta_2 = 360° - (120° - 90°)$$
$$= 360° - 30°$$
$$= 330°$$

$\therefore$ the bearing of C from O is **330°**.

(b) In $\triangle ABO$, using Pythagoras' Theorem,

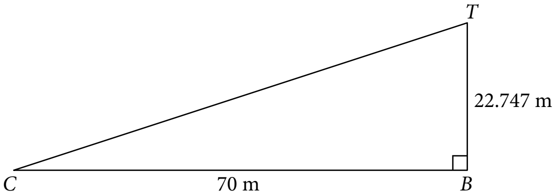
$$AB^2 = OA^2 + OB^2$$
$$= 28^2 + 30^2$$
$$= 784 + 900$$
$$= 1684$$
$$AB = \sqrt{1684}$$
$$= 41.037 \text{ m (to 5 s.f.)}$$



Let T be the point where the bird is.

In $\triangle ABT$, $\angle ABT = 90°$.

$$\tan 29° = \frac{BT}{41.037}$$
$$BT = 41.037 \tan 29°$$
$$= 22.747 \text{ m (to 5 s.f.)}$$



In $\triangle BCT$, $\angle CBT = 90°$.

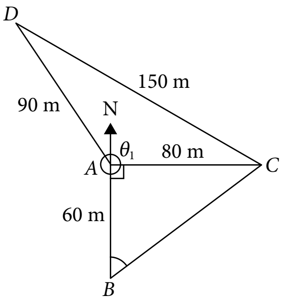
$$\tan \angle BCT = \frac{22.747}{70}$$

$$\angle BCT = \tan^{-1} \frac{22.747}{70}$$

$$= 18.0° \text{ (to 1 d.p.)}$$

$\therefore$ the angle of elevation of the bird from C is **18.0°**.

14. (a)



(i) In $\triangle ABC$,

$$\tan \angle ABC = \frac{80}{60} = \frac{4}{3}$$

$$\angle ABC = \tan^{-1} \frac{4}{3} = 53.1° \text{ (to 1 d.p.)}$$

$\therefore$ the bearing of C from B is **053.1°**.

(ii) Using Cosine Rule,

$$\cos \angle CAD = \frac{90^2 + 80^2 - 150^2}{2 \times 90 \times 80} = -\frac{5}{9}$$

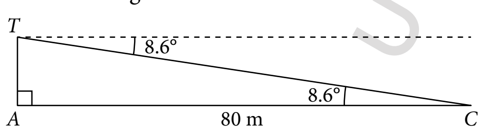
$$\angle CAD = \cos^{-1}\left(-\frac{5}{9}\right) = 123.749° \text{ (to 3 d.p.)}$$

The bearing of D from A is given by the reflex angle θ_1, which is [360° – (123.749° – 90°)].

$$\theta_1 = 360° - (123.749° - 90°)$$
$$= 360° - 33.749°$$
$$= 326.3° \text{ (to 1 d.p.)}$$

$\therefore$ the bearing of D from A is **326.3°**.

(b) (i)



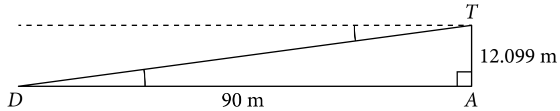
In $\triangle ACT$, $\angle CAT = 90°$.

$$\tan 8.6° = \frac{AT}{80}$$

$$AT = 80 \tan 8.6°$$
$$= 12.099 \text{ m (to 5 s.f.)}$$
$$= 12.1 \text{ m (to 3 s.f.)}$$

$\therefore$ the height of the cliff is **12.1 m**.

(ii)



In $\triangle ADT$, $\angle DAT = 90°$.

$$\tan \angle ADT = \frac{12.099}{90}$$

$$\angle ADT = \tan^{-1} \frac{12.099}{90} = 7.7° \text{ (to 1 d.p.)}$$

$\therefore$ the angle of depression of D from T is **7.7°**.

15. (a) (i) $\angle ACB = 180° - 94° - 47°$ ($\angle$ sum of $\triangle$)

$$= 39°$$

Using Sine Rule,

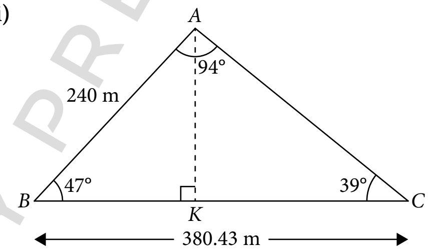
$$\frac{BC}{\sin 94°} = \frac{240}{\sin 39°}$$

$$BC = \frac{240 \sin 94°}{\sin 39°}$$
$$= 380.43 \text{ m (to 5 s.f.)}$$
$$= \mathbf{380\ m} \text{ (to 3 s.f.)}$$

(ii) Area of $\triangle ABC = \frac{1}{2} \times 240 \times 380.43 \times \sin 47°$

$$= \mathbf{33\,400\ m^2} \text{ (to 3 s.f.)}$$

(iii)



The shortest path corresponds to AK, where AK is perpendicular to BC.

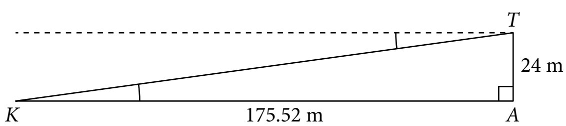
In $\triangle ABK$,

$$\sin 47° = \frac{AK}{240}$$

$$AK = 240 \sin 47°$$
$$= 175.52 \text{ m (to 5 s.f.)}$$
$$= \mathbf{176\ m} \text{ (to 3 s.f.)}$$

(b) The angle of depression of the boat from the top of the tree is greatest when the distance between the boat and A is the shortest.

Let T be the top of the tree.



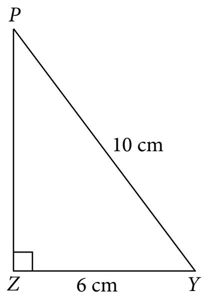
In $\triangle AKT$, $\angle KAT = 90°$.

$$\tan \angle AKT = \frac{24}{175.52}$$

$$\angle AKT = \tan^{-1} \frac{24}{175.52} = 7.8° \text{ (to 1 d.p.)}$$

$\therefore$ the greatest angle of depression of the boat from the top of the tree is **7.8°**.

16. (i)



In $\triangle PYZ$,

$\cos \angle PYZ = \frac{6}{10}$

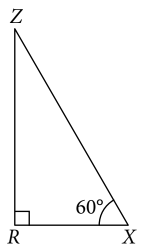
$\angle PYZ = \cos^{-1} \frac{6}{10}$

$= \mathbf{53.1°}$ (to 1 d.p.)

(ii) The required angle is $\angle RZX$.

Since R is the midpoint of XY, $\angle XRZ = 90°$.

Since $\triangle XYZ$ is an equilateral triangle, $\angle RXZ = 60°$.

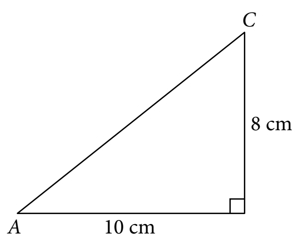


In $\triangle RXZ$,

$\angle RZX = 180° - 90° - 60°$ ($\angle$ sum of $\triangle$)

$= \mathbf{30°}$

17. (i)



Using Pythagoras' Theorem,

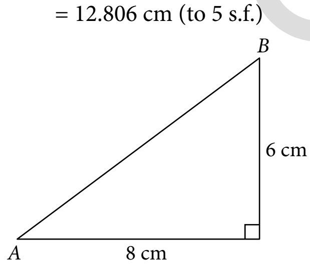
$AC^2 = 10^2 + 8^2$

$= 100 + 64$

$= 164$

$AC = \sqrt{164}$

$= 12.806$ cm (to 5 s.f.)



Using Pythagoras' Theorem,

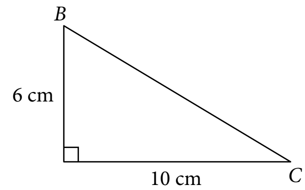
$AB^2 = 8^2 + 6^2$

$= 64 + 36$

$= 100$

$AB = \sqrt{100}$

$= 10$ cm



Using Pythagoras' Theorem,

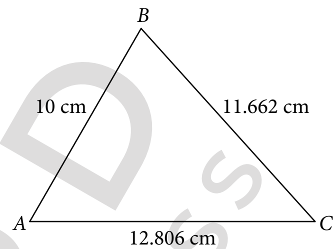
$BC^2 = 10^2 + 6^2$

$= 100 + 36$

$= 136$

$BC = \sqrt{136}$

$= 11.662$ cm (to 5 s.f.)



Using Cosine Rule,

$\cos \angle ABC = \frac{10^2 + 11.662^2 - 12.806^2}{2 \times 10 \times 11.662}$

$= 0.308\ 73$ (to 5 s.f.)

$\angle ABC = \cos^{-1} 0.308\ 73$

$= 72.017°$ (to 3 d.p.)

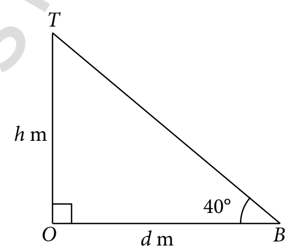
$= \mathbf{72.0°}$ (to 1 d.p.)

(ii) Area of surface to be coated with varnish

$= \frac{1}{2} \times 10 \times 11.662 \times \sin 72.017°$

$= \mathbf{55.5\ cm^2}$ (to 3 s.f.)

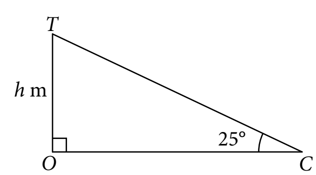
18. Let h m be the height of the tower and d m be the distance OB.



In $\triangle BOT$,

$\tan 40° = \frac{h}{d}$

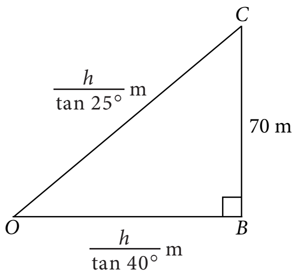
$d = \frac{h}{\tan 40°}$ — (1)



In $\triangle COT$, $\angle COT = 90°$.

$\tan 25° = \frac{h}{OC}$

$OC = \frac{h}{\tan 25°}$ m — (2)



In $\triangle BCO$,
Using Pythagoras' Theorem,

$$OC^2 = OB^2 + BC^2$$

$$\left(\frac{h}{\tan 25°}\right)^2 = \left(\frac{h}{\tan 40°}\right)^2 + 70^2$$

$$4.5989h^2 \approx 1.4203h^2 + 4900$$

$$3.1786h^2 = 4900$$

$$h^2 = \frac{4900}{3.1786}$$

$$h = \sqrt{\frac{4900}{3.1786}}$$

$$= 39.263 \text{ (to 5 s.f.)}$$

$$= 39.3 \text{ (to 3 s.f.)}$$

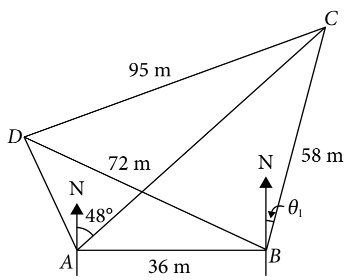
Substitute $h = 39.263$ into (1):

$$d = \frac{39.263}{\tan 40°}$$

$$= 46.8 \text{ (to 3 s.f.)}$$

$\therefore$ the height of the tower is **39.3 m** and the distance OB is **46.8 m**.

19. **(i)**



$$\angle CAB = 90° - 48°$$

$$= 42°$$

Using Sine Rule,

$$\frac{\sin \angle ACB}{36} = \frac{\sin 42°}{58}$$

$$\sin \angle ACB = \frac{36 \sin 42°}{58}$$

$$= 0.415\ 32 \text{ (to 5 s.f.)}$$

$$\angle ACB = \sin^{-1} 0.415\ 32$$

$$= 24.539° \text{ (to 3 d.p.)}$$

$$\angle ABC = 180° - 42° - 24.539° \quad (\angle \text{ sum of } \triangle)$$

$$= 113.461°$$

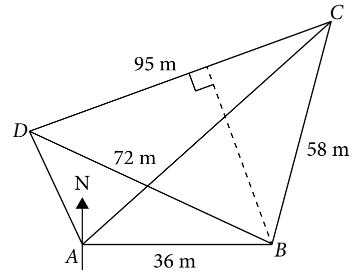
The bearing of C from B is given by the acute angle θ_1, which is $(113.461° - 90°)$.

$$\theta_1 = 113.461° - 90°$$

$$= \mathbf{23.5°} \text{ (to 1 d.p.)}$$

$\therefore$ the bearing of C from B is **023.5°**.

(ii) The angle of depression from the top of the tower is the greatest when it is closest to the tower.



The shortest distance corresponds to BP, where BP is perpendicular to CD.
Using Cosine Rule,

$$\cos \angle CBD = \frac{58^2 + 72^2 - 95^2}{2 \times 58 \times 72}$$

$$= -0.057\ 112 \text{ (to 5 s.f.)}$$

$$\angle CBD = \cos^{-1}(-0.057\ 112)$$

$$= 93.274° \text{ (to 3 d.p.)}$$

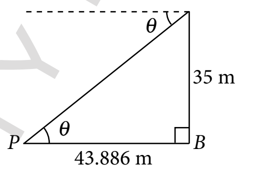
$$\text{Area of } \triangle BCD = \frac{1}{2} \times BD \times BC \times \sin \angle CBD$$

$$\frac{1}{2} \times CD \times BP = \frac{1}{2} \times BD \times BC \times \sin \angle CBD$$

$$\frac{1}{2} \times 95 \times BP = \frac{1}{2} \times 72 \times 58 \times \sin 93.274°$$

$$BP = \frac{4176 \sin 93.274°}{95}$$

$$= 43.886 \text{ m (to 5 s.f.)}$$



$$\tan \theta = \frac{35}{43.886}$$

$$\theta = \tan^{-1} \frac{35}{43.886}$$

$$= 38.6° \text{ (to 1 d.p.)}$$

$\therefore$ the angle of depression of P from the top of the tower is **38.6°**.

Chapter 8 Arc Length and Sector Area

TEACHING NOTES

Suggested Approach

In this chapter, students will be introduced to circles and learn to calculate the arc length and area of the sector of a circle. Teachers may begin the chapter by asking students to identify the different parts of a circle using real-life examples of arcs, sectors and segments of a circle.

Students are expected to know how to apply what they have learnt in trigonometry when solving problems involving arc lengths and sector areas.

Section 8.1: Length of arc

Teachers may begin the chapter by showing students a circle with centre *O* and highlight to students the minor arc, major arc, minor sector, major sector, minor segment and major segment of a circle (see Class Discussion: Understanding the parts of a circle). Once students are familiar with identifying these parts of a circle, teachers can proceed to guide students to derive the formula for the length of an arc of a circle (see Investigation: Discovering how to calculate arc length).

Section 8.2: Area of sector

Teachers can ask students to discover the formula for the area of sector on their own (see Investigation: Discovering how to calculate sector area). When finding the area of a shaded region involving sectors of circles, teachers may wish to take note that some students may need guidance. Teachers can suggest to students that when solving such problems, additional line(s) may need to be drawn on the figures to help students better visualise and find the unknowns.

Introductory Problem

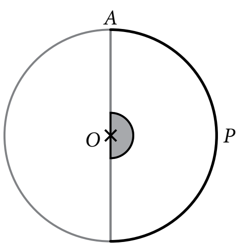
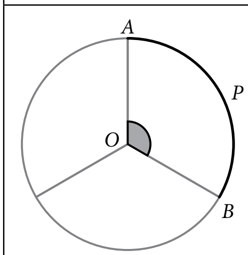
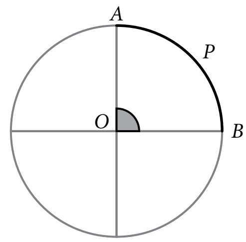
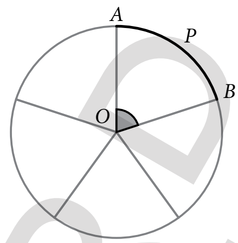
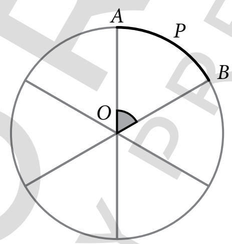
The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 1).

8.1 Length of arc

Class Discussion (Understanding the parts of a circle)

1. The arc *PRQ* is a minor arc because the length of the arc is less than half the circumference.
2. **Yes**, a chord can pass through the centre of a circle. If it passes through, it is called a **diameter**.
3. **Yes**, the diameter is the longest chord because it is the longest distance between two points on the circumference.
4. **Neither**. The arc *PRQ* will just be an arc of a semicircle.
5. The segment *PRQ* is a minor segment because it is smaller than a semicircle.
6. **Yes**, a segment *PRQ* can be a semicircle when the chord *PQ* is a diameter of the circle.
7. The sector *OPRQ* is a minor sector because it is smaller than a semicircle.
8. **Neither**. The sector *OPRQ* will just be a semicircle.
9. **Yes**, a sector *OPRQ* can be a semicircle if *POQ* is a straight line, i.e. a diameter of the circle.

Investigation (Discovering how to calculate arc length)

Diagram	No. of equal parts	$\frac{\text{length of arc } APB}{\text{circumference}}$	$\frac{\angle AOB}{360°}$
	2	$\frac{1}{2}$	$\frac{1}{2}$
	3	$\frac{1}{3}$	$\frac{1}{3}$
	4	$\frac{1}{4}$	$\frac{1}{4}$
	5	$\frac{1}{5}$	$\frac{1}{5}$
	6	$\frac{1}{6}$	$\frac{1}{6}$

1. The values in the last two columns are **equal**.
2. $\frac{\text{length of arc } APB}{\text{circumference}} = \frac{1}{n}$ and $\frac{\angle AOB}{360°} = \frac{1}{n}$
3. **Length of arc** $APB = \frac{\angle AOB}{360°} \times$ **circumference**
4. Since length of arc $APB = \frac{\angle AOB}{360°} \times \text{circumference}$

   $= \frac{\text{circumference}}{360°} \times \angle AOB$,

   where $\frac{\text{circumference}}{360°}$ is a non-zero constant, then the length of arc *APB* is **directly proportional** to $\angle AOB$.

Practise Now 1

1. **(i)** Length of major arc $AYB = \frac{x°}{360°} \times 2\pi r$

   $= \frac{228°}{360°} \times 2\pi \times 25$

   $=$ **99.5 cm** (to 3 s.f.)

   (ii) Method 1:

   $\angle AOB = 360° - 228°$ (∠s at a point)

   $= 132°$

   Perimeter of minor sector *OAXB*

   = length of arc *AXB* + *OA* + *OB*

   $= \frac{132°}{360°} \times 2\pi \times 25 + 25 + 25$

   = **108 cm** (to 3 s.f.)

Method 2:

Length of arc AXB = circumference – length of arc AYB

$= 2\pi \times 25 - 99.484$

$= 57.596$ cm (to 5 s.f.)

Perimeter of minor sector $OAXB$

= length of arc $AXB + OA + OB$

$= 57.596 + 25 + 25$

= **108 cm** (to 3 s.f.)

2. Reflex $\angle POQ = 360° - 50°$ ($\angle$s at a point)

$= 310°$

Length of major arc PXQ = 36 cm

$$\frac{310°}{360°} \times 2\pi r = 36$$

$$r = \frac{36}{2\pi} \times \frac{360°}{310°}$$

= **6.65** (to 3 s.f.)

Introductory Problem Revisited

Length of arc of balcony $= \frac{135°}{360°} \times 2\pi \times 1.6$

= **3.77 m** (to 3 s.f.)

Practise Now 2

1. Length of arc $PR = \frac{x°}{360°} \times 2\pi r$

$= \frac{36.9°}{360°} \times 2\pi \times 8$

$= 1.64\pi$ m

$\tan 36.9° = \frac{\text{opp}}{\text{adj}}$

$= \frac{QR}{OR}$

$= \frac{QR}{8}$

$QR = 8 \tan 36.9°$

$= 6.0066$ m (to 5 s.f.)

$\cos 36.9° = \frac{\text{adj}}{\text{hyp}}$

$= \frac{OR}{OQ}$

$= \frac{8}{OQ}$

$OQ = \frac{8}{\cos 36.9°}$

$= 10.004$ m (to 5 s.f.)

$PQ = OQ - OP$

$= 10.004 - 8$

$= 2.004$ m

$\therefore$ perimeter of shaded region PQR = length of arc $PR + QR + PQ$

$= 1.64\pi + 6.0066 + 2.004$

= **13.2 m** (to 3 s.f.)

2. **(i)** $\sin 59° = \frac{\text{opp}}{\text{hyp}}$

$= \frac{BP}{OB}$

$= \frac{BP}{17}$

$BP = 17 \sin 59°$

$= 14.572$ cm (to 5 s.f.)

$AB = 2BP$

$= 2 \times 14.572$

= **29.1 cm** (to 3 s.f.)

(ii) Length of arc $ARB = \frac{x°}{360°} \times 2\pi r$

$= \frac{1}{2} \times 2\pi \times 14.572$

$= 14.572\pi$ cm

Length of arc $AQB = \frac{x°}{360°} \times 2\pi r$

$= \frac{59° \times 2}{360°} \times 2\pi \times 17$

$= \frac{1003\pi}{90}$ cm

$\therefore$ perimeter of shaded region

= length of arc ARB + length of arc AQB

$= 14.572\pi + \frac{1003\pi}{90}$

= **80.8 cm** (to 3 s.f.)

Exercise 8A

1. **(a)** Length of arc $AXB = \frac{82°}{360°} \times 2\pi \times 8$

= **11.4 cm** (to 3 s.f.)

(b) Length of arc $AXB = \frac{134°}{360°} \times 2\pi \times 14$

= **32.7 cm** (to 3 s.f.)

(c) Length of arc $AXB = \frac{214°}{360°} \times 2\pi \times 17$

= **63.5 cm** (to 3 s.f.)

(d) Reflex $\angle AOB = 360° - 46°$ ($\angle$s at a point)

$= 314°$

Length of arc $AXB = \frac{314°}{360°} \times 2\pi \times 9.8$

= **53.7 cm** (to 3 s.f.)

2. **(a)** **(i)** Length of minor arc $AXB = \frac{76°}{360°} \times 2\pi \times 9$

= **11.9 cm** (to 3 s.f.)

(ii) Reflex $\angle AOB = 360° - 76°$ ($\angle$s at a point)

$= 284°$

Perimeter of major sector $OAYB$

= length of major arc $AYB + OA + OB$

$= \frac{284°}{360°} \times 2\pi \times 9 + 9 + 9$

= **62.6 cm** (to 3 s.f.)

(b) **(i)** Length of minor arc $AXB = \frac{112°}{360°} \times 2\pi \times 16$

= **31.3 cm** (to 3 s.f.)

(ii) Reflex $\angle AOB = 360° - 112°$ ($\angle$s at a point)

$= 248°$

Perimeter of major sector $OAYB$

= length of major arc $AYB + OA + OB$

$= \frac{248°}{360°} \times 2\pi \times 16 + 16 + 16$

= **101 cm** (to 3 s.f.)

(c) **(i)** $\angle AOB = 360° - 215°$ ($\angle$s at a point)

$= 145°$

Length of minor arc $AXB = \frac{145°}{360°} \times 2\pi \times 17.6$

= **44.5 cm** (to 3 s.f.)

(ii) Perimeter of major sector $OAYB$

= length of major arc $AYB + OA + OB$

$= \frac{215°}{360°} \times 2\pi \times 17.6 + 17.6 + 17.6$

= **101 cm** (to 3 s.f.)

3. **(a)** Let the radius of the circle be r cm.
Length of minor arc = 26.53 cm

$$\frac{95°}{360°} \times 2\pi r = 26.53$$
$$r = \frac{26.53}{2\pi} \times \frac{360°}{95°}$$
$$= 16.0 \text{ (to 3 s.f.)}$$

∴ the radius is **16.0 cm**.

(b) Let the radius of the circle be r cm.
Length of major arc = 104.6 cm

$$\frac{214°}{360°} \times 2\pi r = 104.6$$
$$r = \frac{104.6}{2\pi} \times \frac{360°}{214°}$$
$$= 28.0 \text{ (to 3 s.f.)}$$

∴ the radius is **28.0 cm**.

4. **(a)** Let the angle subtended be $x°$.
Length of arc = 12 m

$$\frac{x°}{360°} \times 2\pi \times 14 = 12$$
$$x° = \frac{12 \times 360°}{2\pi \times 14}$$
$$= \mathbf{49°} \text{ (to the nearest degree)}$$

(b) Let the angle subtended be $x°$.
Length of arc = 19.5 m

$$\frac{x°}{360°} \times 2\pi \times 14 = 19.5$$
$$x° = \frac{19.5 \times 360°}{2\pi \times 14}$$
$$= \mathbf{80°} \text{ (to the nearest degree)}$$

(c) Let the angle subtended be $x°$.
Length of arc = 64.2 m

$$\frac{x°}{360°} \times 2\pi \times 14 = 64.2$$
$$x° = \frac{64.2 \times 360°}{2\pi \times 14}$$
$$= \mathbf{263°} \text{ (to the nearest degree)}$$

(d) Let the angle subtended be $x°$.
Length of arc = 84.6 m

$$\frac{x°}{360°} \times 2\pi \times 14 = 84.6$$
$$x° = \frac{84.6 \times 360°}{2\pi \times 14}$$
$$= \mathbf{346°} \text{ (to the nearest degree)}$$

5. Distance travelled by the tip of the hour hand

$$= \frac{45°}{360°} \times 2\pi \times 1.5$$
$$= \mathbf{1.18 \text{ m}} \text{ (to 3 s.f.)}$$

6. Let the angle subtended be $x°$.
Length of wire = 32 cm

$$\frac{x°}{360°} \times 2\pi \times 6 + 6 + 6 = 32$$
$$\frac{\pi x°}{30} + 12 = 32$$
$$\frac{\pi x°}{30} = 20$$
$$x° = 20 \times \frac{30}{\pi}$$
$$= \mathbf{191.0°} \text{ (to 1 d.p.)}$$

7. **(a)** Let the radius of the circle be r cm.
Perimeter of minor sector = 77.91 cm
Length of minor arc $AB + OA + OB$ = 77.91

$$\frac{148°}{360°} \times 2\pi r + r + r = 77.91$$
$$r\left(\frac{37}{45}\pi + 2\right) = 77.91$$
$$r = \frac{77.91}{\frac{37}{45}\pi + 2}$$
$$= 17.0 \text{ (to 3 s.f.)}$$

∴ the radius of the circle is **17.0 cm**.

(b) Let the radius of the circle be r cm.
Reflex $\angle AOB = 360° - 44°$ ($\angle$s at a point)
$= 316°$
Perimeter of major sector = 278.1 cm
Length of major arc $AB + OA + OB$ = 278.1

$$\frac{316°}{360°} \times 2\pi r + r + r = 278.1$$
$$r\left(\frac{79}{45}\pi + 2\right) = 278.1$$
$$r = \frac{278.1}{\frac{79}{45}\pi + 2}$$
$$= 37.0 \text{ (to 3 s.f.)}$$

∴ the radius of the circle is **37.0 cm**.

8. $AP = BQ$
$= 17 - 8$
$= 9$ cm

Perimeter of shaded region
= length of arc AB + length of arc PQ + AP + BQ

$$= \frac{60°}{360°} \times 2\pi \times 8 + \frac{60°}{360°} \times 2\pi \times 17 + 9 + 9$$
$$= \mathbf{\left(18 + \frac{25}{3}\pi\right) \text{ cm}}$$

9. **(i)** Length of minor arc $= \frac{\angle AOB}{360°} \times$ circumference

Since the length of the minor arc is $\frac{7}{24}$ of the circumference of the circle,

$$\frac{\angle AOB}{360°} = \frac{7}{24}$$
$$\angle AOB = \frac{7}{24} \times 360°$$
$$= \mathbf{105°}$$

(ii) Radius of circle $= \frac{14}{2}$
$= 7$ cm

$$\text{Length of minor arc} = \frac{105°}{360°} \times 2\pi \times 7$$
$$= \mathbf{12.8 \text{ cm}} \text{ (to 3 s.f.)}$$

10. **(i)**
$$\tan \angle POA = \frac{\text{opp}}{\text{adj}}$$
$$= \frac{PA}{OA}$$
$$= \frac{14}{7.5}$$
$$\angle POA = \tan^{-1} \frac{14}{7.5}$$
$$= \mathbf{61.8°} \text{ (to 1 d.p.)}$$

(ii) Using Pythagoras' Theorem,

$OP^2 = 14^2 + 7.5^2$
$= 252.25$
$OP = \sqrt{252.25}$
$= 15.882$ cm (to 5 s.f.)
$BP = OP - OB$
$= 15.882 - 7.5$
$= 8.382$ cm

Perimeter of shaded region PBA
= length of arc $AB + BP + AP$
$= \frac{61.821°}{360°} \times 2\pi \times 7.5 + 8.382 + 14$
= **30.5 cm** (to 3 s.f.)

11. (i) Since $\triangle OPR \equiv \triangle OQR$,

$\angle POR = \angle QOR$
$= \frac{138°}{2}$
$= 69°$
$\tan 69° = \frac{\text{opp}}{\text{adj}}$
$= \frac{QR}{OQ}$
$= \frac{QR}{26}$
$QR = 26 \tan 69°$
$=$ **67.7 cm** (to 3 s.f.)

(ii) Perimeter of shaded region
= length of arc $PQ + PR + QR$
$= \frac{138°}{360°} \times 2\pi \times 26 + 67.732 + 67.732$
= **198 cm** (to 3 s.f.)

12. (i) $\cos 36° = \frac{\text{adj}}{\text{hyp}}$

$= \frac{QR}{OQ}$
$= \frac{QR}{35}$
$QR = 35 \cos 36°$
$= 28.316$ cm (to 5 s.f.)
$PQ = 2QR$
$= 2 \times 28.316$
= **56.6 cm** (to 3 s.f.)

(ii) $\angle QOR = 180° - 90° - 36°$ ($\angle$ sum of $\triangle$)

$= 54°$
$\angle POQ = 2 \times 54°$
$= 108°$

Perimeter of shaded region
= length of arc PBQ + length of arc PAQ
$= \frac{1}{2} \times 2\pi \times 28.316 + \frac{108°}{360°} \times 2\pi \times 35$
= **155 cm** (to 3 s.f.)

13. (i) Using Cosine Rule,

$\cos \angle AOB = \frac{13^2 + 13^2 - 22^2}{2 \times 13 \times 13}$
$= -\frac{73}{169}$
$\angle AOB = \cos^{-1}\left(-\frac{73}{169}\right)$
$= 115.6°$ (to 1 d.p.) (shown)

(ii) Perimeter of shaded region = length of arc + AB

$= \frac{115.592°}{360°} \times 2\pi \times 13 + 22$
= **48.2 cm** (to 3 s.f.)

14. (i) Using Cosine Rule,

$\cos \angle APB = \frac{16^2 + 9^2 - 9^2}{2 \times 16 \times 9}$
$= \frac{8}{9}$
$\angle APB = \cos^{-1} \frac{8}{9}$
= **27.3°** (to 1 d.p.)

(ii) $\angle AOB = 27.266° + 27.266°$ (ext. $\angle$ of $\triangle OAP$, base $\angle$s of isos. $\triangle$)

= **54.5°** (to 1 d.p.)

(iii) $\angle APD = 27.266° \times 2$

$= 54.532°$
$\angle AOD = 54.532° \times 2$
$= 109.064°$

Perimeter of shaded region
= length of arc ACD + length of arc ABD
$= \frac{54.532°}{360°} \times 2\pi \times 16 + \frac{109.064°}{360°} \times 2\pi \times 9$
= **32.4 cm** (to 3 s.f.)

15. Let the radius of the circle be r cm.

Using Cosine Rule,
$r^2 = r^2 + \left(\frac{15}{2}\sqrt{3}\right)^2 - 2r\left(\frac{15}{2}\sqrt{3}\right)\cos 30°$
$0 = 168.75 - 22.5r$
$22.5r = 168.75$
$r = 7.5$
$\therefore$ radius = 7.5 cm
$\angle AOB = 180° - 30° - 30°$ ($\angle$ sum of $\triangle$, base $\angle$s of isos. $\triangle$)
$= 120°$
Reflex $\angle AOB = 360° - 120°$ ($\angle$s at a point)
$= 240°$
Perimeter of shaded region = length of major arc $AB + AB$
$= \frac{240°}{360°} \times 2\pi \times 7.5 + \frac{15}{2}\sqrt{3}$
= **44.4 cm** (to 3 s.f.)

16. $\sin 36° = \dfrac{\text{opp}}{\text{hyp}}$

$= \dfrac{OP}{OT}$

$= \dfrac{OP}{14}$

$OP = 14 \sin 36°$

$= 8.2290$ cm (to 5 s.f.)

$\therefore$ radius of semicircle = 8.2290 cm

Radius of sector = 14 + 8.2290

= 22.2290 cm

$\cos 36° = \dfrac{\text{adj}}{\text{hyp}}$

$= \dfrac{PT}{OT}$

$= \dfrac{PT}{14}$

$PT = 14 \cos 36°$

$= 11.326$ cm (to 5 s.f.)

$PQ = QT - PT$

$= 22.2290 - 11.326$

$= 10.903$ cm

$\angle POR = 90° + 36°$ (ext. $\angle$ of $\triangle OPT$)

$= 126°$

Perimeter of shaded region

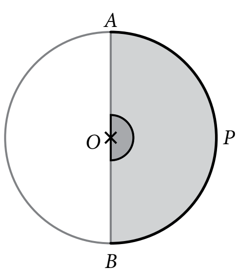
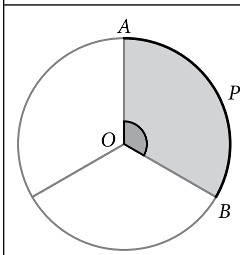
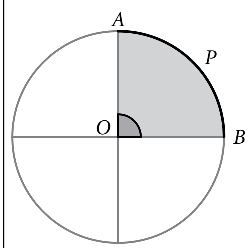
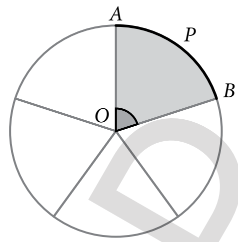
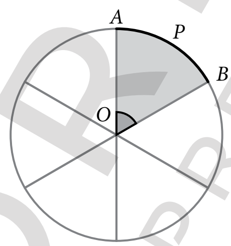
= length of arc QR + length of arc PR + PQ

$= \dfrac{36°}{360°} \times 2\pi \times 22.2290 + \dfrac{126°}{360°} \times 2\pi \times 8.2290 + 10.903$

= **43.0 cm** (to 3 s.f.)

8.2 Area of sector

Investigation (Discovering how to calculate sector area)

Diagram	No. of equal parts	$\dfrac{\text{area of sector } OAPB}{\text{area of circle}}$	$\dfrac{\angle AOB}{360°}$
	2	$\frac{1}{2}$	$\frac{1}{2}$
	3	$\frac{1}{3}$	$\frac{1}{3}$
	4	$\frac{1}{4}$	$\frac{1}{4}$
	5	$\frac{1}{5}$	$\frac{1}{5}$
	6	$\frac{1}{6}$	$\frac{1}{6}$

1. The values in the last two columns are **equal**.
2. $\dfrac{\text{area of sector } OAPB}{\text{area of circle}} = \dfrac{1}{n}$ and $\dfrac{\angle AOB}{360°} = \dfrac{1}{n}$
3. **Area of sector $OAPB$ = $\dfrac{\angle AOB}{360°}$ × area of circle**
4. Since area of sector $OAPB = \dfrac{\angle AOB}{360°} \times$ area of circle

   $= \dfrac{\text{area of circle}}{360°} \times \angle AOB$,

   where $\dfrac{\text{area of circle}}{360°}$ is a non-zero constant, then the area of sector $OAPB$ is **directly proportional** to $\angle AOB$.
5. **No**, because area of sector $OAPB = \dfrac{\angle AOB}{360°} \times$ area of circle

   $= \dfrac{\angle AOB}{360°} \times \pi r^2$, and length of arc $APB = \dfrac{\angle AOB}{360°} \times$ circumference

   $= \dfrac{\angle AOB}{360°} \times 2\pi r^2$, which means for a fixed $\angle AOB$, the area of the sector is directly proportional to r^2 but the length of the arc is directly proportional to r.

Practise Now 3

1. **(i)** Area of minor sector $= \dfrac{x°}{360°} \times \pi r^2$

$= \dfrac{126°}{360°} \times \pi \times 15^2$

= **247 cm²** (to 3 s.f.)

(ii) Method 1:

Reflex $\angle AOB = 360° - 126°$ ($\angle$s at a point)

$= 234°$

Area of major sector $= \frac{x°}{360°} \times \pi r^2$

$= \frac{234°}{360°} \times \pi \times 15^2$

$=$ **459 cm²** (to 3 s.f.)

Method 2:

Area of major sector = area of circle – area of minor sector

$= \pi \times 15^2 - 247.40$

$=$ **459 cm²** (to 3 s.f.)

2. **(i)** Length of minor arc $AQB = 52$ cm

$\frac{\angle AOB}{360°} \times 2\pi \times 26 = 52$

$\angle AOB = \frac{52 \times 360°}{2\pi \times 26}$

$= 114.6°$ (to 1 d.p.) (shown)

(ii) Reflex $\angle AOB = 360° - 114.59°$ ($\angle$s at a point)

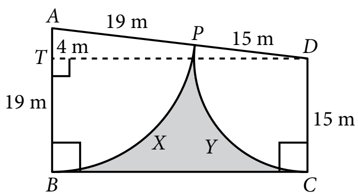
$= 245.41°$

Area of major sector $OAPB = \frac{245.41°}{360°} \times \pi \times 26^2$

$=$ **1450 cm²** (to 3 s.f.)

Practise Now 4

(i) Draw a line DT such that T lies on AB and DT is perpendicular to AB.



$AT = AB - TB$

$= 19 - 15$

$= 4$ m

$AD = AP + PD$

$= 19 + 15$

$= 34$ m

$\sin A\hat{D}T = \frac{AT}{AD}$

$= \frac{4}{34}$

$A\hat{D}T = \sin^{-1} \frac{4}{34}$

$= 6.756°$ (to 3 d.p.)

$A\hat{D}C = 90° + 6.756°$

$= 96.8°$ (to 1 d.p.) (shown)

(ii) Using Pythagoras' Theorem,

$DT^2 = 34^2 - 4^2$

$= 1140$

$DT = \sqrt{1140}$

$= 33.764$ m (to 5 s.f.)

$D\hat{A}T = 180° - 90° - 6.756°$ ($\angle$ sum of $\triangle$)

$= 83.244°$

Area of shaded region

= area of trapezium $ABCD$ – area of sector ABP

– area of sector CDP

$= \frac{1}{2} \times (19 + 15)(33.764) - \frac{83.244°}{360°} \times \pi \times 19^2 - \frac{96.756°}{360°} \times \pi \times 15^2$

$=$ **122 m²** (to 3 s.f.)

Practise Now 5

1. **(i)** Using Cosine Rule,

$\cos \angle HOL = \frac{13^2 + 13^2 - 10^2}{2 \times 13 \times 13}$

$= \frac{119}{169}$

$\angle HOL = \cos^{-1} \frac{119}{169}$

$= 45.2°$ (to 1 d.p.) (shown)

(ii) Area of minor segment HKL

= area of sector $HKLO$ – area of $\triangle HOL$

$= \frac{45.240°}{360°} \times \pi \times 13^2 - \frac{1}{2} \times 13 \times 13 \times \sin 45.240°$

$=$ **6.72 cm²** (to 3 s.f.)

2. Reflex $\angle AOB = 360° - 80°$ ($\angle$s at a point)

$= 280°$

Area of major sector $AQB = \frac{280°}{360°} \times \pi \times 7^2$

$= \frac{343}{9}\pi$ cm²

Area of $\triangle AOB = \frac{1}{2} \times 7 \times 7 \times \sin 80°$

$= 24.128$ cm² (to 5 s.f.)

Required percentage $= \frac{24.128}{\frac{343}{9}\pi + 24.128} \times 100\%$

$=$ **16.8%** (to 3 s.f.)

Exercise 8B

1. **(a)** Arc length $= \frac{72°}{360°} \times 2\pi \times 7$

$=$ **8.80 cm** (to 3 s.f.)

Area $= \frac{72°}{360°} \times \pi \times 7^2$

$=$ **30.8 cm²** (to 3 s.f.)

Perimeter $= 8.7965 + 7 + 7$

$=$ **22.8 cm** (to 3 s.f.)

(b) Arc length $= 136 - 35 - 35$

$=$ **66 mm**

Let the angle at centre be $x°$.

Arc length $= 66$ mm

$\frac{x°}{360°} \times 2\pi \times 35 = 66$

$x° = \frac{66 \times 360°}{2\pi \times 35}$

$= \left(\frac{2376}{7\pi}\right)°$

$=$ **108.0°** (to 1 d.p.)

Area $= \frac{\left(\frac{2376}{7\pi}\right)°}{360°} \times \pi \times 35^2$

$=$ **1155 mm²**

(c) Let the radius be r.

$\text{Area} = 1848 \text{ mm}^2$

$\frac{270°}{360°} \times \pi r^2 = 1848$

$r^2 = \frac{1848 \times 360°}{270°\pi}$

$r = \sqrt{\frac{1848 \times 360°}{270°\pi}}$

$= \textbf{28.0 mm}$ (to 3 s.f.)

Arc length $= \frac{270°}{360°} \times 2\pi \times 28.006$

$= \textbf{132 mm}$ (to 3 s.f.)

Perimeter $= 131.97 + 28.006 + 28.006$

$= \textbf{188 mm}$ (to 3 s.f.)

(d) Let the radius be r.

Arc length $= 220$ cm

$\frac{150°}{360°} \times 2\pi r = 220$

$r = \frac{220 \times 360°}{150° \times 2\pi}$

$= \textbf{84.0 cm}$ (to 3 s.f.)

Area $= \frac{150°}{360°} \times \pi \times 84.034^2$

$= \textbf{9240 cm}^2$ (to 3 s.f.)

Perimeter $= 220 + 84.034 + 84.034$

$= \textbf{388 cm}$ (to 3 s.f.)

(e) Let the angle at centre be $x°$.

Arc length $= 55$ m

$\frac{x°}{360°} \times 2\pi \times 14 = 55$

$x° = \frac{55 \times 360°}{2\pi \times 14}$

$= \textbf{225.1°}$ (to 1 d.p.)

Area $= \frac{225.091°}{360°} \times \pi \times 14^2$

$= \textbf{385 m}^2$ (to 3 s.f.)

Perimeter $= 55 + 14 + 14$

$= \textbf{83 m}$

(f) Let the radius be r.

$\text{Area} = 154 \text{ cm}^2$

$\frac{75°}{360°} \times \pi r^2 = 154$

$r^2 = \frac{154 \times 360°}{75°\pi}$

$r = \sqrt{\frac{154 \times 360°}{75°\pi}}$

$= \textbf{15.3 cm}$ (to 3 s.f.)

Arc length $= \frac{75°}{360°} \times 2\pi \times 15.339$

$= \textbf{20.1 cm}$ (to 3 s.f.)

Perimeter $= 20.079 + 15.339 + 15.339$

$= \textbf{50.8 cm}$ (to 3 s.f.)

2. (a) (i) Perimeter of minor sector $= \frac{30°}{360°} \times 2\pi \times 7 + 7 + 7$

$= \textbf{17.7 cm}$ (to 3 s.f.)

(ii) Area of minor sector $= \frac{30°}{360°} \times \pi \times 7^2$

$= \textbf{12.8 cm}^2$ (to 3 s.f.)

(b) (i) $\angle O = 360° - 340°$ ($\angle$s at a point)

$= 20°$

Perimeter of minor sector

$= \frac{20°}{360°} \times 2\pi \times 3.5 + 3.5 + 3.5$

$= \textbf{8.22 cm}$ (to 3 s.f.)

(ii) Area of minor sector $= \frac{20°}{360°} \times \pi \times 3.5^2$

$= \textbf{2.14 cm}^2$ (to 3 s.f.)

(c) (i) Perimeter of minor sector $= \frac{140°}{360°} \times 2\pi \times 6 + 6 + 6$

$= \textbf{26.7 cm}$ (to 3 s.f.)

(ii) Area of minor sector $= \frac{140°}{360°} \times \pi \times 6^2$

$= \textbf{44.0 cm}^2$ (to 3 s.f.)

3. Let the radius of the circle be r.

Circumference $= 88$ cm

$2\pi r = 88$

$r = \frac{88}{2\pi}$

$= 14.006$ cm (to 5 s.f.)

(a) Length of arc $ACB = \frac{60°}{360°} \times 2\pi \times 14.006$

$= \textbf{14.7 cm}$ (to 3 s.f.)

Area of sector $OACB = \frac{60°}{360°} \times \pi \times 14.006^2$

$= \textbf{103 cm}^2$ (to 3 s.f.)

(b) Length of arc $ACB = \frac{99°}{360°} \times 2\pi \times 14.006$

$= \textbf{24.2 cm}$ (to 3 s.f.)

Area of sector $OACB = \frac{99°}{360°} \times \pi \times 14.006^2$

$= \textbf{169 cm}^2$ (to 3 s.f.)

(c) Length of arc $ACB = \frac{126°}{360°} \times 2\pi \times 14.006$

$= \textbf{30.8 cm}$ (to 3 s.f.)

Area of sector $OACB = \frac{126°}{360°} \times \pi \times 14.006^2$

$= \textbf{216 cm}^2$ (to 3 s.f.)

(d) Length of arc $ACB = \frac{216°}{360°} \times 2\pi \times 14.006$

$= \textbf{52.8 cm}$ (to 3 s.f.)

Area of sector $OACB = \frac{216°}{360°} \times \pi \times 14.006^2$

$= \textbf{370 cm}^2$ (to 3 s.f.)

4. Let the radius of the circle be r.

Area = 3850 cm^2

$\pi r^2 = 3850$

$r^2 = \frac{3850}{\pi}$

$r = \sqrt{\frac{3850}{\pi}}$

$= 35.007$ cm (to 5 s.f.)

(a) Area of sector $OPSQ = \frac{36°}{360°} \times 3850$

$=$ **385 cm²**

Length of arc $PSQ = \frac{36°}{360°} \times 2\pi \times 35.007$

$=$ **22.0 cm** (to 3 s.f.)

(b) Area of sector $OPSQ = \frac{84°}{360°} \times 3850$

$=$ **$898\frac{1}{3}$ cm² or 898 cm²** (to 3 s.f.)

Length of arc $PSQ = \frac{84°}{360°} \times 2\pi \times 35.007$

$=$ **51.3 cm** (to 3 s.f.)

(c) Area of sector $OPSQ = \frac{108°}{360°} \times 3850$

$=$ **1155 cm²**

Length of arc $PSQ = \frac{108°}{360°} \times 2\pi \times 35.007$

$=$ **66.0 cm** (to 3 s.f.)

(d) Area of sector $OPSQ = \frac{198°}{360°} \times 3850$

$=$ **2117.5 cm²**

Length of arc $PSQ = \frac{198°}{360°} \times 2\pi \times 35.007$

$=$ **121 cm** (to 3 s.f.)

5. (a) Let the radius of the circle be r.

Area of minor sector = 114 cm^2

$\frac{150°}{360°} \times \pi r^2 = 114$

$r^2 = \frac{114 \times 360°}{150°\pi}$

$r = \sqrt{\frac{114 \times 360°}{150°\pi}}$

$=$ **9.33 cm** (to 3 s.f.)

(b) Reflex angle at centre of circle = 360° – 66° (∠s at a point)

= 294°

Let the radius of the circle be r.

Area of major sector = 369 cm^2

$\frac{294°}{360°} \times \pi r^2 = 369$

$r^2 = \frac{369 \times 360°}{294°\pi}$

$r = \sqrt{\frac{369 \times 360°}{294°\pi}}$

$=$ **12.0 cm** (to 3 s.f.)

6. Radius $= \frac{18}{2}$

= 9 cm

(a) Let the angle subtended be $x°$.

Area of sector = 42.6 cm^2

$\frac{x°}{360°} \times \pi \times 9^2 = 42.6$

$x° = \frac{42.6 \times 360°}{\pi \times 9^2}$

$=$ **60.3°** (to 1 d.p.)

(b) Let the angle subtended be $x°$.

Area of sector = 117.2 cm^2

$\frac{x°}{360°} \times \pi \times 9^2 = 117.2$

$x° = \frac{117.2 \times 360°}{\pi \times 9^2}$

$=$ **165.8°** (to 1 d.p.)

(c) Let the angle subtended be $x°$.

Area of sector = 214.5 cm^2

$\frac{x°}{360°} \times \pi \times 9^2 = 214.5$

$x° = \frac{214.5 \times 360°}{\pi \times 9^2}$

$=$ **303.5°** (to 1 d.p.)

(d) Let the angle subtended be $x°$.

Area of sector = 18.9 cm^2

$\frac{x°}{360°} \times \pi \times 9^2 = 18.9$

$x° = \frac{18.9 \times 360°}{\pi \times 9^2}$

$=$ **26.7°** (to 1 d.p.)

7. (a) $AD = BC$

$= 20 - 10$

$= 10$ cm

Perimeter of shaded region

= length of arc AB + length of arc CD + AD + BC

$= \frac{45°}{360°} \times 2\pi \times 10 + \frac{45°}{360°} \times 2\pi \times 20 + 10 + 10$

$=$ **43.6 cm** (to 3 s.f.)

Area of shaded region

= area of sector ODC – area of sector OAB

$= \frac{45°}{360°} \times \pi \times 20^2 - \frac{45°}{360°} \times \pi \times 10^2$

$=$ **118 cm²** (to 3 s.f.)

(b) $AD = BC$

$= 8 - 5$

$= 3$ cm

Perimeter of shaded region

= length of arc AB + length of arc CD + AD + BC

$= \frac{120°}{360°} \times 2\pi \times 5 + \frac{120°}{360°} \times 2\pi \times 8 + 3 + 3$

$=$ **33.2 cm** (to 3 s.f.)

Area of shaded region

= area of sector ODC – area of sector OAB

$= \frac{120°}{360°} \times \pi \times 8^2 - \frac{120°}{360°} \times \pi \times 5^2$

$=$ **40.8 cm²** (to 3 s.f.)

(c) $AD = BC$

$= 49 - 35$

$= 14$ cm

Perimeter of shaded region

= length of arc AB + length of arc CD + AD + BC

$= \frac{160°}{360°} \times 2\pi \times 35 + \frac{160°}{360°} \times 2\pi \times 49 + 14 + 14$

$=$ **263 cm** (to 3 s.f.)

Area of shaded region

= area of sector ODC – area of sector OAB

$= \frac{160°}{360°} \times \pi \times 49^2 - \frac{160°}{360°} \times \pi \times 35^2$

$=$ **1640 cm²** (to 3 s.f.)

8. **(i)** Let the radius of the circle be r.

$$\frac{\angle POQ}{360°} \times \pi r^2 = \frac{5}{18} \times \pi r^2$$

$$\frac{\angle POQ}{360°} = \frac{5}{18}$$

$$\angle POQ = \frac{5}{18} \times 360°$$

$$= \mathbf{100°}$$

(ii) Area of shaded sector = 385 cm^2

$$\frac{100°}{360°} \times \pi r^2 = 385$$

$$r^2 = \frac{385 \times 360°}{100°\pi}$$

$$r = \sqrt{\frac{385 \times 360°}{100°\pi}}$$

$$= 21.004 \text{ cm (to 5 s.f.)}$$

Diameter = 21.004 × 2

= **42.0 cm** (to 3 s.f.)

9. Let the angle subtended at the centre of the sector be $x°$.

Perimeter of sector = 38 cm

$$\frac{x°}{360°} \times 2\pi \times 12 + 12 + 12 = 38$$

$$\frac{x°}{15°}\pi + 24 = 38$$

$$\frac{x°}{15°}\pi = 14$$

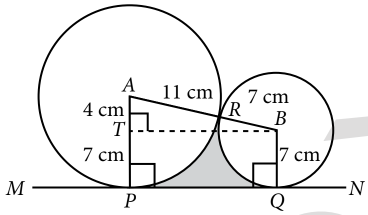
$$x° = \frac{14 \times 15°}{\pi}$$

$$= \frac{210°}{\pi}$$

$$\text{Area of paper used} = \frac{\frac{210°}{\pi}}{360°} \times \pi \times 12^2$$

$$= \mathbf{84\ cm^2}$$

10. **(i)** Draw a line BT such that T lies on AP and BT is perpendicular to AP.



$AT = AP - TP$

$= 11 - 7$

$= 4$ cm

$AB = AR + BR$

$= 11 + 7$

$= 18$ cm

$$\cos P\hat{A}B = \frac{AT}{AB}$$

$$= \frac{4}{18}$$

$$P\hat{A}B = \cos^{-1} \frac{4}{18}$$

$= 77.2°$ (to 1 d.p.) (shown)

(ii) Using Pythagoras' Theorem,

$BT^2 = 18^2 - 4^2$

$= 308$

$BT = \sqrt{308}$

$= 17.550$ cm (to 5 s.f.)

$A\hat{B}T = 180° - 90° - 77.160°$ (∠ sum of △)

$= 12.840°$

$A\hat{B}Q = 90° + 12.840°$

$= 102.840°$

Area of shaded region

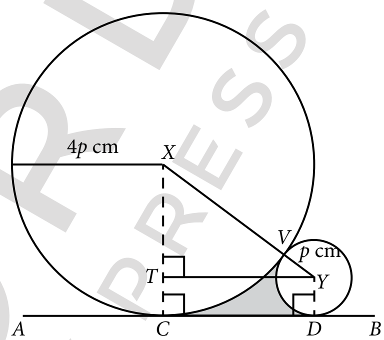
= area of trapezium $ABQP$ – area of sector APR

– area of sector BQR

$$= \frac{1}{2} \times (11 + 7)(17.550) - \frac{77.160°}{360°} \times \pi \times 11^2 - \frac{102.840°}{360°} \times \pi \times 7^2$$

$= \mathbf{32.5\ cm^2}$ (to 3 s.f.)

11. Draw a line YT such that T lies on CX and YT is perpendicular to CX.



$TX = CX - CT$

$= 4p - p$

$= 3p$ cm

$XY = VX + VY$

$= 4p + p$

$= 5p$ cm

$$\cos \angle CXY = \frac{TX}{XY}$$

$$= \frac{3p}{5p}$$

$$= \frac{3}{5}$$

$$\angle CXY = \cos^{-1} \frac{3}{5}$$

$= 53.130°$ (to 3 d.p.)

$\angle XYT = 180° - 90° - 53.130°$ (∠ sum of △)

$= 36.870°$

$\angle DYX = 90° + 36.870°$

$= 126.870°$

Using Pythagoras' Theorem,

$TY^2 = (5p)^2 - (3p)^2$

$= 25p^2 - 9p^2$

$= 16p^2$

$TY = \sqrt{16p^2}$

$= 4p$ cm

Area of enclosed region

= area of trapezium $CDYX$ – area of sector XCV

– area of sector YDV

$$= \frac{1}{2} \times (4p + p)(4p) - \frac{53.130°}{360°} \times \pi \times (4p)^2 - \frac{126.870°}{360°} \times \pi \times p^2$$

$= \mathbf{1.47p^2\ cm^2}$ (to 3 s.f.)

12. Since $ABCD$ is a rectangle with $BC = 2AB$ and $BF = CF$, then $AB = BF = CF = CD = x$ units.

Using Pythagoras' Theorem,

$$AF^2 = x^2 + x^2$$
$$= 2x^2$$
$$AF = \sqrt{2x^2}$$
$$= \sqrt{2}x \text{ units}$$

$\therefore$ radius of sector $FAD = \sqrt{2}x$ units

$AD = BC = 2x$ units

$\therefore$ radius of semicircle $AODE = \frac{2x}{2}$

$= x$ units

Area of shaded figure = area of sector FAD – area of $\triangle ADF$

$$= \frac{90°}{360°} \times \pi \times \left(\sqrt{2}x\right)^2 - \frac{1}{2} \times \sqrt{2}x \times \sqrt{2}x$$
$$= \frac{\pi x^2}{2} - x^2$$
$$= \frac{\frac{22}{7}x^2}{2} - x^2$$
$$= \frac{4}{7}x^2 \text{ units}^2$$

Area of entire figure = area of semicircle $AODE$ + area of rectangle $ABCD$

$$= \frac{1}{2} \times \pi x^2 + (2x)(x)$$
$$= \frac{\pi x^2}{2} + 2x^2$$
$$= \frac{\frac{22}{7}x^2}{2} + 2x^2$$
$$= \frac{25}{7}x^2 \text{ units}^2$$

Fraction of figure that is shaded $= \dfrac{\frac{4}{7}x^2}{\frac{25}{7}x^2}$

$$= \mathbf{\frac{4}{25}}$$

13. Since $DB = \frac{2}{3}BE$, $DB : BE = 2 : 3$.

$$DB = \frac{2}{2+3} \times 1.5$$
$$= 0.6 \text{ m}$$
$$OB = DB - OD$$
$$= 0.6 - 0.4$$
$$= 0.2 \text{ m}$$
$$\cos \angle AOB = \frac{OB}{OA}$$
$$= \frac{0.2}{0.4}$$
$$\angle AOB = \cos^{-1} \frac{0.2}{0.4}$$
$$= 60°$$

Since the sign is symmetrical about DE,

$$\angle AOC = 60° \times 2$$
$$= 120°$$

Reflex $\angle AOC = 360° - 120°$ ($\angle$s at a point)

$= 240°$

Using Pythagoras' Theorem,

$$AB^2 = 0.4^2 - 0.2^2$$
$$= 0.12$$
$$AB = \sqrt{0.12}$$
$$= 0.346\ 41 \text{ m (to 5 s.f.)}$$
$$AC = 0.346\ 41 \times 2$$
$$= 0.692\ 82 \text{ m}$$
$$BE = DE - DB$$
$$= 1.5 - 0.6$$
$$= 0.9 \text{ m}$$

Area of sign

= area of major sector $OADC$ + area of $\triangle OAC$ + area of $\triangle ACE$

$= \frac{240°}{360°} \times \pi \times 0.4^2 + \frac{1}{2} \times 0.692\ 82 \times 0.2 + \frac{1}{2} \times 0.692\ 82 \times 0.9$

= **0.716 m²** (to 3 s.f.)

14. **(i)** $\angle OQA = \angle OAQ$ (base $\angle$s of isos. $\triangle$)

$= 66°$

$\angle BOQ = 66° + 66°$ (ext. $\angle$ of $\triangle OAQ$)

$= \mathbf{132°}$

(ii) Since OX divides $\triangle OAQ$ into two congruent triangles, $\triangle OAX \equiv \triangle OQX$.

Since $\triangle OAX \equiv \triangle OQX$,

$\angle OXA = \angle OXQ$

$= \frac{180°}{2}$ (adj. $\angle$s on a str. line)

$= 90°$

$$\cos 66° = \frac{AX}{OA}$$
$$= \frac{AX}{16}$$
$$AX = 16 \cos 66°$$
$$= 6.5078 \text{ cm (to 5 s.f.)}$$
$$AQ = 6.5078 \times 2$$

$= \mathbf{13.0\ cm}$ (to 3 s.f.)

(iii) $PQ = AP - AQ$

$$= 32 - 13.0156$$
$$= 18.9844 \text{ cm}$$

Perimeter of shaded region

= length of arc BYP + length of arc BQ + PQ

$= \frac{66°}{360°} \times 2\pi \times 32 + \frac{132°}{360°} \times 2\pi \times 16 + 18.9844$

= **92.7 cm** (to 3 s.f.)

(iv) Area of shaded region

= area of sector $APYB$ – area of sector OBQ – area of $\triangle OAQ$

$= \frac{66°}{360°} \times \pi \times 32^2 - \frac{132°}{360°} \times \pi \times 16^2 - \frac{1}{2} \times 13.0156 \times 16 \times \sin 66°$

= **200 cm²** (to 3 s.f.)

15. **(i)** Since B is the midpoint of the arc AC, $\angle AOB = \angle BOD = 45°$.

$$\sin \angle BOD = \frac{BD}{OB}$$

$$\sin 45° = \frac{BD}{12}$$

$$BD = 12 \sin 45°$$
$$= 8.4853 \text{ cm (to 5 s.f.)}$$
$$= \mathbf{8.49\ cm} \text{ (to 3 s.f.)}$$

(ii) Using Pythagoras' Theorem,

$OD^2 = 12^2 - 8.4853^2$

$OD = \sqrt{12^2 - 8.4853^2}$

$= 8.4853$ cm (to 5 s.f.)

$CD = OC - OD$

$= 12 - 8.4853$

$= 3.5147$ cm

Length of arc $BC = \frac{1}{2} \times$ length of arc AC

$= \frac{1}{2} \times \frac{1}{4} \times 2\pi \times 12$

$= 3\pi$ cm

Perimeter of shaded region = length of arc $BC + BD + CD$

$= 3\pi + 8.4853 + 3.5147$

$= \mathbf{21.4\ cm}$ (to 3 s.f.)

(iii) Area of shaded region

= area of quadrant $OABC$ – area of sector OAB – area of $\triangle OBD$

$= \frac{1}{4} \times \pi \times 12^2 - \frac{45°}{360°} \times \pi \times 12^2 - \frac{1}{2} \times 8.4853 \times 8.4853$

$= \mathbf{20.5\ cm^2}$ (to 3 s.f.)

16. **(a)** Since M is the midpoint of AB, $\angle AMO = \angle BMO = 90°$.

$$\cos \angle AOM = \frac{\text{adj}}{\text{hyp}}$$
$$= \frac{OM}{OA}$$
$$= \frac{5}{7.5}$$

$$\angle AOM = \cos^{-1} \frac{5}{7.5}$$
$$= 48.190° \text{ (to 5 s.f.)}$$

$\angle AOB = \angle AOM \times 2$

$= 48.190° \times 2$

$= 96.4°$ (to 3 s.f.) (shown)

(b) Using Pythagoras' Theorem,

$AM^2 = 7.5^2 - 5^2$

$= 31.25$

$AM = \sqrt{31.25}$

$= 5.5902$ cm (to 5 s.f.)

$AB = AM \times 2$

$= 5.5902 \times 2$

$= 11.1804$ cm

Shaded area = area of sector OAB – area of $\triangle OAB$

$= \frac{96.380°}{360°} \times \pi \times 7.5^2 - \frac{1}{2} \times 11.1804 \times 5$

$= \mathbf{19.4\ cm^2}$ (to 3 s.f.)

(c) **(i)** Volume of chocolate mousse layer = 19.359×32

$= \mathbf{619\ cm^3}$ (to 3 s.f.)

(ii) Volume of dark chocolate in 1 chocolate mousse layer

$= 45\% \times 619.488$

$= 278.7696$ cm^3

10 litres = 10×1000 cm^3 (1 litre = 1000 cm^3)

$= 10\,000$ cm^3

Number of cakes that can be made $= \frac{10\,000}{278.7696}$

$\approx \mathbf{35}$

Chapter 9 Geometrical Properties of Circles

TEACHING NOTES

Suggested Approach

Teachers may begin the topic by asking students how the centre of a circle can be determined, or how a circle can be constructed given only three points that pass through the circle. With this, teachers may introduce both the symmetric and angle properties of circles and build up on past knowledge, i.e. Pythagoras' Theorem, Basic Geometry, Triangles learnt by students when covering this chapter.

Section 9.1: Symmetric properties of circles

Teachers may recall with students what the perpendicular bisector of a chord is before introducing circle symmetric property 1 through exploring three conditions and using a cut-out piece of circle. This will help students to better visualise and understand the symmetric properties of circles (see Investigation: Discovering circle symmetric property 1).

For circle symmetric property 2, teachers may ask students what equal chords are and the properties of equal chords of a circle (see Investigation: Discovering circle symmetric property 2).

For circle symmetric property 3 and circle symmetric property 4, these two properties involve the radius of a circle and tangent to a circle (see Investigation: Discovering circle symmetric property 3 and Investigation: Discovering circle symmetric property 4). It is important for students to differentiate between a secant and a tangent.

Section 9.2: Angle properties of circles

After students have learnt the symmetric properties of circles, teachers may then guide students on angle properties of circles. It is important for students to be able to recognise the angle that is subtended an arc (see Class Discussion: Identifying angles at centre and at circumference) before they explore the relationship between the angle at the centre of a circle and the angle at the circumference subtended by the same arc (see Investigation: Discovering circle angle property 1).

For Circle Angle Property 2, teachers can ask students what the angle at the circumference subtended by an arc is when the angle subtended at the centre of the circle is 180° (see Investigation: Discovering circle angle property 2).

Teachers should guide students along on identifying angles in the same or opposite segments of a circle before teaching Circle Angle Property 3 (see Class Discussion: Angles in same or in opposite segments, Investigation: Discovering circle angle property 3 and Investigation: Discovering circle angle property 4).

Teachers should note that students may have difficulty in recognising the properties of circles when the diagrams are not drawn and given in an obvious manner. As such, teachers may get students to observe the diagram from a different orientation or to add extra lines to the diagram to help them.

Section 9.3: Alternate Segment Theorem

Before the start of this section, it is very important for students to be able to recognise parts of a circle such as chords, tangents and radii. To help students identify alternate segments, when students are investigating the Alternate Segment Theorem (see Investigation: Alternate Segment Theorem), teachers may also use an example to show students where the 2 segments of a circle are when there is a chord.

At the end of this chapter, it is likely that students will get the theorems and properties of a circle mixed up. As such, teachers should show them a summary of what students have learnt in this chapter (see Summary on pages 307 and 308 of the textbook).

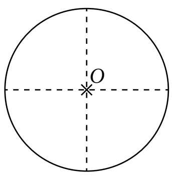
Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 2).

9.1 Symmetric properties of circles

Investigation (Discovering circle symmetric property 1)

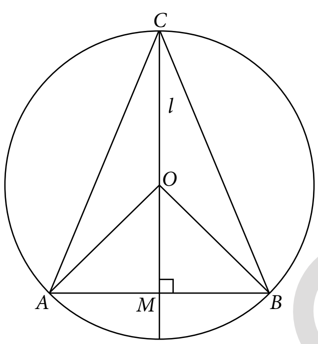
1. Conditions ***A* and *B***
2. **(a)** ***AM* = *MB***
   (b) *M* is the **midpoint of *AB***.
3. Conditions ***A* and *C***
4. $\boldsymbol{\angle AMO = \angle BMO = 90°}$
5. The intersection point of the fold lines is the centre of the circle. It is because a diameter of the circle always passes through the centre of the circle.



6. **(a)** Conditions ***B* and *C***
   (b) **Yes**

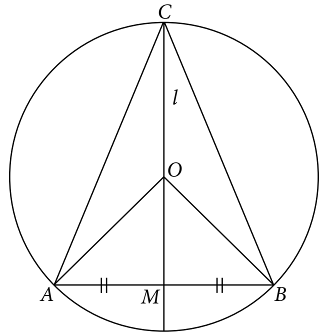
Thinking Time (Page 271)

1.



$A\hat{M}O = B\hat{M}O = 90°$ (adj. ∠s on a str. line)
$M\hat{A}O = M\hat{B}O$ (base ∠ of isos. △)
$\therefore A\hat{O}M = B\hat{O}M$
Hence $\triangle AMO$ is similar to $\triangle BMO$.
Since radius of circle = $OA = OB$, $AM = BM$ and the line l bisects the chord AB. (proven)

2.



$AO = BO$ (radii)
$OM = OM$ (common side)
$AM = BM$ (given)
$\therefore \triangle AMO \equiv \triangle BMO$
Since $\triangle AMO \equiv \triangle BMO$, $A\hat{M}O = B\hat{M}O$.
$A\hat{M}O + B\hat{M}O = 180°$ (adj. ∠s on a str. line)
$A\hat{M}O + A\hat{M}O = 180°$
$2A\hat{M}O = 180°$
$A\hat{M}O = 90°$
Hence $A\hat{M}O = B\hat{M}O = 90°$ and the line l is perpendicular to the chord AB. (proven)

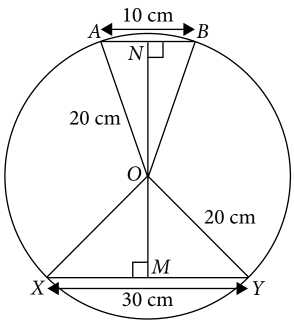
Practise Now 1

1. $MP = MQ$ (⊥ bisector of chord)
$= \frac{24}{2}$
$= 12$ cm
Using Pythagoras' Theorem,
$OP^2 = MP^2 + OM^2$
$= 12^2 + 5^2$
$= 169$
$OP = \sqrt{169}$ (since length $OP > 0$)
$= 13$ cm
$\therefore$ radius = **13 cm**

2. $A\hat{X}O = B\hat{X}O = 90°$ (⊥ bisector of chord)
Using Pythagoras' Theorem,
$OA^2 = AX^2 + OX^2$
$7^2 = AX^2 + 3^2$
$49 = AX^2 + 9$
$AX^2 = 40$
$AX = \sqrt{40}$ (since length $AX > 0$)
$= 6.3246$ cm (to 5 s.f.)
$AB = 2AX$
$= 2(6.3246)$
$=$ **12.6 cm** (to 3 s.f.)

Practise Now 2

1. **Case 1: The chords are on opposite sides of the centre *O*.**



$AN = NB$ (⊥ bisector of chord)

$= \frac{AB}{2}$

$= \frac{10}{2}$

$= 5$ cm

Using Pythagoras' Theorem,

$ON^2 = 20^2 - 5^2$

$= 375$

$ON = \sqrt{375}$ (since length $ON > 0$)

$= 19.365$ cm (to 5 s.f.)

$XM = MY$ (⊥ bisector of chord)

$= \frac{XY}{2}$

$= \frac{30}{2}$

$= 15$ cm

Using Pythagoras' Theorem,

$OM^2 = 20^2 - 15^2$

$= 175$

$OM = \sqrt{175}$ (since length $OM > 0$)

$= 13.229$ cm (to 5 s.f.)

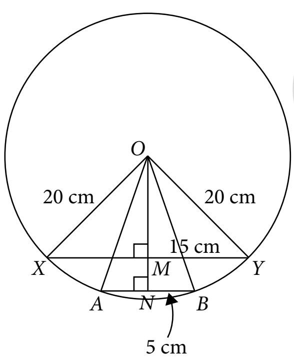
Shortest distance between the chords $= MN$

$= ON + OM$

$= 19.365 + 13.229$

$= 32.6$ cm (to 3 s.f.)

Case 2: The chords are on the same side of the centre *O*.



Shortest distance between the chords $= MN$

$= ON - OM$

$= 19.365 - 13.229$

$= 6.14$ cm (to 3 s.f.)

∴ the two possible distances between the chords are **6.14 cm or 32.6 cm**.

2. $AS = AT$ (⊥ bisector of chord)

$= \frac{20}{2}$

$= 10$ cm

Using Pythagoras' Theorem,

$OT^2 = 8^2 + 10^2$

$= 164$

$OT = \sqrt{164}$ (since length $OT > 0$)

$= 12.806$ cm (to 5 s.f.)

∴ radius = 12.806 cm

Using Pythagoras' Theorem,

$BV^2 = 12.806^2 - 6^2$

$= 127.993\ 636$

$BV = \sqrt{127.993\ 636}$ (since length $BV > 0$)

$= 11.313$ cm (to 5 s.f.)

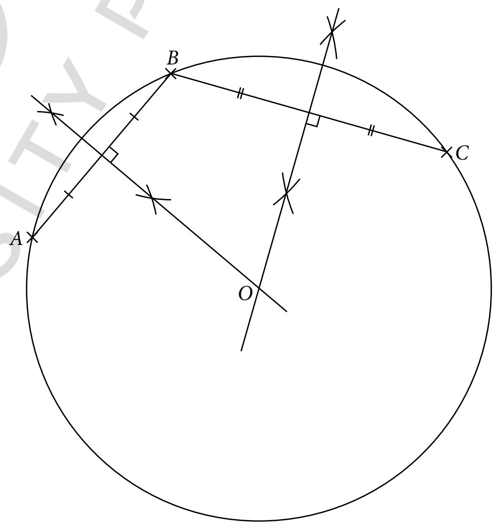
$UV = 2BV$

$= 2(11.313)$

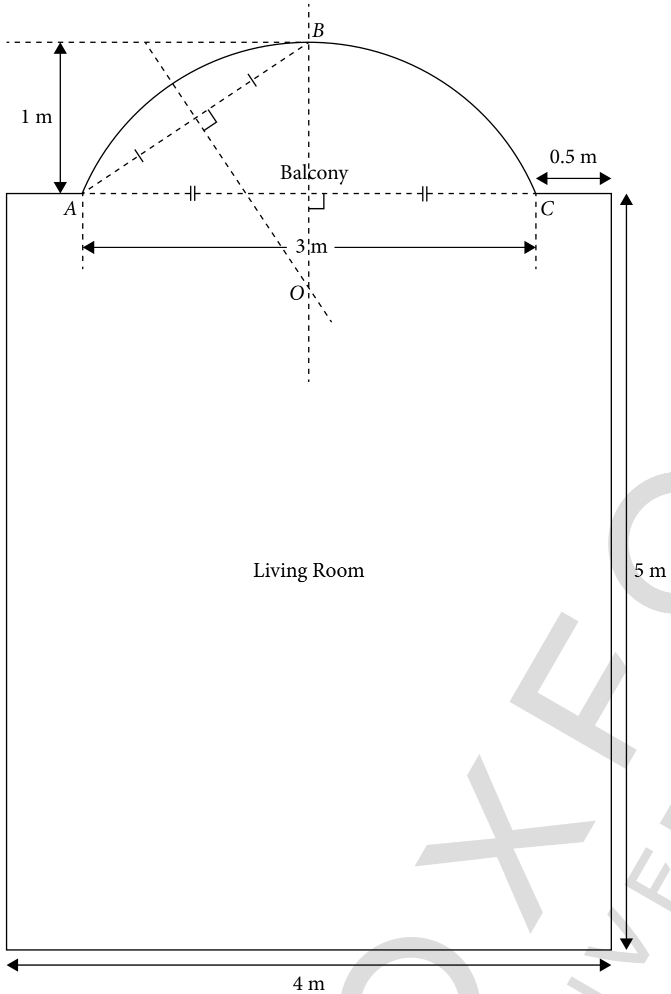
$=$ **22.6 cm** (to 3 s.f.)

Introductory Problem Revisited

1. Construct the perpendicular bisector of *AB*.
Construct the perpendicular bisector of *BC*.
The point of intersection of the two perpendicular bisectors will be the centre *O* of the circle passing through the three points *A*, *B* and *C*.
Construct the circle with centre *O* and radius *OA*.



2. Draw a dotted line segment AC of length 6 cm.
Construct the perpendicular bisector of AC.
Mark the point 2 cm above the midpoint of AC that lies on the perpendicular bisector as B.
Construct the perpendicular bisector of AB (or BC).
Mark the point of intersection between the perpendicular bisectors of AB and AC as O.
Draw the arc ABC of a circle with centre O.
Construct the remaining parts of the room using 2 cm to represent 1 m.



Investigation (Discovering circle symmetric property 2)

2. The distances of both chords from the centre O are **equal**.
3. In general, equal chords of a circle are **equidistant** from the centre of the circle.
5. The lengths of both chords are **equal**.
6. In general, chords that are equidistant from the centre of a circle are **equal** (in length).

Practise Now 3

1. $\angle OFH = 90°$ ($\perp$ bisector of chord)
$\therefore OF$ = distance of HK from O
= distance of XY from O (equal chords)
$= OE$
= **16 cm**
2. $\angle OAM = 90°$ ($\perp$ bisector of chord)
$\therefore CD = MN$ (equal chords)
= **14 cm**

Investigation (Discovering circle symmetric property 3)

3. **(a)** The secant has become the **tangent** at the point of contact P.
(b) The angle is **90°**.
4. In general, the tangent at the point of contact is **perpendicular** to the radius of the circle.

Practise Now 4

1. **(i)** $\angle OAP = 90°$ (tangent $\perp$ radius)

$$\tan \angle OPA = \frac{\text{opp}}{\text{adj}} = \frac{OA}{PA} = \frac{4.5}{10.5}$$

$$\angle OPA = \tan^{-1} \frac{4.5}{10.5} = 23.199° \text{ (to 3 d.p.)} = \mathbf{23.2°} \text{ (to 1 d.p.)}$$

(ii) Method 1:
Using Pythagoras' Theorem,

$$OP^2 = 10.5^2 + 4.5^2 = 130.5$$

$$OP = \sqrt{130.5} \text{ (since length } OP > 0)$$

$$= \mathbf{11.4\ cm} \text{ (to 3 s.f.)}$$

Method 2:

$$\cos \angle OPA = \frac{\text{adj}}{\text{hyp}} = \frac{AP}{OP}$$

$$\cos 23.199° = \frac{10.5}{OP}$$

$$OP = \frac{10.5}{\cos 23.199°} = \mathbf{11.4\ cm} \text{ (to 3 s.f.)}$$

(iii) Area of $\triangle OPA = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times PA \times OA = \frac{1}{2} \times 10.5 \times 4.5 = \mathbf{23.625\ cm^2}$$

2. **(i)** $\angle OAB = 90°$ (tangent $\perp$ radius)
$OC = OA$ (radii)
$= x$ cm
Using Pythagoras' Theorem,

$$OB^2 = AB^2 + OA^2$$

$$(5 + x)^2 = 8^2 + x^2$$

$$25 + 10x + x^2 = 64 + x^2$$

$$10x = 39$$

$$x = \mathbf{3.9}$$

(ii)

$$\tan \angle AOB = \frac{\text{opp}}{\text{adj}} = \frac{AB}{OA} = \frac{8}{3.9}$$

$$\angle AOB = \tan^{-1} \frac{8}{3.9} = 64.011° \text{ (to 3 d.p.)} = \mathbf{64.0°} \text{ (to 1 d.p.)}$$

(iii) Area bounded by AB, BC and the minor arc AC

$= \text{area of } \triangle OAB - \text{area of sector } OAC$

$= \frac{1}{2} \times 8 \times 3.9 - \frac{64.011°}{360°} \times \pi \times 3.9^2$

$=$ **7.10 cm²** (to 3 s.f.)

Investigation (Discovering circle symmetric property 4)

2. **(a)** $\boldsymbol{AP = BP}$

(b) $\boldsymbol{\angle OPA = \angle OPB}$

(c) $\boldsymbol{\angle AOP = \angle BOP}$

3. In general,

(a) the tangents from an external point are **equal** (in length);

(b) the line from the centre of a circle to an external point **bisects** the angle between the two tangents;

(c) the line from the centre of a circle to an external point **bisects** the angle between the radii OA and OB, where A and B are the points of contact between the two tangents and the circle.

Practise Now 5

1. **(i)** $\angle OYP = 90°$ (tangent ⊥ radius)

$\angle POY = 180° - 90° - 38°$ (∠ sum of △)

$= 52°$

$\angle XOY = 52° \times 2$ (tangents from ext. pt.)

$=$ **104°**

(ii) $\tan \angle OPX = \frac{\text{opp}}{\text{adj}}$

$= \frac{OX}{PX}$

$\tan 38° = \frac{11}{PX}$

$PX = \frac{11}{\tan 38°}$

$= 14.079$ cm (to 5 s.f.)

$=$ **14.1 cm** (to 3 s.f.)

(iii) Area of $\triangle OPX = \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times PX \times OX$

$= \frac{1}{2} \times 14.079 \times 11$

$= 77.4345 \text{ cm}^2$

Area of quadrilateral $OXPY = 2 \times$ Area of $\triangle OPX$

$= 2 \times 77.4345$

$=$ **155 cm²** (to 3 s.f.)

2. $\angle PQT = \angle PTQ$ (tangents from ext. pt.)

$\angle PQT + \angle PTQ + \angle QPT = 180°$ (∠ sum of △)

$2\angle PQT + 64° = 180°$

$2\angle PQT = 116°$

$\angle PQT = 58°$

$\angle OQP = 90°$ (tangent ⊥ radius)

$\angle SQT = \angle OQP - \angle PQT$

$= 90° - 58°$

$=$ **32°**

Exercise 9A

1. $PX = PY$ (⊥ bisector of chord)

$= \frac{8}{2}$

$= 4$ cm

Using Pythagoras' Theorem,

$OX^2 = OP^2 + PX^2$

$= 3^2 + 4^2$

$= 25$

$OX = \sqrt{25}$ (since length $OX > 0$)

$= 5$ cm

∴ radius = **5 cm**

2. $\angle OQJ = 90°$ (⊥ bisector of chord)

Using Pythagoras' Theorem,

$OJ^2 = OQ^2 + JQ^2$

$16^2 = 7^2 + JQ^2$

$256 = 49 + JQ^2$

$JQ^2 = 207$

$JQ = \sqrt{207}$ (since length $JQ > 0$)

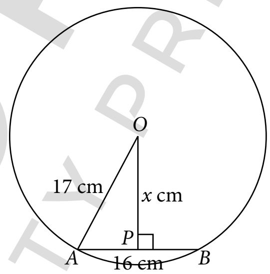
$= 14.387$ cm (to 5 s.f.)

$JK = 2JQ$

$= 2(14.387)$

$=$ **28.8 cm** (to 3 s.f.)

3.



$AP = BP$ (⊥ bisector of chord)

$= \frac{16}{2}$

$= 8$ cm

Using Pythagoras' Theorem,

$x^2 = 17^2 - 8^2$

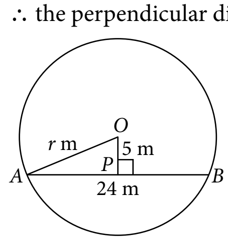
$= 225$

$x = \sqrt{225}$ (since $x > 0$)

$= 15$

∴ the perpendicular distance from O to AB is **15 cm**.

4.



$AP = BP$ (⊥ bisector of chord)

$= \frac{24}{2}$

$= 12$ m

Using Pythagoras' Theorem,

$r^2 = 12^2 + 5^2$

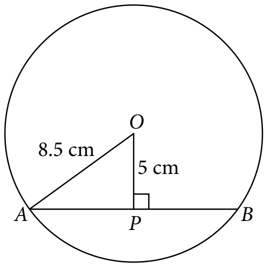
$= 169$

$r = \sqrt{169}$ (since $r > 0$)

$= 13$

∴ radius = **13 m**

5.



$AP = BP$ (⊥ bisector of chord)

Using Pythagoras' Theorem,

$AP^2 = 8.5^2 - 5^2$

$= 47.25$

$AP = \sqrt{47.25}$ (since length $AP > 0$)

$= 6.8739$ cm (to 5 s.f.)

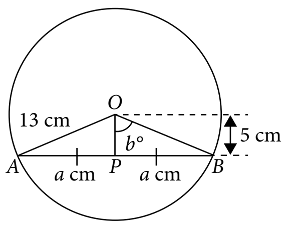
$AB = 2AP$

$= 2(6.8739)$

$= 13.7$ cm (to 3 s.f.)

$\therefore$ the length of the chord is **13.7 cm**.

6. (a)



$\angle OPA = \angle OPB = 90°$ (⊥ bisector of chord)

Using Pythagoras' Theorem,

$a^2 = 13^2 - 5^2$

$= 144$

$a = \sqrt{144}$ (since $a > 0$)

$= \mathbf{12}$

$OB = OA$ (radii)

$= 13$ cm

$\cos b° = \frac{\text{adj}}{\text{hyp}}$

$= \frac{OP}{OB}$

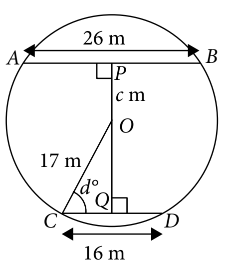
$= \frac{5}{13}$

$b° = \cos^{-1}\frac{5}{13}$

$= 67.4°$ (to 1 d.p.)

$\therefore$ ***b* = 67.4**

(b)



$AP = BP$ (⊥ bisector of chord)

$= \frac{26}{2}$

$= 13$ m

$OA = OC$ (radii)

$= 17$ m

Using Pythagoras' Theorem,

$c^2 = 17^2 - 13^2$

$= 120$

$c = \sqrt{120}$ (since $c > 0$)

$= \mathbf{11.0}$ (to 3 s.f.)

$CQ = DQ$ (⊥ bisector of chord)

$= \frac{16}{2}$

$= 8$ m

$\cos d° = \frac{\text{adj}}{\text{hyp}}$

$= \frac{CQ}{OC}$

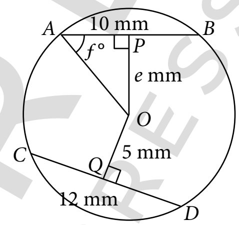
$= \frac{8}{17}$

$d° = \cos^{-1}\frac{8}{17}$

$= 61.9°$ (to 1 d.p.)

$\therefore$ ***d* = 61.9**

(c)



$CQ = DQ$ (⊥ bisector of chord)

$= \frac{12}{2}$

$= 6$ mm

Using Pythagoras' Theorem,

$OC^2 = 5^2 + 6^2$

$= 61$

$OC = \sqrt{61}$ mm (since length $OC > 0$)

$OA = OC$ (radii)

$= \sqrt{61}$ mm

$AP = BP$ (⊥ bisector of chord)

$= \frac{10}{2}$

$= 5$ mm

Using Pythagoras' Theorem,

$e^2 = \left(\sqrt{61}\right)^2 - 5^2$

$= 36$

$e = \sqrt{36}$ (since $e > 0$)

$= \mathbf{6}$

$\tan f° = \frac{\text{opp}}{\text{adj}}$

$= \frac{OP}{AP}$

$= \frac{6}{5}$

$f° = \tan^{-1}\frac{6}{5}$

$= 50.2°$ (to 1 d.p.)

$\therefore$ ***f* = 50.2**

7. $\angle AMO = 90°$ (⊥ bisector of chord)

$OM = ON$ (equal chords)

$= \mathbf{8\ cm}$

8. $\angle RIO = 90°$ ($\perp$ bisector of chord)
$RS = UV$ (equal chords)
$= \mathbf{11\ cm}$

9. **(a)** $\boldsymbol{a} = \mathbf{12}$ (equal chords)
(b) $y° = 90°$ ($\perp$ bisector of chord)
$\boldsymbol{y} = \mathbf{90}$
$2x = 22$ (equal chords)
$\boldsymbol{x} = \mathbf{11}$

10. $\angle OPB = 90°$ (tangent $\perp$ radius)
$\angle APB = 33° + 90°$
$= 123°$
Since $OA = OP$ (radii), $\triangle OAP$ is an isosceles $\triangle$.
$\angle OAP = \angle OPA$ (base $\angle$s of isos. $\triangle$)
$= 33°$
$\angle PBA = 180° - 33° - 123°$ ($\angle$ sum of $\triangle$)
$= \mathbf{24°}$

11. **(i)** $\angle OAT = 90°$ (tangent $\perp$ radius)
$\angle ATB = 180° - 90° - 64°$ ($\angle$ sum of $\triangle$)
$= \mathbf{26°}$
(ii) $\angle AOB = 180° - 64°$ (adj. $\angle$s on a str. line)
$= 116°$
Since $OA = OB$ (radii), $\triangle OAB$ is an isosceles $\triangle$.
$\angle OAB = \angle OBA$ (base $\angle$s of isos. $\triangle$)
$\angle OAB + \angle OBA + \angle AOB = 180°$ ($\angle$ sum of $\triangle$)
$\angle OAB + \angle OAB + 116° = 180°$
$2\angle OAB + 116° = 180°$
$2\angle OAB = 64°$
$\angle OAB = 32°$
$\angle TAB = \angle OAT + \angle OAB$
$= 90° + 32°$
$= \mathbf{122°}$

12. $\angle AOB = 180° - x - 90°$ ($\angle$ sum of $\triangle$)
$= 90° - x$
$\angle ODB = \angle ODC = 90°$ (adj. $\angle$s on a str. line)
$OB = OC$ (radii)
$OD = OD$ (common side)
$\therefore \triangle OBD \equiv \triangle OCD$ (RHS Congruence Test)
Since $\triangle OBD \equiv \triangle OCD$,
$\angle BOD = \angle COD$
$\angle AOB + \angle BOD + \angle COD = 180°$ (adj. $\angle$s on a str. line)
$90° - x + \angle COD + \angle COD = 180°$
$2\angle COD + 90° - x = 180°$
$2\angle COD = 90° + x$
$\angle COD = \mathbf{\dfrac{90° + \boldsymbol{x}}{2}}$

13. **(a)** $a° = 49°$ (tangents from ext. pt.)
$\boldsymbol{a} = \mathbf{49}$
$\boldsymbol{b} = \mathbf{14}$ (tangents from ext. pt.)
(b) $\angle OPA = \angle OPB$ (tangents from ext. pt.)
$= 32°$
$\angle OAP = 90°$ (tangent $\perp$ radius)
$c° = 180° - 90° - 32°$ ($\angle$ sum of $\triangle$)
$= 58°$
$\boldsymbol{c} = \mathbf{58}$
$\boldsymbol{d} = \mathbf{15}$ (tangents from ext. pt.)
(c) $\angle OBP = \angle OAP$
$= 90°$ (tangent $\perp$ radius)
$\angle OPB = e°$ (tangents from ext. pt.)
$\angle BOP = \dfrac{112°}{2}$ (tangents from ext. pt.)
$= 56°$
$e° = 180° - 56° - 90°$ ($\angle$ sum of $\triangle$)
$= 34°$
$\boldsymbol{e} = \mathbf{34}$
$\tan 56° = \dfrac{\text{opp}}{\text{adj}}$
$= \dfrac{BP}{OB}$
$= \dfrac{f}{10}$
$f = 10 \tan 56°$
$= \mathbf{14.8}$ (to 3 s.f.)
(d) $\angle OBP = \angle OAP = 90°$ (tangent $\perp$ radius)
$\angle BOP = 180° - 90° - 35°$ ($\angle$ sum of $\triangle$)
$= 55°$
$\angle AOP = \angle BOP$ (tangents from ext. pt.)
$= 55°$
$\angle OAB = \angle OBA$ (base $\angle$s of isos. $\triangle$)
$= g°$
$g° + g° + 55° + 55° = 180°$ ($\angle$ sum of $\triangle$)
$2g° + °110 = 180°$
$2g° = 70°$
$g° = 35°$
$\boldsymbol{g} = \mathbf{35}$
$\angle OAB + \angle BAP = \angle OAP$
$35° + h° = 90°$
$h° = 55°$
$\boldsymbol{h} = \mathbf{55}$
(e) $\angle OBP = 90°$ (tangent $\perp$ radius)
Using Pythagoras' Theorem,
$OP^2 = 5^2 + 12^2$
$= 169$
$OP = \sqrt{169}$ (since length $OP > 0$)
$= 13$ m
$i = 13 - 5$
$= \mathbf{8}$
$\tan j° = \dfrac{\text{opp}}{\text{adj}}$
$= \dfrac{BP}{OB}$
$= \dfrac{12}{5}$
$j° = \tan^{-1} \dfrac{12}{5}$
$= 67.4°$ (to 1 d.p.)
$\boldsymbol{j} = \mathbf{67.4}$

(f) $\angle OBP = 90°$ (tangent $\perp$ radius)

Using Pythagoras' Theorem,

$$k^2 + 15^2 = (k + 7)^2$$
$$= k^2 + 14k + 49$$
$$14k + 49 = 225$$
$$14k = 176$$
$$k = 12.571 \text{ (to 5 s.f.)}$$
$$= \mathbf{12.6} \text{ (to 3 s.f.)}$$

$$\tan l° = \frac{\text{opp}}{\text{adj}}$$
$$= \frac{BP}{OB}$$
$$= \frac{15}{12.571}$$
$$l° = \tan^{-1} \frac{15}{12.571}$$
$$= 50.0° \text{ (to 1 d.p.)}$$
$$\boldsymbol{l = 50.0}$$

14. (i) $\angle OBP = \angle OAP = 90°$ (tangent $\perp$ radius)

$\angle OPB = 180° - 90° - 62°$ ($\angle$ sum of $\triangle$)

$= \mathbf{28°}$

(ii) $\angle AOC = \angle BOC$ (tangents from ext. pt.)

$= 62°$

Since $OA = OC$ (radii), $\triangle OAC$ is an isosceles $\triangle$.

$\angle OAC = \angle OCA$ (base $\angle$s of isos. $\triangle$)

$\angle OAC + \angle OCA + \angle AOC = 180°$ ($\angle$ sum of $\triangle$)

$$\angle OAC + \angle OAC + 62° = 180°$$
$$2\angle OAC = 118°$$
$$\angle OAC = \mathbf{59°}$$

(iii) $OB = OA$ (radii)

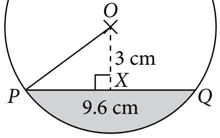
$= 14$ cm

$$\tan 62° = \frac{\text{opp}}{\text{adj}}$$
$$= \frac{BP}{OB}$$
$$= \frac{BP}{14}$$
$$BP = 14 \tan 62°$$
$$= 26.330 \text{ cm (to 5 s.f.)}$$
$$= \mathbf{26.3\ cm} \text{ (to 3 s.f.)}$$

(iv) Area of quadrilateral $OAPB = 2 \times$ area of $\triangle OBP$

$$= 2 \times \frac{1}{2} \times OB \times BP$$
$$= OB \times BP$$
$$= 14 \times 26.330$$
$$= \mathbf{369\ cm^2} \text{ (to 3 s.f.)}$$

15.



$PX = QX$ ($\perp$ bisector of chord)

$$= \frac{9.6}{2}$$
$$= 4.8 \text{ cm}$$

Using Pythagoras' Theorem,

$$OP^2 = 3^2 + 4.8^2$$
$$= 32.04$$
$$OP = \sqrt{32.04} \text{ (since length } OP > 0)$$
$$= 5.6604 \text{ cm (to 5 s.f.)}$$

$\therefore$ radius $= 5.6604$ cm

$$\tan \angle POX = \frac{\text{opp}}{\text{adj}}$$
$$= \frac{PX}{OX}$$
$$= \frac{4.8}{3}$$
$$= 1.6$$
$$\angle POX = \tan^{-1} 1.6$$
$$= 57.995° \text{ (to 3 d.p.)}$$

$\angle OXQ = \angle OXP = 90°$ (adj. $\angle$s on a str. line)

$OP = OQ$ (radii)

$OX = OX$ (common side)

$\therefore \triangle OPX \equiv \triangle OQX$ (RHS Congruence Test)

Since $\triangle OPX \equiv \triangle OQX$,

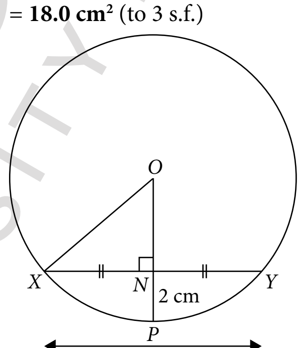
$$\angle QOX = \angle POX$$
$$= 57.995°$$
$$\angle POQ = 2 \times 57.995°$$
$$= 115.99°$$

Cross-sectional area of water in pipe

$=$ area of sector OPQ $-$ area of $\triangle OPQ$

$$= \frac{115.99°}{360°} \times \pi(5.6604)^2 - \frac{1}{2} \times 9.6 \times 3$$
$$= \mathbf{18.0\ cm^2} \text{ (to 3 s.f.)}$$

16.



$XN = NY$ ($\perp$ bisector of chord)

$$= \frac{16}{2}$$
$$= 8 \text{ cm}$$

Let the radius of the circle be r cm.

Using Pythagoras' Theorem,

$$OX^2 = NX^2 + ON^2$$
$$r^2 = 8^2 + (r - 2)^2$$
$$= 64 + r^2 - 4r + 4$$
$$= r^2 - 4r + 68$$
$$4r = 68$$
$$r = 17$$

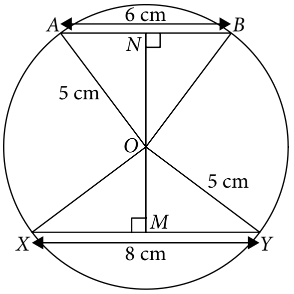
$\therefore$ the radius of the circle is **17 cm**.

17. (i) $\angle OCA = \angle OCE = 90°$ ($\perp$ bisector of chord)
Using Pythagoras' Theorem,
$OC^2 = 9^2 - 7^2$
$= 32$
$CE = AC$
$= 6 + 7$
$= 13$ cm
Using Pythagoras' Theorem,
$OE^2 = 32 + 13^2$
$= 201$
$OE = \sqrt{201}$ (since length $OE > 0$)
$= 14.177$ cm (to 5 s.f.)
$=$ **14.2 cm** (to 3 s.f.)

(ii) Shaded area $= \pi(201) - \pi(9)^2$
$=$ **377 cm²** (to 3 s.f.)

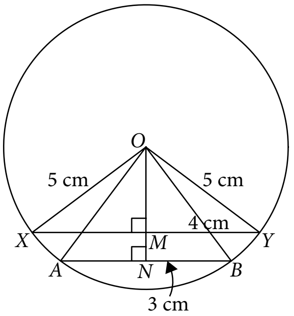
18. $AP = BP$
$= \frac{AB}{2}$
$= \frac{22}{2}$
$= 11$ cm
Using Pythagoras' Theorem,
$OA^2 = 9^2 + 11^2$
$= 202$
$OC^2 = OA^2$ (radii)
$= 202$
Using Pythagoras' Theorem,
$CQ^2 = 202 - 7^2$
$= 153$
$CQ = \sqrt{153}$ (since length $CQ > 0$)
$= 12.369$ cm (to 5 s.f.)
$CD = 2CQ$ ($\perp$ bisector of chord)
$= 2(12.369)$
$=$ **24.7 cm** (to 3 s.f.)

19. Case 1: The chords are on opposite sides of the centre *O*.



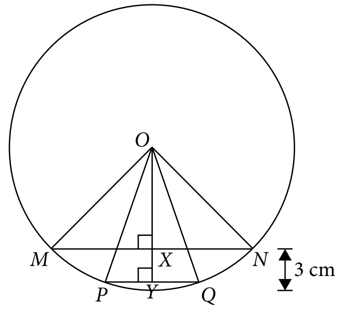
$AN = NB$ ($\perp$ bisector of chord)
$= \frac{AB}{2}$
$= \frac{6}{2}$
$= 3$ cm
Using Pythagoras' Theorem,
$ON^2 = 5^2 - 3^2$
$= 16$
$ON = \sqrt{16}$ (since length $ON > 0$)
$= 4$ cm
$XM = MY$ ($\perp$ bisector of chord)
$= \frac{XY}{2}$
$= \frac{8}{2}$
$= 4$ cm
Using Pythagoras' Theorem,
$OM^2 = 5^2 - 4^2$
$= 9$
$OM = \sqrt{9}$ (since length $OM > 0$)
$= 3$ cm
Shortest distance between the chords $= MN$
$= ON + OM$
$= 4 + 3$
$= 7$ cm

Case 2: The chords are on the same side of the centre *O*.



Shortest distance between the chords $= MN$
$= ON - OM$
$= 4 - 3$
$= 1$ cm
$\therefore$ the two possible distances between the chords are **1 cm or 7 cm**.

20.



$MX = NX$ (⊥ bisector of chord)

$= \frac{MN}{2}$

$= \frac{14}{2}$

$= 7$ cm

$PY = QY$ (⊥ bisector of chord)

$= \frac{PQ}{2}$

$= \frac{7}{2}$

$= 3.5$ cm

Let r cm be the radius of the circle and x cm be the length of OX.

Using Pythagoras' Theorem,

$OM^2 = MX^2 + OX^2$

$r^2 = 7^2 + x^2$

$r^2 = 49 + x^2$ — (1)

$OP^2 = PY^2 + OY^2$

$r^2 = 3.5^2 + (x + 3)^2$

$r^2 = 12.25 + x^2 + 6x + 9$

$r^2 = x^2 + 6x + 21.25$ — (2)

Substitute (2) into (1):

$x^2 + 6x + 21.25 = 49 + x^2$

$6x + 21.25 = 49$

$6x = 27.75$

$x = 4.625$

Substitute $x = 4.625$ into (1):

$r^2 = 49 + 4.625^2$

$= 70.390\ 625$

$r = \sqrt{70.390\ 625}$ (since length $r > 0$)

$= 8.39$ (to 3 s.f.)

∴ the radius of the circle is **8.39 cm**.

21. (i) $\angle OAP = \angle OAT = 90°$ (tangent ⊥ radius)

Since $OA = OB$ (radii), $\triangle OAB$ is an isosceles △.

$\angle OAB = \angle OBA$ (base ∠s of isos. △)

$= 46°$

$\angle BAT = 90° - 46°$

$=$ **44°**

(ii) $\angle CAT = 180° - 69°$ (adj. ∠s on a str. line)

$= 111°$

$\angle PTC = 180° - 44° - 111°$ (∠ sum of △)

$=$ **25°**

22. (i) $\angle OAC = 90°$ (tangent ⊥ radius)

Let the radius of the circle be r cm.

Using Pythagoras' Theorem,

$OC^2 = OA^2 + AC^2$

$(r + 12)^2 = r^2 + 18^2$

$r^2 + 24r + 144 = r^2 + 324$

$24r + 144 = 324$

$24r = 180$

$r = 7.5$

∴ the radius of the circle is **7.5 cm**.

(ii) $\tan \angle AOB = \frac{\text{opp}}{\text{adj}}$

$= \frac{AC}{OA}$

$= \frac{18}{7.5}$

$\angle AOB = \tan^{-1} \frac{18}{7.5}$

$= 67.380°$ (to 3 d.p.)

$=$ **67.4°** (to 1 d.p.)

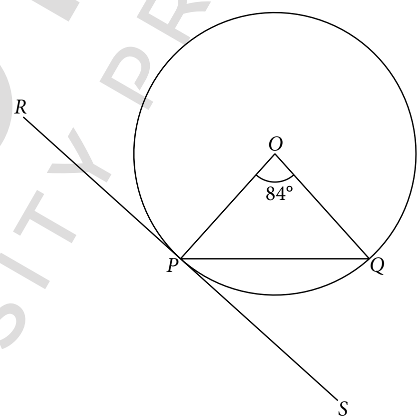
(iii) Area of shaded region

$=$ area of $\triangle OAC$ – area of sector OAB

$= \frac{1}{2} \times 18 \times 7.5 - \frac{67.380°}{360°} \times \pi(7.5)^2$

$=$ **34.4 cm²** (to 3 s.f.)

23.



$\angle OPR = \angle OPS = 90°$ (tangent ⊥ radius)

Since $OP = OQ$ (radii), $\triangle OPQ$ is an isosceles △.

$\angle OPQ = \angle OQP$ (base ∠s of isos. △)

$= \frac{180° - 84°}{2}$

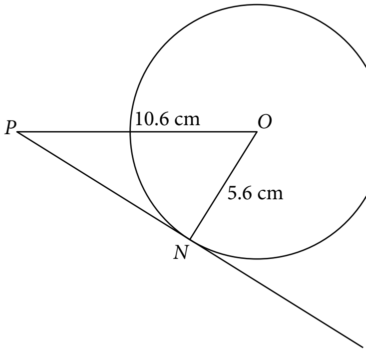
$= 48°$

$\angle QPR = 90° + 48°$

$= 138°$

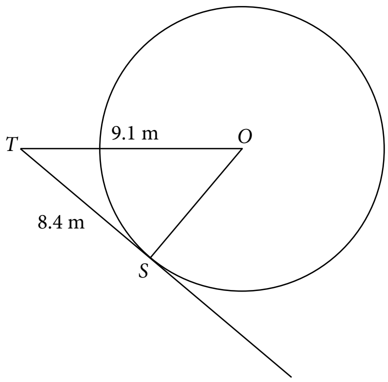
∴ the obtuse angle is **138°**.

24.



$\angle ONP = 90°$ (tangent $\perp$ radius)
Using Pythagoras' Theorem,
$PN^2 = 10.6^2 - 5.6^2$
$= 81$
$PN = \sqrt{81}$ (since length $PN > 0$)
$=$ **9 cm**

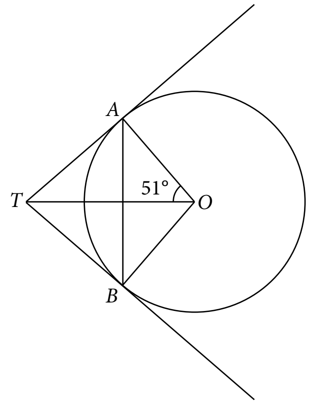
25.



$\angle OST = 90°$ (tangent $\perp$ radius)
Using Pythagoras' Theorem,
$OS^2 = 9.1^2 - 8.4^2$
$= 12.25$
$OS = \sqrt{12.25}$ (since length $OS > 0$)
$= 3.5$ m
$\therefore$ radius $= 3.5$ m
Diameter $= 3.5 \times 2$
$=$ **7 m**

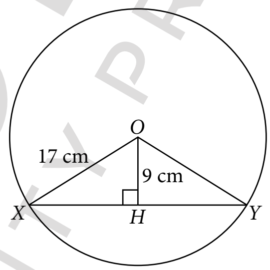
26. $\angle AOC = 180° - 122°$ (adj. $\angle$s on a str. line)
$= 58°$
$\angle AOB = \angle AOC$ (tangents from ext. pt.)
$= 58°$
$\angle OBA = 90°$ (tangent $\perp$ radius)
$\angle OAB = 180° - 58° - 90°$ ($\angle$ sum of $\triangle$)
$= 32°$
$\angle BAC = 2 \times \angle OAB$ (tangents from ext. pt.)
$= 2 \times 32°$
$=$ **64°**

27.



$\angle OAT = 90°$ (tangent $\perp$ radius)
$\angle OTA = 180° - 90° - 51°$ ($\angle$ sum of $\triangle$)
$= 39°$
$\angle BTA = 2 \times \angle OTA$ (tangents from ext. pt.)
$= 2 \times 39°$
$= 78°$
Since $AT = BT$ (tangents from ext. pt.),
$\angle BAT = \angle ABT$ (base $\angle$s of isos. $\triangle$)
$= \dfrac{180° - 78°}{2}$
$=$ **51°**

28.

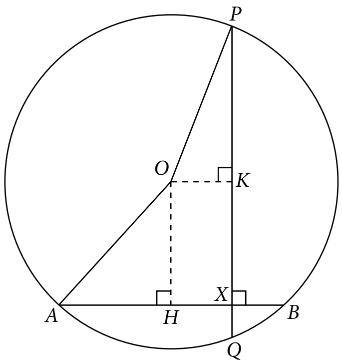


$HX = HY$ ($\perp$ bisector of chord)
Using Pythagoras' Theorem,
$HX^2 = 17^2 - 9^2$
$= 208$
$HX = \sqrt{208}$ (since length $HX > 0$)
$= 14.422$ cm (to 5 s.f.)

$\cos \angle HOX = \dfrac{\text{adj}}{\text{hyp}}$
$= \dfrac{OH}{OX}$
$= \dfrac{9}{17}$

$\angle HOX = \cos^{-1} \dfrac{9}{17}$
$= 58.034°$ (to 3 d.p.)
Similarly, $\angle HOY = 58.034°$.
$\angle XOY = 2 \times 58.034°$
$= 116.068°$
Perimeter of minor segment
$= XY +$ minor arc XY
$= (2 \times 14.422) + \left[\dfrac{116.068°}{360°} \times 2\pi(17)\right]$
$=$ **63.3 cm** (to 3 s.f.)

29.



$AH = BH$ ($\perp$ bisector of chord)

$= \frac{AB}{2}$

$= \frac{11}{2}$

$= 5.5$ cm

$PK = KQ$ ($\perp$ bisector of chord)

$= \frac{PQ}{2}$

$= \frac{13}{2}$

$= 6.5$ cm

$OA = OP$ (radii)

$= 7$ cm

Using Pythagoras' Theorem,

$OH^2 = 7^2 - 5.5^2$

$= 18.75$

$OK^2 = 7^2 - 6.5^2$

$= 6.75$

Since $OKXH$ is a rectangle,

$HX^2 = OK^2$

$= 6.75$

Using Pythagoras' Theorem,

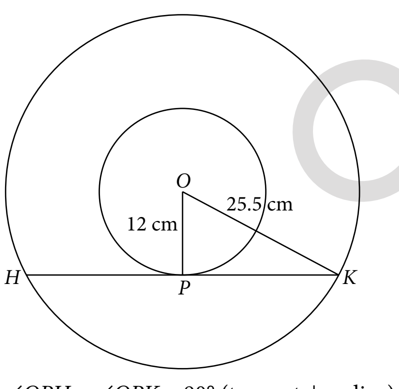
$OX^2 = 18.75 + 6.75$

$= 25.5$

$OX = \sqrt{25.5}$ (since length $OX > 0$)

$=$ **5.05 cm** (to 3 s.f.)

30.



$\angle OPH = \angle OPK = 90°$ (tangent $\perp$ radius)

$HP = KP$ ($\perp$ bisector of chord)

Using Pythagoras' Theorem,

$KP^2 = 25.5^2 - 12^2$

$= 506.25$

$KP = \sqrt{506.25}$ (since length $KP > 0$)

$= 22.5$ cm

$HK = 2 \times KP$

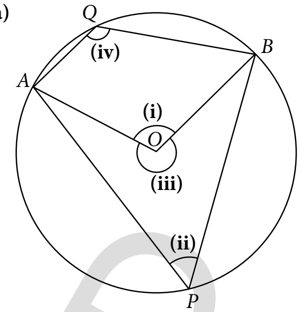
$= 2 \times 22.5$

$=$ **45 cm**

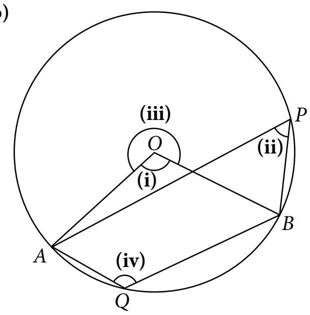
9.2 Angle properties of circles

Class Discussion (Identifying angles at centre and at circumference)

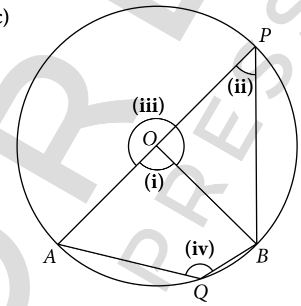
(a)



(b)



(c)



Investigation (Discovering circle angle property 1)

2.

$\angle AOB$	60°	90°	120°	145°	250°	320°
$\angle APB$	**30°**	**45°**	**60°**	**72.5°**	**125°**	**160°**
$\frac{\angle AOB}{\angle APB}$	**2**	**2**	**2**	**2**	**2**	**2**

3. $\angle AOB = 2\angle APB$

4. If the point P does not lie on the circumference of the circle, $\angle AOB \neq 2\angle APB$.

5. In general, an angle subtended at the centre of a circle is **twice** that of any angle subtended at the circumference by the same arc.

Class Discussion (Finding unknown angle using circle angle property 1)

(a) $x = 2 \times \angle APB$ ($\angle$ at centre $= 2\ \angle$ at $\odot^{ce}$)

$= 2 \times 50°$

$=$ **100°**

(b) $x = \frac{\angle AOB}{2}$ ($\angle$ at centre $= 2\ \angle$ at $\odot^{ce}$)

$= \frac{42°}{2}$

$=$ **21°**

(c) $x = 2 \times \angle APB$ ($\angle$ at centre $= 2\ \angle$ at $\odot^{ce}$)

$= 2 \times 35°$

$=$ **70°**

(d) $x = \frac{\text{reflex } \angle AOB}{2}$ ($\angle$ at centre $= 2\ \angle$ at $\odot^{ce}$)

$= \frac{276°}{2}$

$=$ **138°**

Practise Now 6

1. **(i)** $\angle SOR = 2 \times \angle SPR$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 2 \times 28°$
$= \mathbf{56°}$

(ii) Reflex $\angle POR = 180° + 56°$
$= 236°$
$\angle PQR = \dfrac{\text{reflex } \angle POR}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= \dfrac{236°}{2}$
$= \mathbf{118°}$

2. Since $OA = OB$ (radii), $\triangle OAB$ is an isosceles $\triangle$.
$\angle OAB = \angle OBA$ (base $\angle$s of isos. $\triangle$)
$= 35°$
$x = 180° - 35° - 35°$ ($\angle$ sum of $\triangle$)
$= \mathbf{110°}$
$y = \dfrac{\angle AOB}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= \dfrac{110°}{2}$
$= \mathbf{55°}$

3. $\angle ABC = 180° - 73°$ (adj. $\angle$s on a str. line)
$= 107°$
Reflex $\angle AOC = 2 \times \angle ABC$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 2 \times 107°$
$= 214°$
$\angle AOC = 360° - 214°$ ($\angle$s at a point)
$= \mathbf{146°}$

Investigation (Discovering circle angle property 2)

2. **(a)** $\angle APB = \mathbf{90°}$ **(b)** **Semicircle**
3. If the point P does not lie on the circumference of the circle, $\angle APB \neq 90°$ when $\angle AOB = 180°$.
4. In general, an angle in a semicircle is always equal to **90°**.
5. $\angle AOB = 180°$
$\angle APB = \dfrac{\angle AOB}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= \dfrac{180°}{2}$
$= 90°$ (proven)

Practise Now 7

1. $\angle PSQ = \angle RQS$ (alt. $\angle$s, SP // RQ)
$= 35°$
$\angle PQS = 90°$ (rt. $\angle$ in semicircle)
$x = 180° - 90° - 35°$ ($\angle$ sum of $\triangle$)
$= \mathbf{55°}$
$y = 2 \times \angle RQS$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 2 \times 35°$
$= \mathbf{70°}$
$\angle OSR = \angle ORS$ (base $\angle$s of isos. $\triangle$)
$= \dfrac{180° - 70°}{2}$
$= 55°$
$z = \angle OSR - \angle PSQ$
$= 55° - 35°$
$= \mathbf{20°}$

2. **(i)** $\angle PRS = 90°$ (rt. $\angle$ in semicircle)
$\angle ORP = 90° - 65°$
$= 25°$
$\angle OPR = \angle ORP$ (base $\angle$s of isos. $\triangle$)
$= \mathbf{25°}$

(ii) $\angle QPR = \angle ORP$ (alt. $\angle$s, PQ // OR)
$= 25°$
$\angle QOR = 2 \times \angle QPR$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 2 \times 25°$
$= \mathbf{50°}$

(iii) $\angle PQO = \angle QOR$ (alt. $\angle$s, PQ // OR)
$= 50°$
$\angle PXQ = 180° - 25° - 50°$ ($\angle$ sum of $\triangle$)
$= \mathbf{105°}$

Class Discussion (Angles in same segment or in opposite segments)

1. $\angle x$ **and** $\angle y$ are angles in the same segment (formed by the chord ***BE***).
$\angle w$ **and** $\angle z$ are angles in opposite segments (formed by the chord ***BF***).
2. $\angle w$ and $\angle y$ are **not** angles in opposite segments because the two segments in which the two angles lie are not formed by the same chord, i.e. the two segments are just different segments.

Investigation (Discovering circle angle property 3)

2. $\angle APB = \angle AQB$
3. In general, angles in the same segment are **equal**.
4. $\angle AOB = 2 \times \angle APB$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$\angle AOB = 2 \times \angle AQB$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$\therefore \angle APB = \angle AQB$ (proven)

Practise Now 8

1. **(i)** $\angle CDB = \angle CAB$ ($\angle$s in same segment)
$= \mathbf{44°}$

(ii) $\angle ABX = \angle ACD$ ($\angle$s in same segment)
$= \mathbf{25°}$

(iii) $\angle CXB = \angle CDX + \angle DCX$ (ext. $\angle$ of $\triangle CDX$)
$= 44° + 25°$
$= \mathbf{69°}$

2. $\angle APB = 90°$ (rt. $\angle$ in semicircle)
$\angle ABP = 180° - 45° - 90°$ ($\angle$ sum of $\triangle$)
$= 45°$
$\angle ABR = 25° + 45°$ (ext. $\angle$ of $\triangle ABQ$)
$= 70°$
$\angle PBR = \angle ABP + \angle ABR$
$= 45° + 70°$
$= 115°$
$\angle BRP = \angle BAP$ ($\angle$s in same segment)
$= 45°$
$\angle BPR = 180° - 115° - 45°$ ($\angle$ sum of $\triangle$)
$= \mathbf{20°}$

Investigation (Discovering circle angle property 4)

2. $\angle APB + \angle AQB = 180°$
3. In general, angles in opposite segments are supplementary, i.e. they add up to **180°**.

4. $\angle AOB = 2 \times \angle APB$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
Reflex $\angle AOB = 2 \times \angle AQB$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$\angle AOB$ + reflex $\angle AOB = 360°$ ($\angle$s at a point)
$2 \times \angle APB + 2 \times \angle AQB = 360°$
$2 \times (\angle APB + \angle AQB) = 360°$
$\angle APB + \angle AQB = 180°$ (proven)

Practise Now 9

1. **(i)** $\angle WZY = 180° - 67°$ ($\angle$s in opp. segments)
$= \mathbf{113°}$
(ii) $\angle WPZ = 113° - 92°$ (ext. $\angle$ of $\triangle$)
$= \mathbf{21°}$

2. $\angle POR = 180° - 42°$ ($\angle$s in opp. segments)
$= 138°$
Reflex $\angle POR = 360° - 138°$ ($\angle$s at a point)
$= 222°$
$\angle PSR = \frac{222°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= \mathbf{111°}$

Practise Now 10

1. **(i)** $\angle CBE = 180° - \angle ABE$ (adj. $\angle$s on a str. line)
$\angle ADC = 180° - (180° - \angle ABE)$ ($\angle$s in opp. segments)
$= \angle ABE$ (shown)
$\angle BED = 180° - \angle AEB$ (adj. $\angle$s on a str. line)
$\angle ACD = 180° - (180° - \angle AEB)$ ($\angle$s in opp. segments)
$= \angle AEB$ (shown)
(ii) Since $\triangle ABE$ is similar to $\triangle ADC$,
$\frac{AC}{AE} = \frac{AD}{AB}$
$\frac{3.8 + BC}{3.9} = \frac{3.9 + 7.4}{3.8}$
$= \frac{11.3}{3.8}$
$14.44 + 3.8BC = 44.07$
$3.8BC = 29.63$
$BC = \mathbf{7.80\ cm}$ (to 3 s.f.)

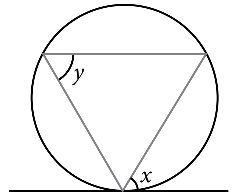
2. **(i)** $\angle QAX = \angle BPX$ ($\angle$s in same segment) (shown)
(ii) Since $\triangle AXQ$ is similar to $\triangle PXB$,
$\frac{BX}{QX} = \frac{PX}{AX}$
$\frac{BX}{10.5} = \frac{3.4}{5}$
$BX = \frac{3.4}{5} \times 10.5$
$= \mathbf{7.14\ cm}$

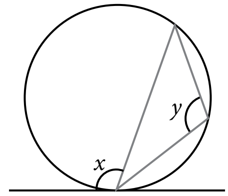
9.3 Alternate Segment Theorem

Investigation (Alternate Segment Theorem)

1. $\angle x = \angle y$
2. $\angle OAB = 90° - x$
$\angle AOB = 180° - 2(90° - x)$
$= 2x°$ ($\angle$ sum of $\triangle$)
$\angle APB = \frac{1}{2} \angle AOB$ ($\angle$ at centre = 2 $\angle$ at O^{ce})
$= x°$ (proven)

Thinking Time (Page 300)

(a)


(b)


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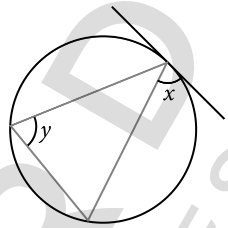

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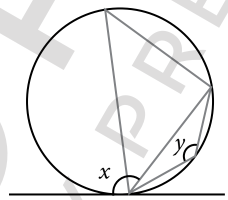


Practise Now 11

(i) $\angle BAC = 90°$ (rt. $\angle$ in semicircle)
$\angle CAT = \angle ABC$ ($\angle$s in alt. segments)
$= \mathbf{33°}$
(ii) $\angle ATC = 180° - 33° - (33° + 90°)$ ($\angle$ sum of $\triangle$)
$= \mathbf{24°}$

Exercise 9B

1. **(a)** $a° = 2 \times 40°$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 80°$
$\therefore a = \mathbf{80}$
(b) $b° = \frac{60°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 30°$
$\therefore b = \mathbf{30}$
(c) $c° = 40°$ ($\angle$s in same segment)
$\therefore c = \mathbf{40}$
(d) $d° = \frac{230°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 115°$
$\therefore d = \mathbf{115}$
(e) Reflex $\angle O = 360° - 110°$ ($\angle$s at a point)
$= 250°$
$e° = \frac{250°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)
$= 125°$
$\therefore e = \mathbf{125}$

(f) $\angle O = 2 \times 40°$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)

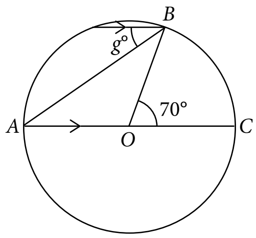
$= 80°$

$f° = \dfrac{180° - 80°}{2}$ (base $\angle$s of isos. $\triangle$)

$= 50°$

$\therefore f =$ **50**

(g)

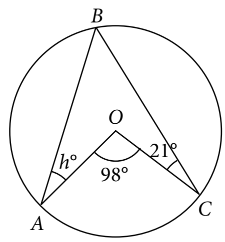


$\angle BAC = \dfrac{70°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)

$= 35°$

Since $\angle BAC = g°$ (alt. $\angle$s, // lines), $g =$ **35**.

(h)



$\angle ABC = \dfrac{98°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)

$= 49°$

Reflex $\angle AOC = 360° - 98°$ ($\angle$s at a point)

$= 262°$

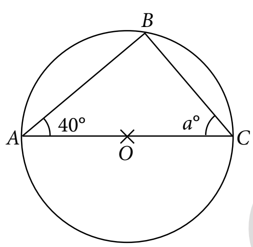
$\angle BAO$ + reflex $\angle AOC + \angle BCO + \angle ABC = 360°$ ($\angle$ sum of quad.)

$h° + 262° + 21° + 49° = 360°$

$h° = 28°$

$\therefore h =$ **28**

2. (a)



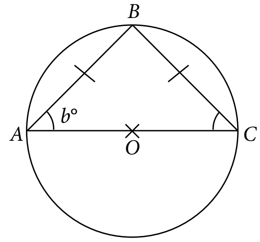
$\angle ABC = 90°$ (rt. $\angle$ in semicircle)

$a° = 180° - 90° - 40°$ ($\angle$ sum of $\triangle$)

$= 50°$

$\therefore a =$ **50**

(b)



$\angle ABC = 90°$ (rt. $\angle$ in semicircle)

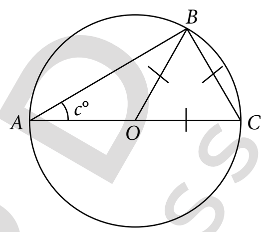
$\angle BAC = \angle BCA$ (base $\angle$s of isos. $\triangle$)

$b° = \dfrac{180° - 90°}{2}$ ($\angle$ sum of $\triangle$)

$= 45°$

$\therefore b =$ **45**

(c)



$\angle ABC = 90°$ (rt. $\angle$ in semicircle)

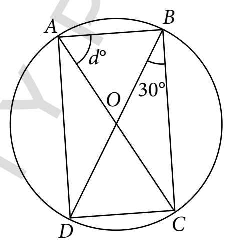
$\angle BCO = 60°$ ($\angle$s of equilateral $\triangle$)

$c° = 180° - 90° - 60°$ ($\angle$ sum of $\triangle$)

$= 30°$

$\therefore c =$ **30**

(d)



$\angle CAD = \angle CBD$ ($\angle$s in same segment)

$= 30°$

$\angle BAD = 90°$ (rt. $\angle$ in semicircle)

$d° = 90° - 30°$

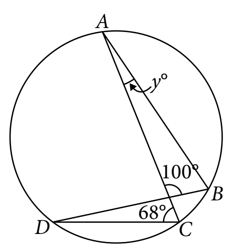
$= 60°$

$\therefore d =$ **60**

3. (a) $x° = 50°$ ($\angle$s in same segment)

$\therefore x =$ **50**

(b)



$\angle ABD = \angle ACD$ ($\angle$s in same segment)

$= 68°$

$y° = 180° - 100° - 68°$ ($\angle$ sum of $\triangle$)

$= 12°$

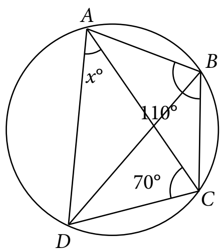
$\therefore y =$ **12**

4. $x = 2 \times 25°$ (∠ at centre = 2 ∠ at ⊙ce)
$= \mathbf{50°}$
$\mathbf{y = 25°}$ (∠s in same segment)

5. $\angle PTQ = \angle PSQ$ (∠s in same segment)
$= 20°$
$\angle PQT = 180° - 100° - 20°$ (∠ sum of △)
$= \mathbf{60°}$

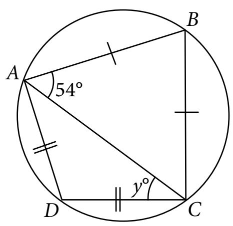
6. $\angle ADC = 180° - 65°$ (adj. ∠s on a str. line)
$= 115°$
$\angle ABC = 180° - 115°$ (∠s in opp. segments)
$= \mathbf{65°}$

7. (a)



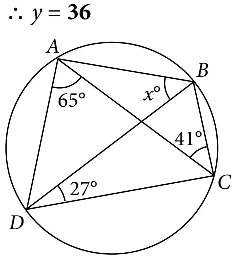
$\angle ADC = 180° - 110°$ (∠s in opp. segments)
$= 70°$
$x° = 180° - 70° - 70°$ (∠ sum of △)
$= 40°$
$\therefore x = \mathbf{40}$

(b)



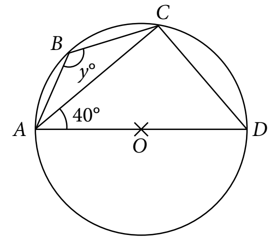
$\angle ACB = \angle BAC$ (base ∠s of isos. △)
$= 54°$
$\angle ABC = 180° - 54° - 54°$ (∠ sum of △)
$= 72°$
$\angle ADC = 180° - 72°$ (∠s in opp. segments)
$= 108°$
$\angle ACD = \angle CAD$ (base ∠s of isos. △)
$y° = \dfrac{180° - 108°}{2}$ (∠ sum of △)
$= 36°$
$\therefore y = \mathbf{36}$

(c)



$\angle ADB = \angle ACB$ (∠s in same segment)
$= 41°$
$\angle ACD = 180° - 65° - 41° - 27°$ (∠ sum of △)
$= 47°$
$x° = \angle ACD$ (∠s in same segment)
$= 47°$
$\therefore x = \mathbf{47}$

(d)



$\angle ACD = 90°$ (rt. ∠ in semicircle)
$\angle ADC = 180° - 40° - 90°$ (∠ sum of △)
$= 50°$
$y° = 180° - 50°$ (∠s in opp. segments)
$= 130°$
$\therefore y = \mathbf{130}$

8. (i) $\angle BAD = 180° - 80° - 30°$ (∠ sum of △)
$= \mathbf{70°}$

(ii) $\angle BCD = 180° - 70°$ (∠s in opp. segments)
$= 110°$
$\angle XCD = 180° - 110°$ (adj. ∠s on a str. line)
$= \mathbf{70°}$

9. $\angle PQR = 90°$ (rt. ∠ in semicircle)
$\angle PTS + \angle PRS = 180°$ (∠s in opp. segments)
$\therefore \angle PQR + \angle PRS + \angle PTS = 90° + 180°$
$= \mathbf{270°}$

10. Reflex $\angle AOC = 360° - 144°$ (∠s at a point)
$= 216°$
$\angle ADC = \dfrac{216°}{2}$ (∠ at centre = 2 ∠ at ⊙ce)
$= 108°$
$\angle BAD = 145° - 108°$ (ext. ∠ of △ADP)
$= \mathbf{37°}$

11. (i) $\angle BAC = 180° - 43° - 28°$ (∠ sum of △)
$= 109°$
Reflex $\angle BOC = 2 \times 109°$ (∠ at centre = 2 ∠ at ⊙ce)
$= 218°$
$\angle BOC = 360° - 218°$ (∠s at a point)
$= 142°$
$\angle OBC = \dfrac{180° - 142°}{2}$ (base ∠s of isos. △)
$= 19°$
$\angle OBA = 43° + 19°$
$= \mathbf{62°}$

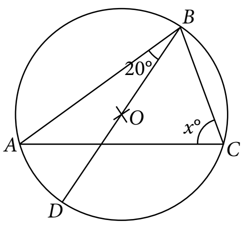
(ii) $\angle OCB = \angle OBC$ (base ∠s of isos. △)
$= 19°$
$\angle OCA = 28° + 19°$
$= \mathbf{47°}$

12. (i) $\angle PWS = 90°$ (rt. ∠ in semicircle)
$\angle PWR = 90° - 26°$
$= \mathbf{64°}$

(ii) $\angle RPW = 90°$ (rt. ∠ in semicircle)
$\angle RPS = 26°$ (∠s in same segment)
$\angle SPW = 90° - 26°$
$= \mathbf{64°}$

13. $\angle BCD = 180° - 90°$ (∠s in opp. segments)
$= 90°$
Since $\angle BCD = 90°$, BD is the diameter of the circle (rt. ∠ in semicircle).
Using Pythagoras' Theorem,
$BD^2 = 6^2 + 8^2$
$= 100$
$BD = \sqrt{100}$ (since length $BD > 0$)
$= 10$ cm
$\therefore$ radius $= \frac{10}{2}$
$= 5$ cm
Area of circle $= \pi(5)^2$
$=$ **78.5 cm²** (to 3 s.f.)

14.



$\angle BAD = 90°$ (rt. ∠ in semicircle)
$\angle ADB = 180° - 20° - 90°$ (∠ sum of △)
$= 70°$
$x° = \angle ADB$ (∠s in same segment)
$= 70°$
$\therefore x =$ **70**

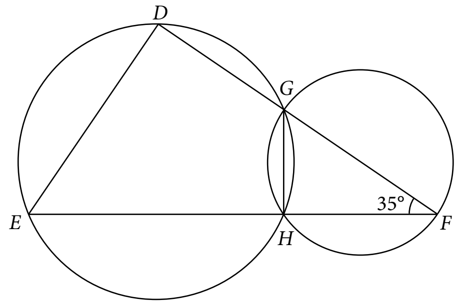
15. $\angle BXQ = 24°$ (∠s in same segment)
$\angle AQB = 90°$ (rt. ∠ in semicircle)
$\angle BQP = 180° - 90°$ (adj. ∠s on a str. line)
$= 90°$
$\angle PQX = 180° - 24° - 35°$ (∠ sum of △)
$= 121°$
$\angle BQX = 121° - 90°$
$=$ **31°**

16. $\angle ACB = 54°$ (∠s in same segment)
$\angle BCP = 180° - 58° - 54°$ (adj. ∠s on a str. line)
$= 68°$
$\angle APD = 180° - 68° - 80°$ (∠ sum of △)
$=$ **32°**

17. $\angle ADC = 90°$ (rt. ∠ in semicircle)
$\angle CDE = 180° - 90°$ (adj. ∠s on a str. line)
$= 90°$
$\angle DCE = \frac{180° - 90°}{2}$ (∠ sum of △, base ∠s of isos. △)
$= 45°$
$\angle BCD = 180° - 45°$ (adj. ∠s on a str. line)
$= 135°$
$\angle BAD = 180° - 135°$ (∠s in opp. segments)
$=$ **45°**

18. $\angle ACB = 90°$ (rt. ∠ in semicircle)
$\angle ABC = 180° - 90° - 35°$ (∠ sum of △)
$= 55°$
$\angle ADC = 180° - 55°$ (∠s in opp. segments)
$=$ **125°**

19.



(i) $\angle FHG = 90°$ (rt. ∠ in semicircle)
$\angle EHG = 180° - 90°$ (adj. ∠s on a str. line)
$= 90°$
$\angle EDG = 180° - 90°$ (∠s in opp. segments)
$=$ **90°**

(ii) $\angle DEF = 180° - 35° - 90°$ (∠ sum of △)
$=$ **55°**

20. Reflex $\angle QOR = 2 \times 110°$ (∠ at centre = 2 ∠ at ⊙ce)
$= 220°$
$\angle QOR = 360° - 220°$ (∠s at a point)
$= 140°$
$\angle QPS = 180° - 140°$ (∠s in opp. segments)
$=$ **40°**

21. **(i)** $\angle PAX = 26°$ (∠s in same segment)
$\angle BAY = 180° - 58°$ (adj. ∠s on a str. line)
$= 122°$
$\angle BQY = 180° - 122°$ (∠s in opp. segments)
$= 58°$
$\angle AQY = 23°$ (∠s in same segment)
$\angle AQB = 58° - 23°$
$=$ **35°**

(ii) $\angle BAQ = 180° - 26° - 58°$ (adj. ∠s on a str. line)
$= 96°$
$\angle ABQ = 180° - 96° - 35°$ (∠ sum of △)
$= 49°$
$\angle AYQ = 180° - 49°$ (∠s in opp. segments)
$=$ **131°**

22. **(i)** $\angle BAQ = 180° - 49° - x°$ (∠ sum of △)
$=$ **(131 – x)°**

(ii) $\angle PCB = 180° - 31° - x°$ (∠ sum of △)
$=$ **(149 – x)°**

(iii) $(131 - x)° + (149 - x)° = 180°$ (∠s in opp. segments)
$280° - 2x° = 180°$
$-2x° = -100°$
$x =$ **50**

(iv) $\angle BAQ = (131 - 50)°$
$= 81°$
$\angle PAD = 180° - 81°$ (adj. ∠s on a str. line)
$=$ **99°**

23. **(a)** $\angle OBA = \angle DCA = 90°$ (rt. ∠ in semicircle) (shown)

(b) **(i)** Diameter of smaller circle, $OA = 2 \times AP$
$= 2 \times 4$
$= 8$ cm
$OC = OA$ (radii)
$=$ **8 cm**

(ii) Since the radius of the larger circle, OA, is the diameter of the smaller circle, the length of the diameter of the larger circle is twice the length of the diameter of the smaller circle, i.e. $\frac{DA}{OA} = 2$.

Since $\triangle ABO$ is similar to $\triangle ACD$,

$$\frac{CD}{BO} = \frac{DA}{OA}$$
$$\frac{CD}{4.5} = 2$$
$$CD = 2 \times 4.5$$
$$= \mathbf{9\ cm}$$

24. (i) $\angle DAP = \angle QBP$ (corr. $\angle$s, DA // QB) (shown)

(ii) $\angle BAD = 180° - \angle DAP$ (adj. $\angle$s on a str. line)

$\angle BCP = 180° - (180° - \angle DAP)$ ($\angle$s in opp. segments)
$= \angle DAP$ (shown)

25. (i) $\angle ADC = 180° - \angle ADP$ (adj. $\angle$s on a str. line)

$\angle CBP = 180° - (180° - \angle ADP)$ ($\angle$s in opp. segments)
$= \angle ADP$ (shown)

(ii) Since $\triangle PAD$ is similar to $\triangle PCB$,

$$\frac{BC}{DA} = \frac{PC}{PA}$$
$$\frac{BC}{7} = \frac{28}{12}$$
$$BC = \frac{28}{12} \times 7$$
$$= \mathbf{16\frac{1}{3}\ cm\ or\ 16.3\ cm}\ \text{(to 3 s.f.)}$$

26. $\angle DAT = 46°$ ($\angle$s in alt. segments)

$\angle CAT = \angle CBA$ ($\angle$s in alt. segments)
$= 180° - 2 \times 41°$
$= 98°$

$\angle ATC = 180° - 98° - 46°$ ($\angle$ sum of $\triangle$)
$= \mathbf{36°}$

27. $\angle ACD = 180° - 18° - 90°$ ($\angle$ sum of $\triangle$)
$= 72°$

$\angle AEB = 72°$ ($\angle$s in same segment)

$\angle ABE = 90°$ (rt. $\angle$ in semicircle)

$\angle BAE = 180° - 90° - 72°$ ($\angle$ sum of $\triangle$)
$= \mathbf{18°}$

28. $\angle APQ = \angle ABQ$
$= \frac{70°}{2}$ ($\angle$s in same segment)
$= 35°$

$\angle APR = \angle ACR$
$= \frac{60°}{2}$ ($\angle$s in same segment)
$= 30°$

$\angle P = 35° + 30°$
$= \mathbf{65°}$

$\angle BQR = \angle BCR$
$= \frac{60°}{2}$ ($\angle$s in same segment)
$= 30°$

$\angle BQP = \angle BAP$
$= \frac{50°}{2}$ ($\angle$s in same segment)
$= 25°$

$\angle Q = 30° + 25°$
$= \mathbf{55°}$

$\angle R = 180° - 65° - 55°$ ($\angle$ sum of $\triangle$)
$= \mathbf{60°}$

29. (i) $\angle ACB = 90°$ (rt. $\angle$ in semicircle)

$\angle ACK = 90° - \angle BCK$
$= \angle CBK$ ($\angle$ sum of $\triangle BCK$) (shown)

(ii) Since $\triangle ACK$ is similar to $\triangle CBK$,

$$\frac{BK}{CK} = \frac{CK}{AK}$$
$$\frac{BK}{10} = \frac{10}{12}$$
$$BK = \frac{10}{12} \times 10$$
$$= 8\frac{1}{3}\ \text{cm}$$
$$AB = 12 + 8\frac{1}{3}$$
$$= 20\frac{1}{3}\ \text{cm}$$
$$\text{Radius} = \frac{20\frac{1}{3}}{2}$$
$$= \mathbf{10\frac{1}{6}\ cm\ or\ 10.2\ cm}\ \text{(to 3 s.f.)}$$

Chapter 10 Geometrical Transformation

TEACHING NOTES

Suggested Approach

This topic deals with spatial visualisation and teachers would be able to find many examples in the surroundings. Teachers should make use of these everyday examples to help students understand transformations.

Section 10.1: Reflection

Students who study Physics would be familiar with the idea of reflection in terms of light. Teachers may further explore their understanding of it and introduce mathematical terms pertaining to motion geometry.

An interesting method to initiate the notion of an invariant point would be to ask students to picture themselves standing in front of the mirror. Teachers can prompt them by asking them to think about how far their images behind the mirror is compared to how far they are from the mirror; and what happens if they move closer to the mirror. Teachers can further suggest that students hold their palms in front of their faces and simulate it as a mirror, and ask what would happen when their noses touch the mirror i.e. pulling their palms to touch the tip of their noses. Students should be able to deduce that the image of the nose tip is the nose tip itself.

Section 10.2: Rotation

Teachers should highlight the importance of providing exact specifications for transformations. In the case of a rotation, the centre of rotation needs to be specified. The importance of specifying the centre can be illustrated by calling up students to stretch out an arm each, and rotate it 90 clockwise. A few possibilities would arise as some might rotate their arms such that the centre of rotation is the shoulder, or at the elbow joints, or with their wrists at the centre of rotation. Students should be able to conclude that a rotation must include the centre, direction and angle of rotation.

Section 10.3: Translation

Teachers can arouse the curiosity of students by highlighting the applications of translation in art, which in turn link back to Mathematics (see Journal Writing on page 321 of the textbook). Students should also be aware that a single transformation can produce the equivalence of two or more successive transformations (see Thinking Time on page 321 of the textbook). To help students appreciate the application of translation, teachers can also encourage students to contribute ideas and share their insights with their peers.

Section 10.4: Enlargement

Similar to the previous section, teachers may get students to list real-life applications of enlargement, so as to have students appreciate the applications of enlargement. Teachers can revise the construction steps needed for the enlargement of a given figure with a positive scale factor, followed by those of a negative scale factor, and the construction steps involved in finding the centre of enlargement as well as the scale factor if given the original figure and its image.

Section 10.5: Combined transformations

In this section, students will learn to apply what they have learnt in this chapter to perform a series of transformations in succession. They will also learn to describe successive transformations which will map one shape to another.

Some students may also incorrectly believe that transformations are commutative in general. As such, teachers should give students an opportunity to explore and find out (see Thinking Time on page 333).

Introductory Problem

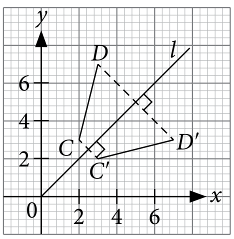
The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 2).

10.1 Reflection

Practise Now 1A

$$\text{Midpoint of } CC' = \left(\frac{2+3}{2}, \frac{3+2}{2}\right) = (2.5, 2.5)$$

$$\text{Midpoint of } DD' = \left(\frac{3+7}{2}, \frac{7+3}{2}\right) = (5, 5)$$



$$\text{Gradient of } l,\ m = \frac{5-2.5}{5-2.5} = 1$$

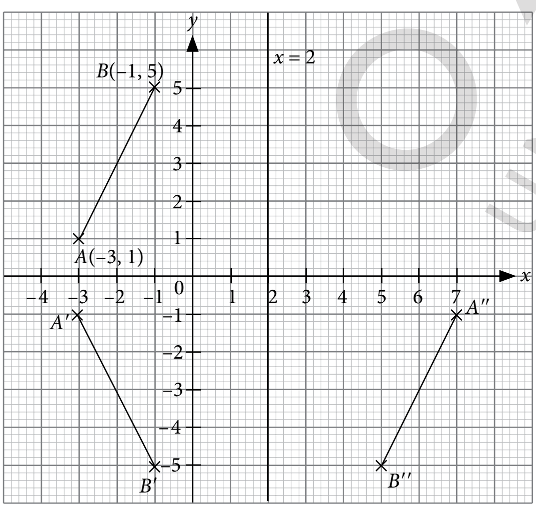
y-intercept, $c = 0$

$\therefore$ the equation of the line of reflection is $\boldsymbol{y = x}$.

Thinking Time (Page 312)

The two points $M_mM_l(A)$ and $M_lM_m(A)$ are **different**.

Practise Now 1B



(i) From the graph, the coordinates are ***A*′(−3, −1)** and ***B*′(−1, −5)**.

(ii) From the graph, the coordinates are ***A*″(7, −1)** and ***B*″(5, −5)**.

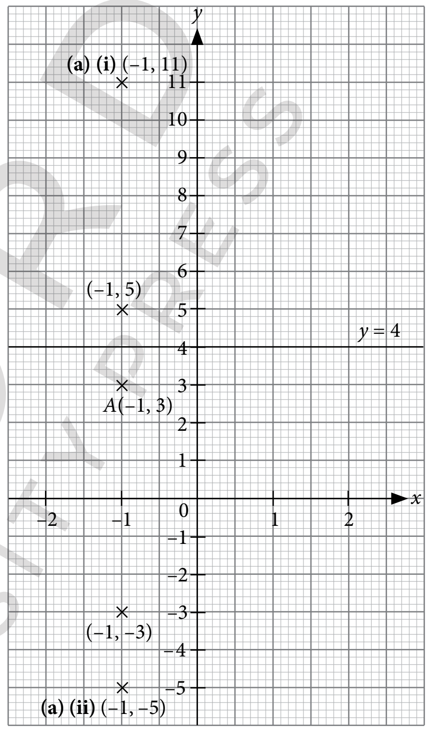
Exercise 10A

1. A reflection of a point in the x-axis brings about the negative of the original y-coordinate. A reflection in the y-axis results in the negative of the original x-coordinate. A reflection in the line $y = x$ results in the value of the x-coordinate becoming the y-coordinate and vice versa.

   (a) **(i)** **(3, −4)** **(ii)** **(−1, −3)** **(iii)** **(3, −3)**
   (iv) **(−3, 4)** **(v)** **(3, 2)** **(vi)** **(*p*, −*q*)**

   (b) **(i)** **(−3, 4)** **(ii)** **(1, 3)** **(iii)** **(−3, 3)**
   (iv) **(3, −4)** **(v)** **(−3, −2)** **(vi)** **(−*p*, *q*)**

   (c) **(i)** **(4, 3)** **(ii)** **(3, −1)** **(iii)** **(3, 3)**
   (iv) **(−4, −3)** **(v)** **(−2, 3)** **(vi)** **(*q*, *p*)**

2. 

   (a) **(i)** **(−1, 11)** **(ii)** **(−1, −5)**

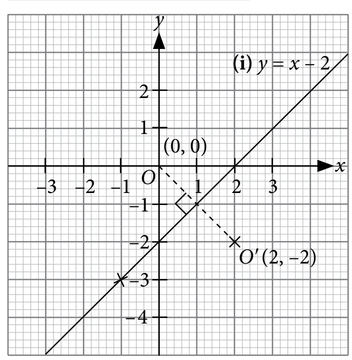
   (b) **No**

3. Reflection in the line $x = 2$ results in the x-coordinate changing by $2 - 3 = -1$.

   $\therefore$ coordinates of the reflection of the point are **(1, 2)**.

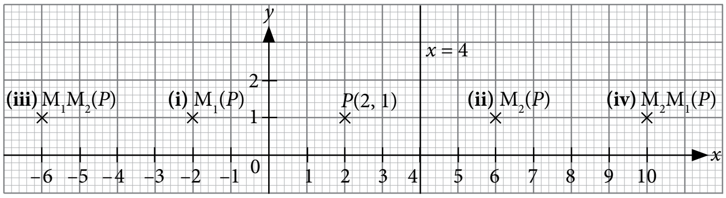
4. **(i)** $y = x - 2$

x	−1	0	2
y	−3	−2	0



   (ii) **(2, −2)**

5.



(i) $M_1(P)$ is obtained through a reflection in the y-axis. From the graph, the coordinates are **(–2, 1)**.

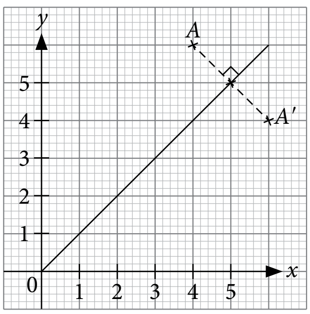
(ii) $M_2(P)$ is obtained through a reflection in the line $x = 4$. From the graph, the coordinates are **(6, 1)**.

(iii) $M_1M_2(P)$ is obtained through a reflection in the line $x = 4$ followed by a reflection in the y-axis. From the graph, the coordinates are **(–6, 1)**.

(iv) $M_2M_1(P)$ is obtained through a reflection in the y-axis followed by a reflection in the line $x = 4$. From the graph, the coordinates are **(10, 1)**.

6. **(i)** Let (3, 5) be the point A, and (5, 3) be the point A'.

$$\text{Midpoint of } AA' = \left(\frac{3+5}{2}, \frac{5+3}{2}\right) = (4, 4)$$



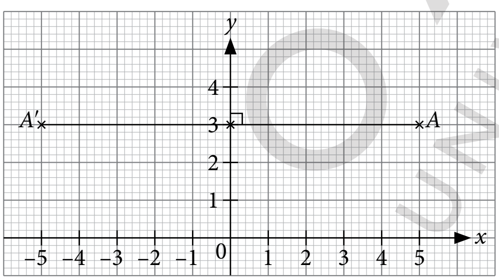
Gradient of line = 1

y-intercept = 0

∴ the equation of the line of reflection is $\boldsymbol{y = x}$.

(ii) Let (5, 3) be the point A, and (–5, 3) be the point A'.

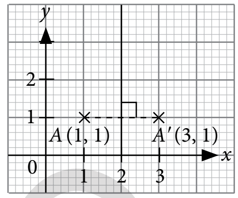
$$\text{Midpoint of } AA' = \left(\frac{5+(-5)}{2}, \frac{3+3}{2}\right) = (0, 3)$$



Gradient of line = undefined

∴ the equation of the line of reflection is $\boldsymbol{x = 0}$.

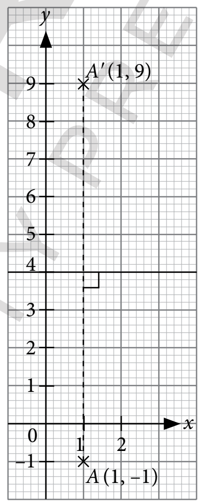
7. **(a)** $\text{Midpoint of } AA' = \left(\frac{1+3}{2}, \frac{1+1}{2}\right) = (2, 1)$



Gradient of line = undefined

∴ the equation of the line of reflection is $\boldsymbol{x = 2}$.

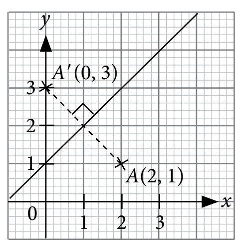
(b) $\text{Midpoint of } AA' = \left(\frac{1+1}{2}, \frac{-1+9}{2}\right) = (1, 4)$



Gradient of line = 0

∴ the equation of the line of reflection is $\boldsymbol{y = 4}$.

(c) $\text{Midpoint of } AA' = \left(\frac{2+0}{2}, \frac{1+3}{2}\right) = (1, 2)$



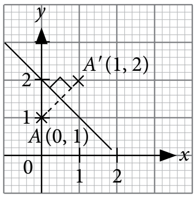
Gradient of line = 1

y-intercept = 1

∴ the equation of the line of reflection is $\boldsymbol{y = x + 1}$.

(d) Midpoint of $AA' = \left(\frac{0+1}{2}, \frac{1+2}{2}\right)$

$= (0.5, 1.5)$



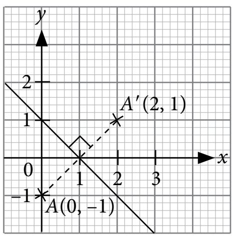
Gradient of line = –1

y-intercept = 2

$\therefore$ the equation of the line of reflection is $\boldsymbol{y = -x + 2}$.

(e) Midpoint of $AA' = \left(\frac{0+2}{2}, \frac{-1+1}{2}\right)$

$= (1, 0)$



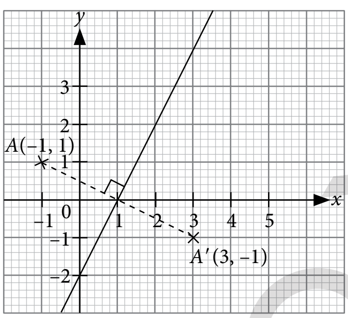
Gradient of line = –1

y-intercept = 1

$\therefore$ the equation of the line of reflection is $\boldsymbol{y = -x + 1}$.

(f) Midpoint of $AA' = \left(\frac{-1+3}{2}, \frac{1+(-1)}{2}\right)$

$= (1, 0)$

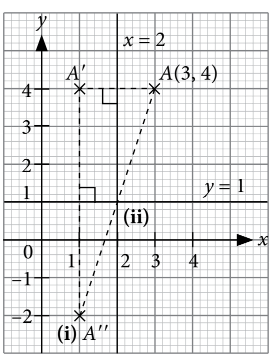


Gradient of line = 2

y-intercept = –2

$\therefore$ the equation of the line of reflection is $\boldsymbol{y = 2x - 2}$.

8. **(i)**

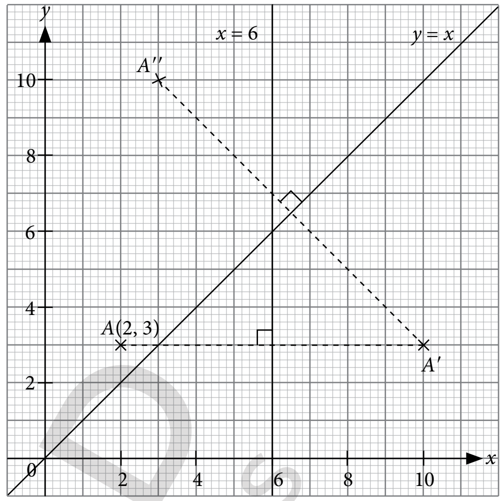


From the graph, the coordinates of the image are **(1, –2)**.

(ii) The point which remains invariant lies on both lines of reflection.

$\therefore$ the coordinates of the point are **(2, 1)**.

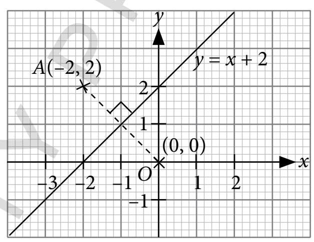
9.



From the graph, the coordinates of the image are **(3, 10)**.

10. **(i)** $y = x + 2$

x	–2	0	1
y	0	2	3

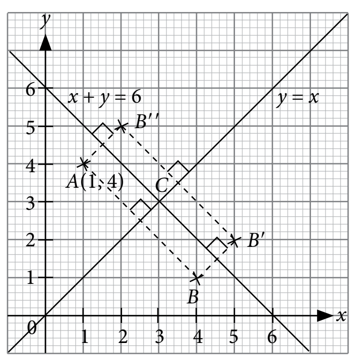


(ii) From the graph, the coordinates of A are **(–2, 2)**.

11. $x + y = 6$

$y = -x + 6$

(i)



(a) From the graph, the coordinates of the image are **(5, 2)**.

(b) From the graph, the coordinates of the image are **(5, 2)**.

(ii) **Yes**

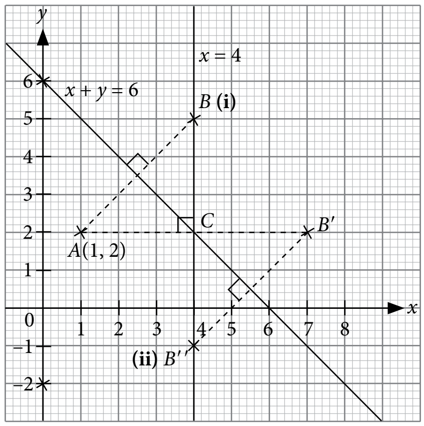
(iii) The point which remains invariant lies on both lines of reflection.

$\therefore$ the coordinates of the point are **(3, 3)**.

12. $x + y = 6$

$y = -x + 6$

(i)



(a) From the graph, the coordinates of the image are **(4, 5)**.

(b) From the graph, the coordinates of the image are **(4, –1)**.

(ii) **No**

(iii) The point which remains invariant lies on both lines of reflection.

$\therefore$ the coordinates of the point are **(4, 2)**.

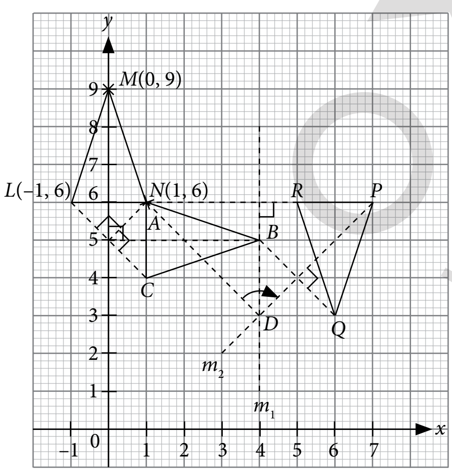
10.2 Rotation

Thinking Time (Page 316)

No. Reflection does not preserve orientation, and in this case, the rotation preserves orientation. A reflection of R in the line $x = 0$ gives the same R', but the results for Q and S are incorrect.

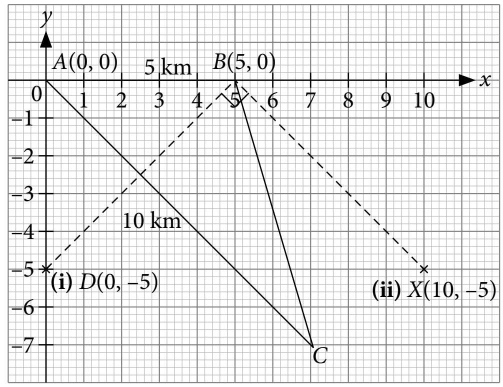
Practise Now 2

(a)



(b) From the graph, the centre of rotation is **(4, 3)** and the angle of rotation is **90° clockwise**.

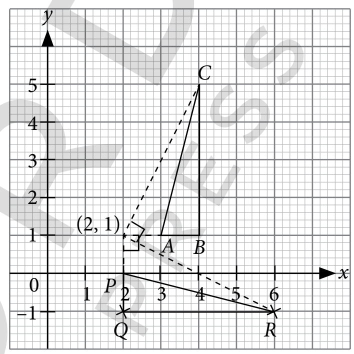
Introductory Problem Revisited



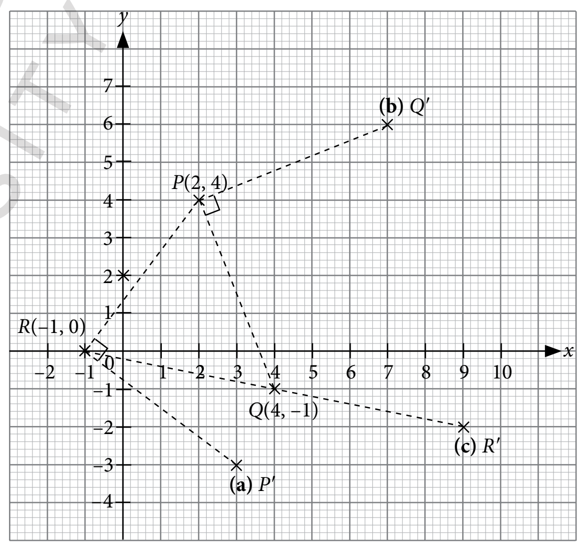
$\therefore$ the treasure is located at **(10, –5)**.

Exercise 10B

1.



2.

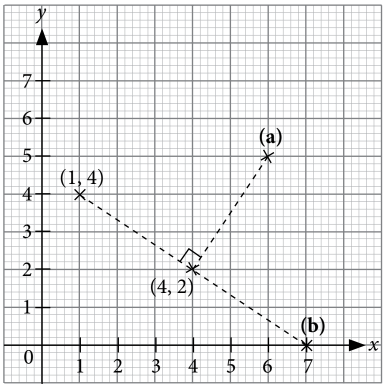


(a) From the graph, the coordinates of the image are **(3, –3)**.

(b) From the graph, the coordinates of the image are **(7, 6)**.

(c) From the graph, the coordinates of the image are **(9, –2)**.

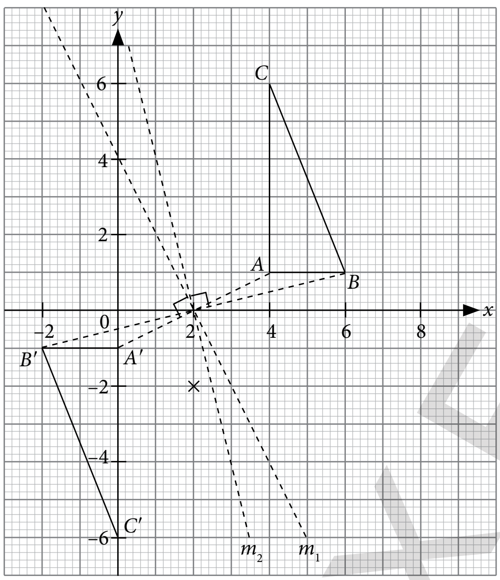
3.



(a) From the graph, the coordinates of the image are **(6, 5)**.
(b) From the graph, the coordinates of the image are **(7, 0)**.

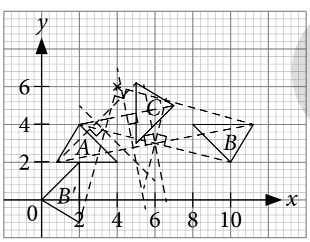
4. Since R represents an anticlockwise rotation of 240° about the origin, R^2 represents an $(240° \times 2) - 360° = 480° - 360°$ $=$ **120° anticlockwise rotation about the origin**. Then, R^4 is an $120° \times 2 =$ **240° anticlockwise rotation about the origin**.

5. **(i)**



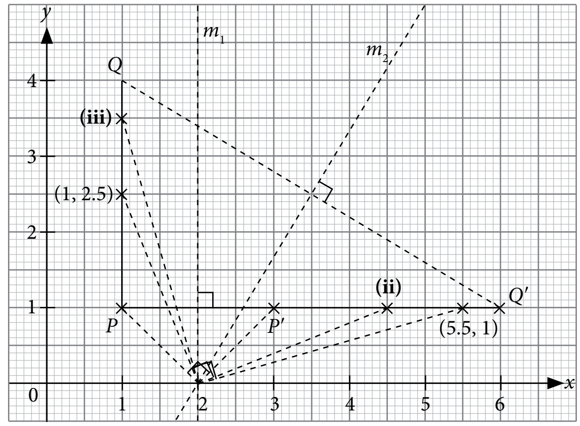
(ii) From the graph, the centre of rotation is **(2, 0)** and the angle of rotation is **180°**.

6.



(a) **(i)** From the graph, the centre of rotation is **(6, 3)**.
(ii) From the graph, the angle of rotation is **180°**.
(b) **(i)** From the graph, the centre of rotation is **(5, 2)**.
(ii) From the graph, the angle of rotation is **90° anticlockwise**.
(c) From the graph, the coordinates of the image are **(2, 2)**, **(0, 0)** and **(2, –1.2)**.

7.



(i) From the graph, the coordinates of Q' are (6, 1).
$\therefore k =$ **6**

(ii) From the graph, the line m_1 is the perpendicular bisector of PP' while the line m_2 is the perpendicular bisector of QQ'. The point of intersection of these two perpendicular bisectors gives the centre of rotation, (2, 0). The angle of rotation is 90° clockwise.
$\therefore$ the coordinates of the image are **(4.5, 1)**.

(iii) From the graph, the coordinates of the point are **(1, 3.5)**.

8. **(i)** $AC = A'C$
$\therefore \triangle CAA'$ is an isosceles triangle.
$A'\hat{C}A = 25°$
$C\hat{A}A' = \dfrac{180° - 25°}{2}$ (base ∠s of isos. △)
$= \mathbf{78°}$ (to the nearest degree)

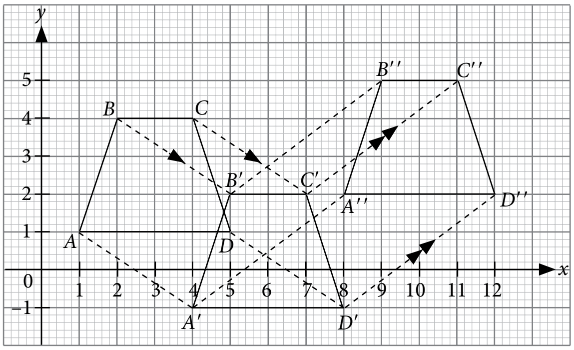
(ii) $A'\hat{C}B' = A\hat{C}B$
$= \tan^{-1} \dfrac{60}{40}$
$= 56.310°$ (to 3 d.p.)
$A\hat{C}B' = 56.310° - 25°$
$= \mathbf{31°}$ (to the nearest degree)

10.3 Translation

Thinking Time (Page 319)

No, there would be no invariant points as each point is translated in the same distance.

Practise Now 3



$$T_1(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$T_1(B) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$T_1(C) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$T_1(D) = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

∴ the vertices of the image are $A'(4, -1)$, $B'(5, 2)$, $C'(7, 2)$ and $D'(8, -1)$.

$$T_2(A') = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

$$T_2(B') = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$$

$$T_2(C') = \begin{pmatrix} 7 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \end{pmatrix}$$

$$T_2(D') = \begin{pmatrix} 8 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}$$

∴ the vertices of the image are $\boldsymbol{A''(8, 2)}$, $\boldsymbol{B''(9, 5)}$, $\boldsymbol{C''(11, 5)}$ and $\boldsymbol{D''(12, 2)}$.

Thinking Time (Page 321)

(a) **Yes**

(b) $\begin{pmatrix} \mathbf{6} \\ \mathbf{3} \end{pmatrix}$

Journal Writing (Page 321)

Teachers may show students some art pieces of Escher's tessellations and illustrate how transformation was used. Students can also find out more on their own through books in the library or the Internet, where there are many documented records on his impressive tessellations.

Practise Now 4

(i) $A' = T_1(A)$

$$= \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

∴ the coordinates of the image are **(6, 9)**.

(ii) $T_2 = \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{5} \end{pmatrix}$

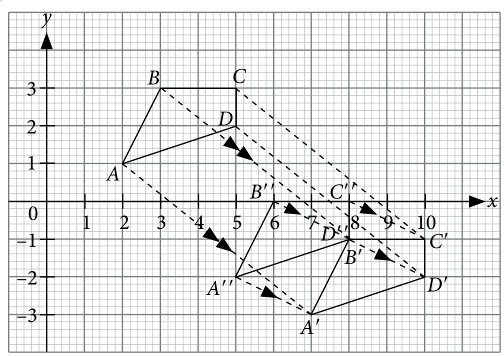
(iii) $B' = T_2(B)$

$$= \begin{pmatrix} 6 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

∴ the coordinates of the image are **(8, 2)**.

Exercise 10C

1.



∴ the coordinates are $\boldsymbol{A''(5, -2)}$, $\boldsymbol{B''(6, 0)}$, $\boldsymbol{C''(8, 0)}$ and $\boldsymbol{D''(8, -1)}$.

2. $P' = T(P)$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$Q' = T(Q)$

$= \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 10 \\ 3 \end{pmatrix}$

$R' = T(R)$

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

∴ the coordinates are ***P*′(4, 1)**, ***Q*′(10, 3)** and ***R*′(5, –2)**.

3. $T = \begin{pmatrix} 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} \mathbf{-4} \\ \mathbf{5} \end{pmatrix}$

$P = \begin{pmatrix} -1 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

∴ the coordinates of *P* are **(–5, 0)**.

4. $T_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$T_2 = \begin{pmatrix} 4 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -10 \end{pmatrix}$

(a) Image under $T_1 = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

∴ the coordinates of the image are **(4, 10)**.

(b) Image under $T_2 = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ -10 \end{pmatrix}$

$= \begin{pmatrix} 13 \\ -4 \end{pmatrix}$

∴ the coordinates of the image are **(13, –4)**.

(c) Image under T_1T_2 = Image under $T_2 + \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 13 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

∴ the coordinates of the image are **(10, 0)**.

(d) Image under T_2T_1 = Image under $T_1 + \begin{pmatrix} 6 \\ -10 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 10 \end{pmatrix} + \begin{pmatrix} 6 \\ -10 \end{pmatrix}$

$= \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

∴ the coordinates of the image are **(10, 0)**.

(e) Image under $T_1^2 = \begin{pmatrix} 7 \\ 6 \end{pmatrix} + 2\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 6 \end{pmatrix} + \begin{pmatrix} -6 \\ 8 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 14 \end{pmatrix}$

∴ the coordinates of the image are **(1, 14)**.

5. **(i)** $A' = T(A)$

$= \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

∴ the coordinates of the image are **(4, 8)**.

(ii) $T(B) = A$

$\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

∴ $p = \mathbf{0}$ and $q = \mathbf{0}$

(iii) $T^2(A) = C$

$\begin{pmatrix} 2 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix}$

$\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix}$

$\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$

∴ $h = \mathbf{6}$ and $k = \mathbf{12}$

(iv) Let the coordinates of *D* be (*x*, *y*).

$T^2(D) = A$

$\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

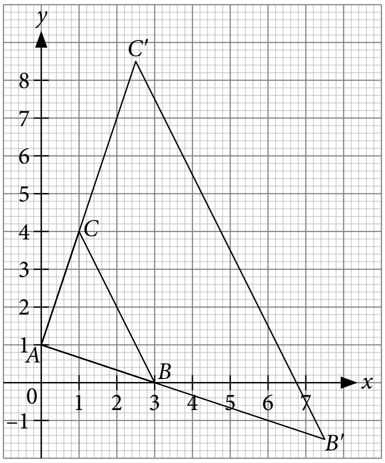
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

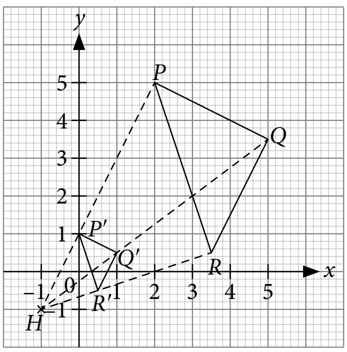
∴ the coordinates are ***D*(–2, –4)**.

10.4 Enlargement

Practise Now 5



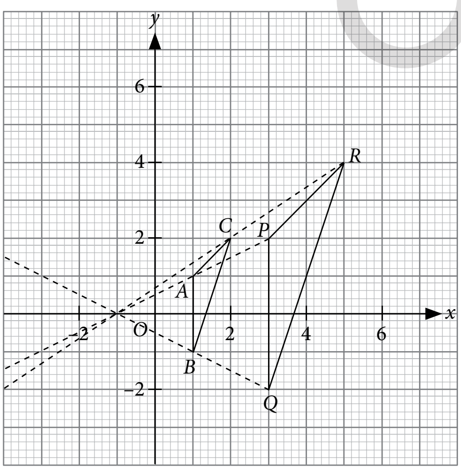
Practise Now 6



Class Discussion (Enlargement in our surroundings)

Teachers can come up with an example of enlargement in the classroom, for example, an A5 notebook compared to an A4 one, before getting students to build on this and discuss with each other.

Practise Now 7



(a) From the graph, the coordinates of the centre of enlargement are **(–1, 0)**.

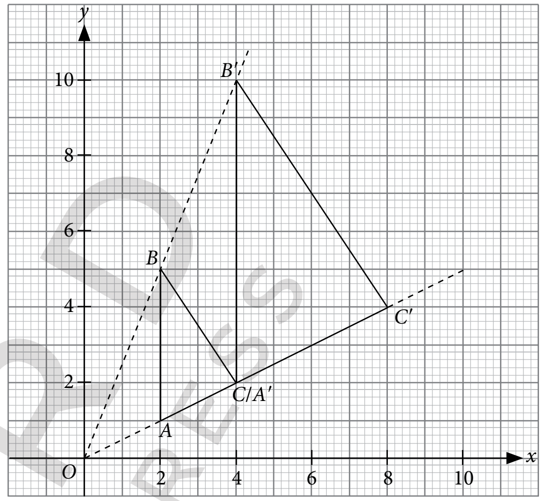
(b) Scale factor $= \dfrac{PQ}{AB}$

$= \dfrac{4}{2}$

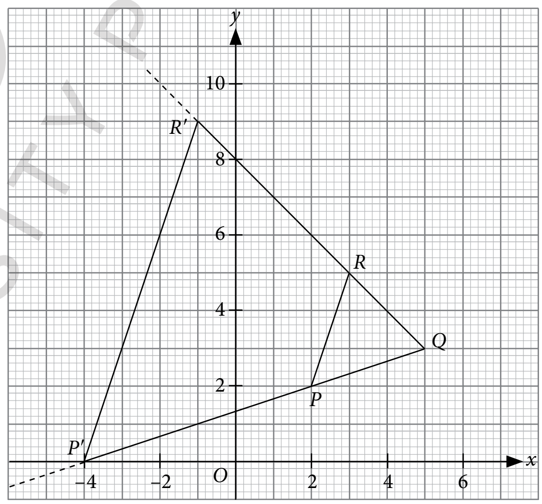
$= \mathbf{2}$

Exercise 10D

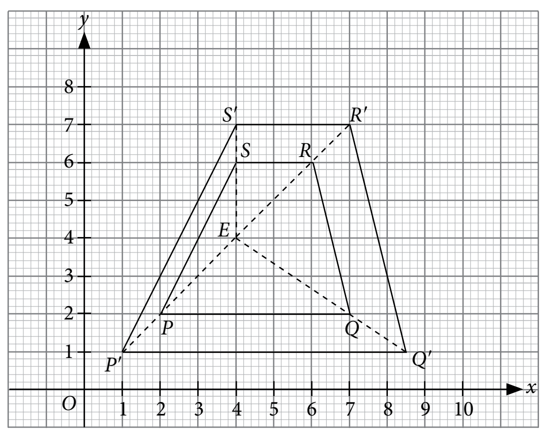
1.



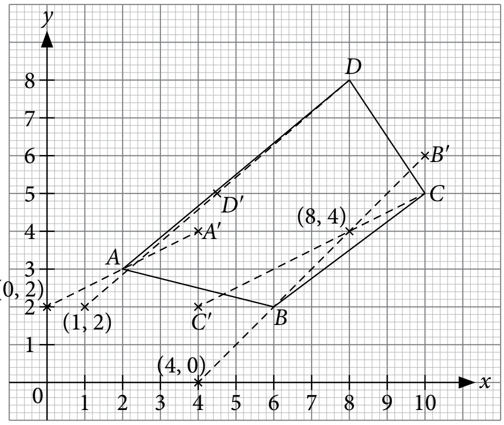
2.



3.



4.



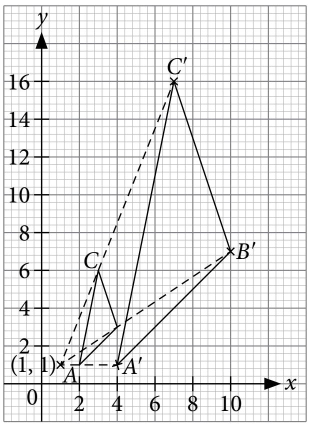
(a) From the graph, the coordinates of the image are **(4, 4)**.

(b) From the graph, the coordinates of the image are **(10, 6)**.

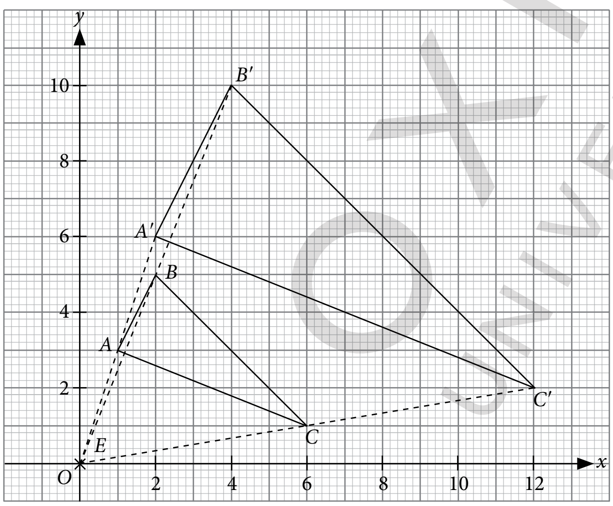
(c) From the graph, the coordinates of the image are **(4, 2)**.

(d) From the graph, the coordinates of the image are **(4.5, 5)**.

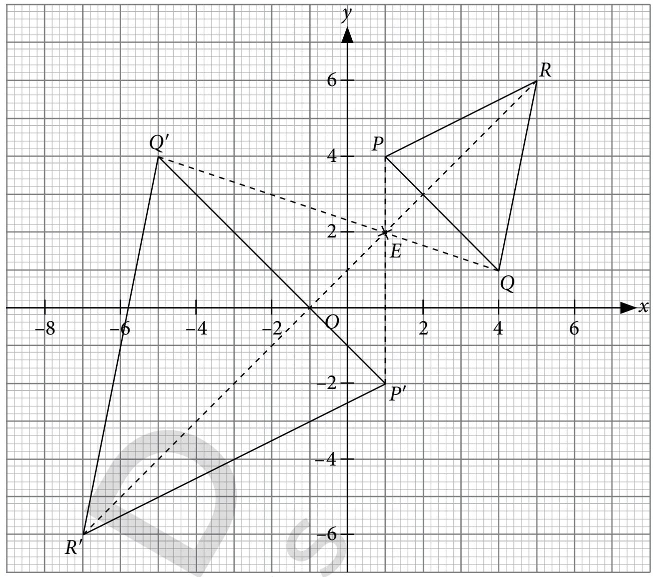
5.



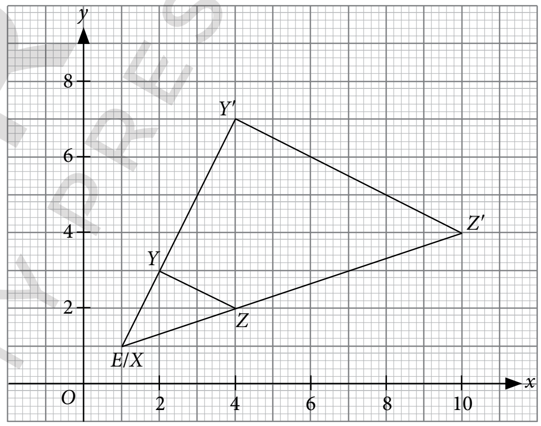
6. **(a)**



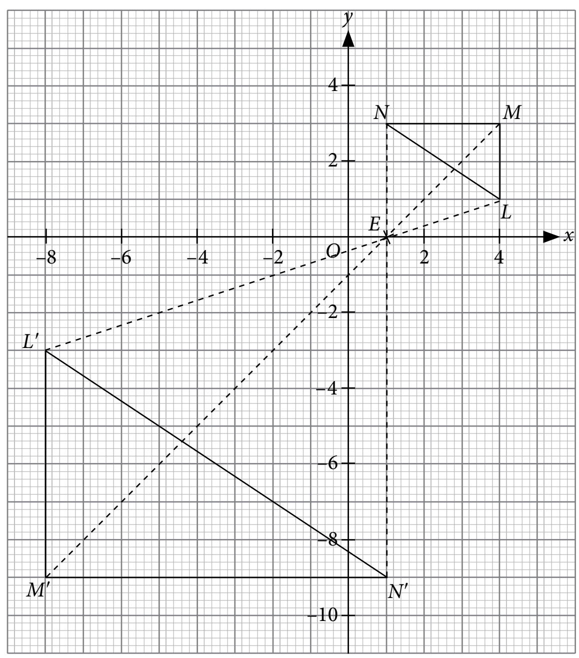
(b)



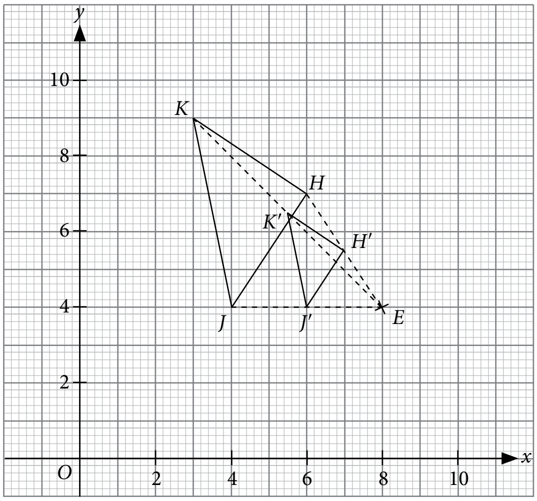
(c)



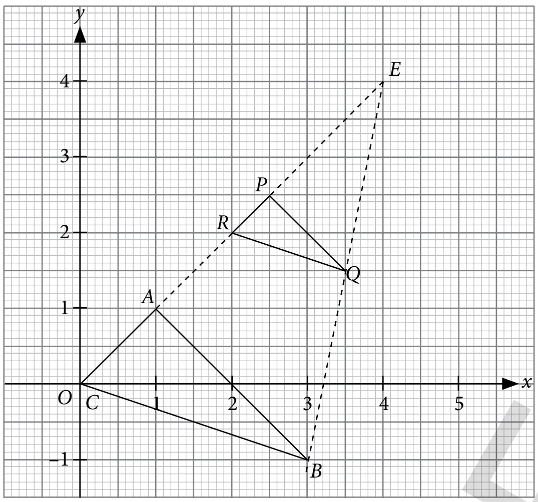
(d)



(e)

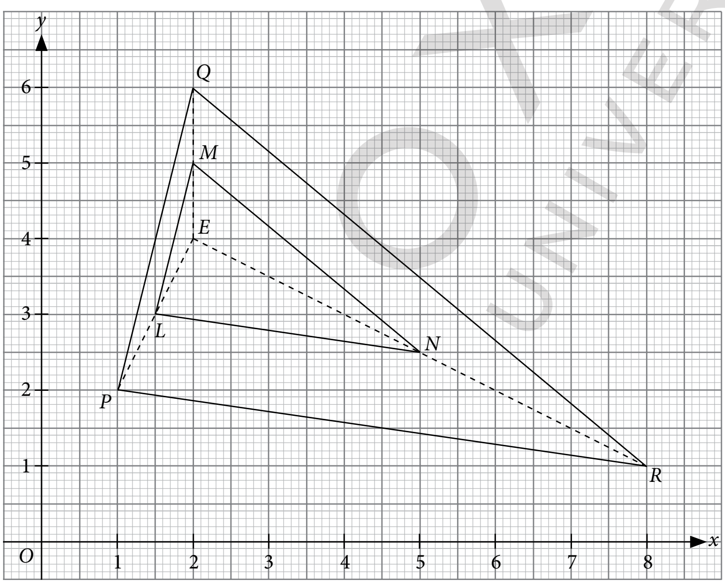


7.



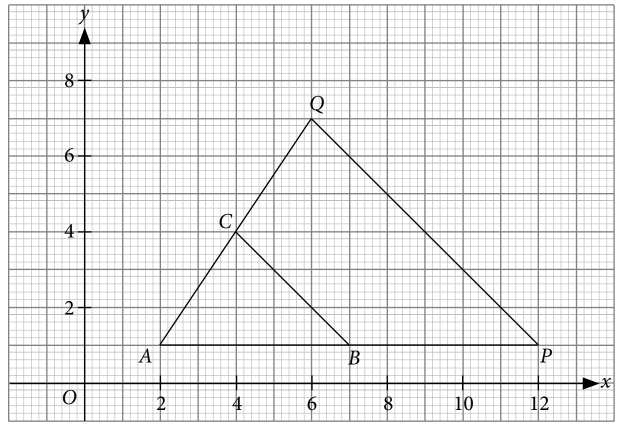
From the graph, the coordinates are ***P*****(2.5, 2.5)**, ***Q*****(3.5, 1.5)** and ***R*****(2, 2)**.

8.



From the graph, the coordinates are ***L*****(1.5, 3)**, ***M*****(2, 5)** and ***N*****(5, 2.5)**.

9.



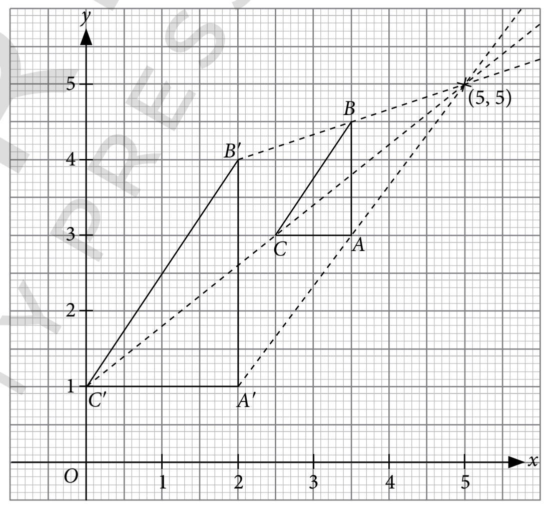
(i) From the graph, the coordinates of the centre of enlargement are **(2, 1)**.

(ii) From the graph, the coordinates are ***P*****(12, 1)** and ***Q*****(6, 7)**.

10. **(a)** $\triangle ABC$

(b) ***PBQR***

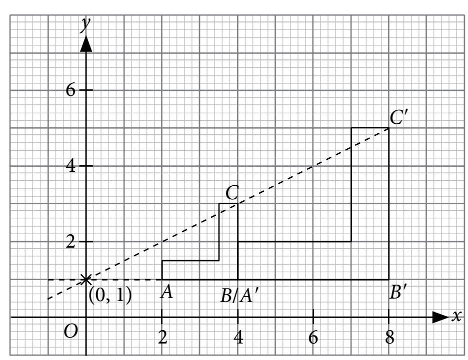
11. **(a)**



From the graph, the coordinates of the centre of enlargement are **(5, 5)**.

$$\text{Scale factor} = \frac{A'B'}{AB} = \frac{3}{1.5} = \mathbf{2}$$

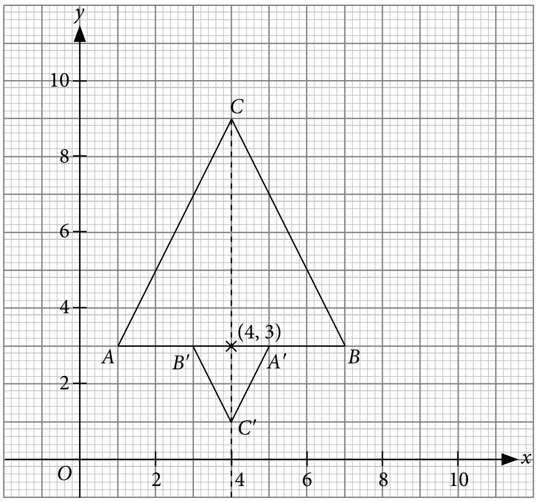
(b)



From the graph, the coordinates of the centre of enlargement are **(0, 1)**.

$$\text{Scale factor} = \frac{C'B'}{CB} = \frac{4}{2} = \mathbf{2}$$

(c)



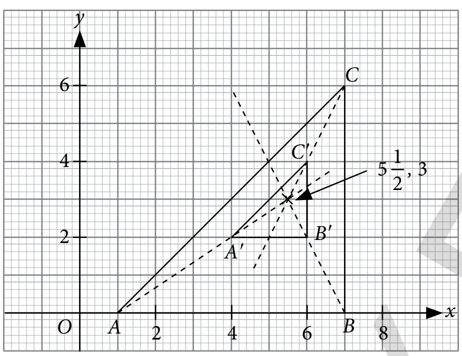
From the graph, the coordinates of the centre of enlargement are **(4, 3)**.

Scale factor $= -\frac{A'B'}{AB}$

$= -\frac{2}{6}$

$= \mathbf{-\frac{1}{3}}$

(d)



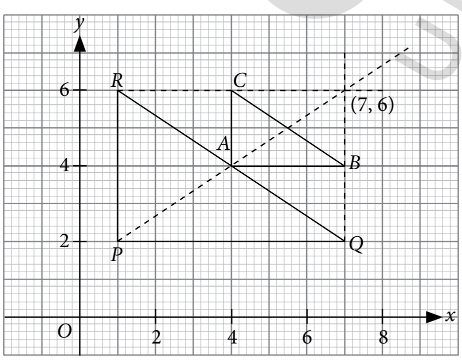
From the graph, the coordinates of the centre of enlargement are **(5.5, 3)**.

Scale factor $= \frac{A'B'}{AB}$

$= \frac{2}{6}$

$= \mathbf{\frac{1}{3}}$

12. (a)



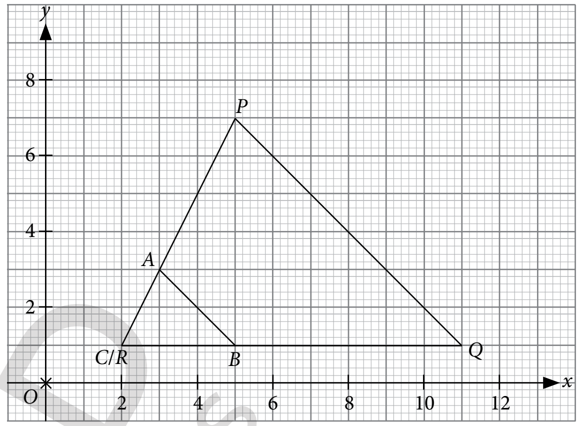
From the graph, the coordinates of the centre of enlargement are **(7, 6)**.

Scale factor $= \frac{PQ}{AB}$

$= \frac{6}{3}$

$= \mathbf{2}$

(b)



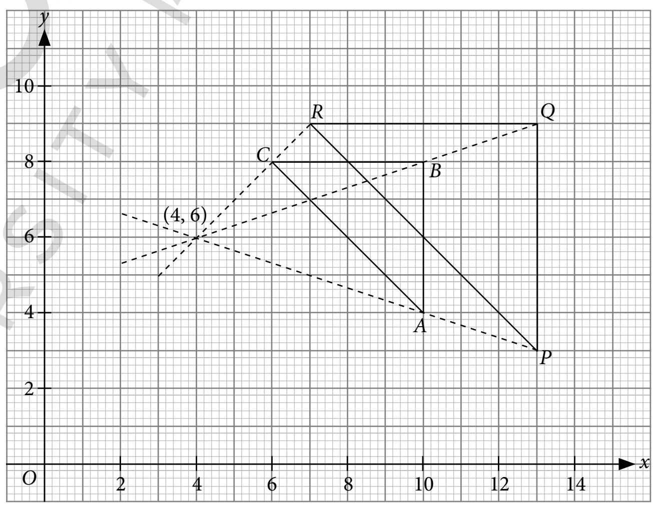
From the graph, the coordinates of the centre of enlargement are **(2, 1)**.

Scale factor $= \frac{RQ}{CB}$

$= \frac{9}{3}$

$= \mathbf{3}$

(c)



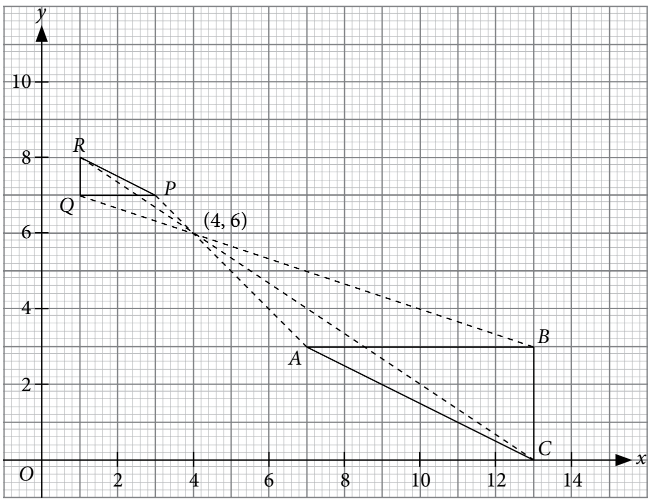
From the graph, the coordinates of the centre of enlargement are **(4, 6)**.

Scale factor $= \frac{RQ}{CB}$

$= \frac{6}{4}$

$= \mathbf{1.5}$

(d)



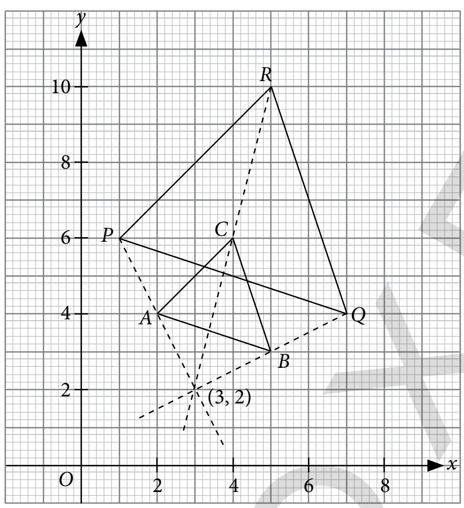
From the graph, the coordinates of the centre of enlargement are **(4, 6)**.

$$\text{Scale factor} = -\frac{PQ}{AB}$$

$$= -\frac{2}{6}$$

$$= \mathbf{-\frac{1}{3}}$$

(e)



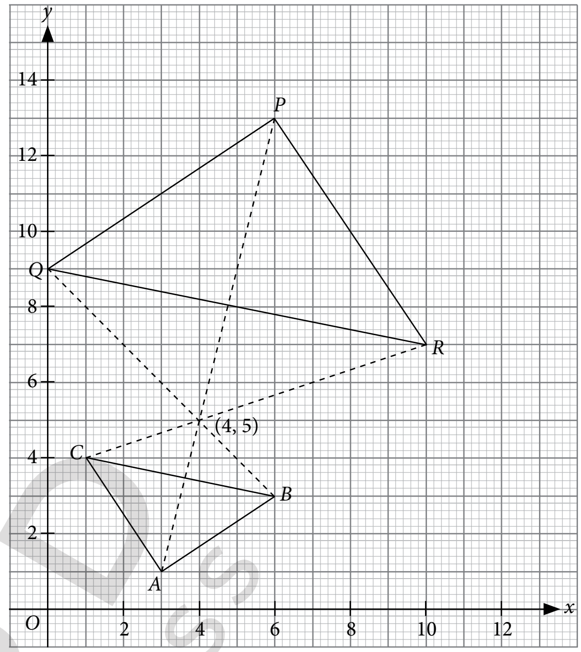
From the graph, the coordinates of the centre of enlargement are **(3, 2)**.

$$\text{Scale factor} = \frac{PQ}{AB}$$

$$= \frac{3}{1.5}$$

$$= \mathbf{2}$$

(f)



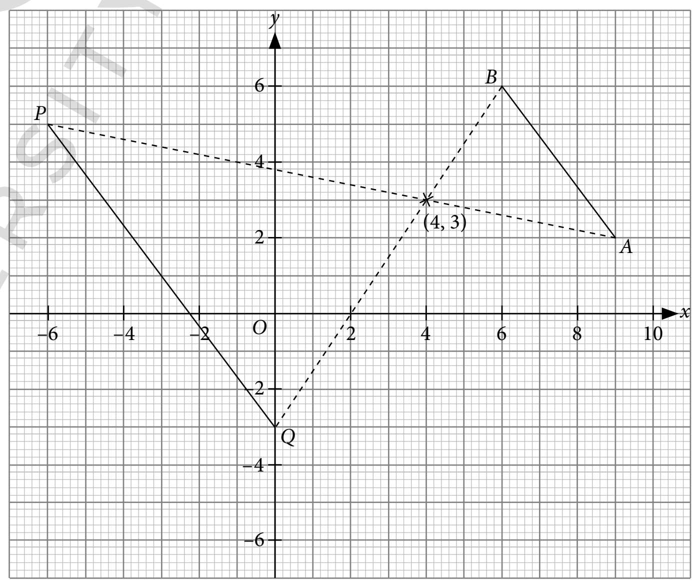
From the graph, the coordinates of the centre of enlargement are **(4, 5)**.

$$\text{Scale factor} = -\frac{PQ}{AB}$$

$$= -\frac{3.6}{1.8}$$

$$= \mathbf{-2}$$

13.



(i) From the graph, the coordinates are $\boldsymbol{P(-6, 5)}$ and $\boldsymbol{Q(0, -3)}$.

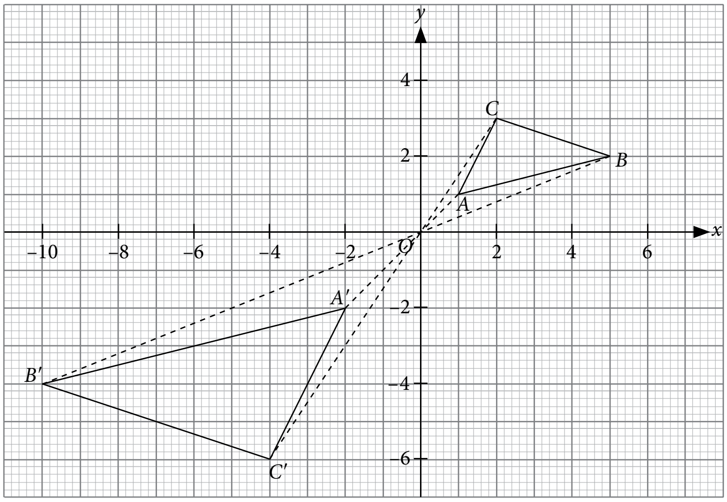
(ii) $PQ = \sqrt{[(0 - (-6))]^2 + (-3 - 5)^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

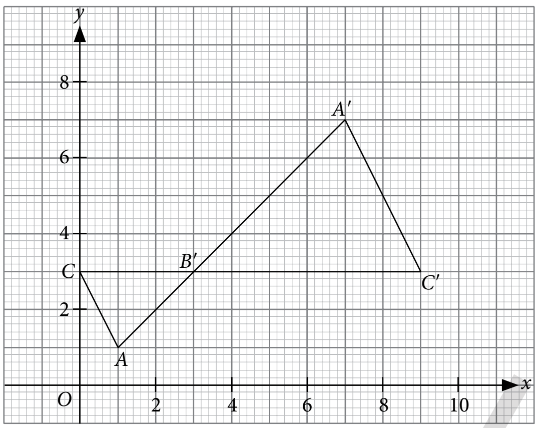
= 10 units

14.



From the graph, the coordinates are ***A*(1, 1)**, ***B*(5, 2)** and ***C*(2, 3)**.

15.



From the graph, the coordinates are ***A*(1, 1)** and ***C*(0, 3)**.

16. Since *PQRS* is a square, $PQ = RQ = 20$ cm.

$$\text{Scale factor} = \frac{PQ}{AB} = \frac{20}{10} = \mathbf{2}$$

Let the distance of the centre of enlargement from point *A* be *x*. Since the scale factor is 2, the distance of the centre of enlargement from point *P* is twice that between the centre of enlargement and point *A*.

$x + 10 + 10 = 2x$

$x + 20 = 2x$

$x = 20$

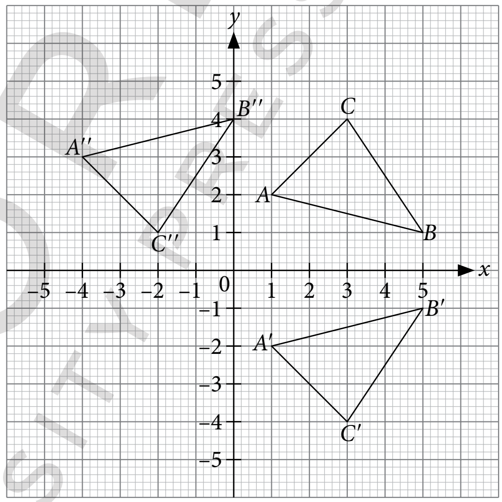
∴ the distance of the centre of enlargement from point *A* is **20 cm**.

10.5 Combined transformation

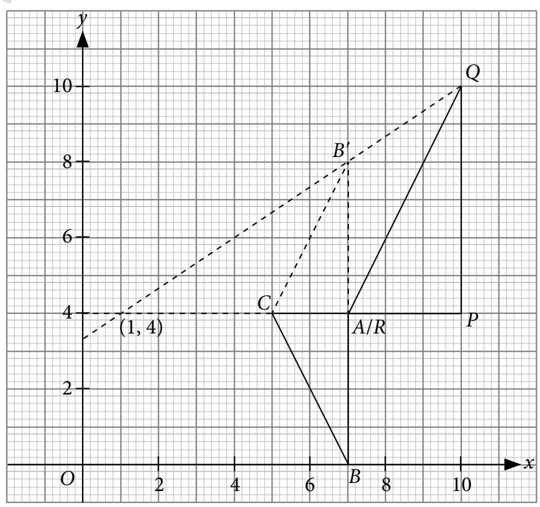
Thinking Time (Page 333)

No, the order of transformation matters.

Practise Now 8



Practise Now 9

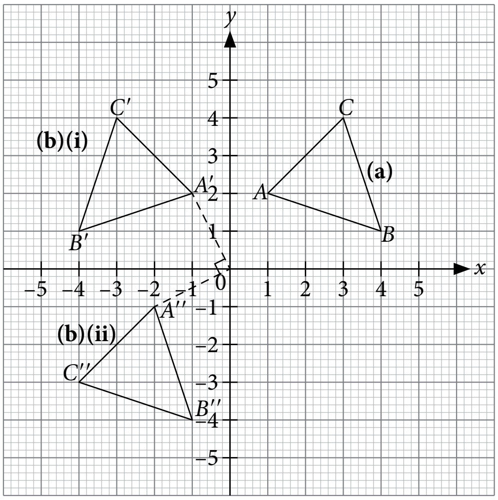


Step 1: $\triangle ABC$ is reflected in the line $y = 4$ to get $\triangle AB'C$

Step 2: $\triangle AB'C$ is enlarged by a scale factor of $\frac{PQ}{AB} = 1\frac{1}{2}$ with enlargement centre (1, 4) to get $\triangle PQR$.

Practise Now 10

(a)(b)



(c) The single transformation is a **reflection in the line $y = -x$**.

Exercise 10E

1. **(a)** Image under T $= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

Image under RT $= \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

$\therefore$ the coordinates of the image are **(–3, 7)**.

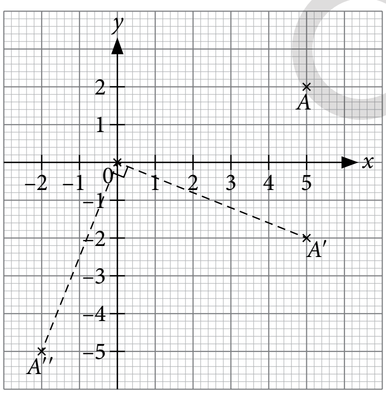
(b) Image under R $= \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Image under TR $= \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

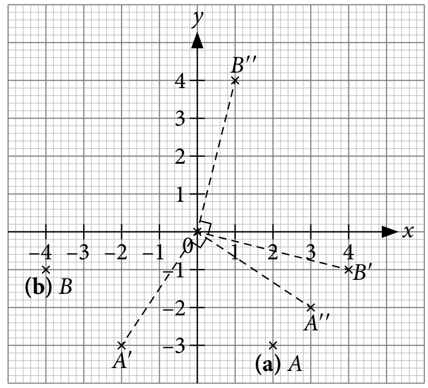
$\therefore$ the coordinates of the image are **(1, 7)**.

2. Let the coordinates (5, 2) be A.



From the graph, the coordinates of the image are **(–2, –5)**.

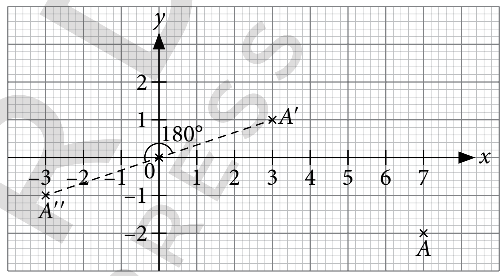
3. Let the coordinates (2, –3) and (–4, –1) be A and B respectively.



(a) From the graph, the coordinates of the image are **(3, –2)**.

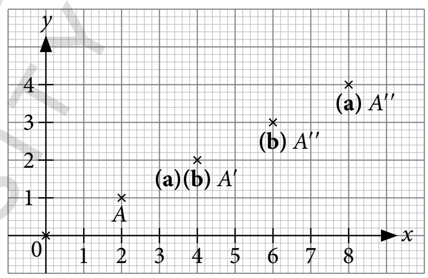
(b) From the graph, the coordinates of the image are **(1, 4)**.

4. Let the coordinates (7, –2) be A.



From the graph, the coordinates of the image are **(–3, –1)**.

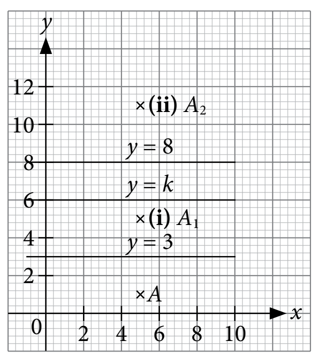
5. Let the coordinates (2, 1) be A.



(a) From the graph, the coordinates of the image are **(8, 4)**.

(b) From the graph, the coordinates of the image are **(6, 3)**.

6.



(i) From the graph, the coordinates of A_1 are **(5, 5)**.

(ii) From the graph, the coordinates of A_2 are **(5, 11)**.

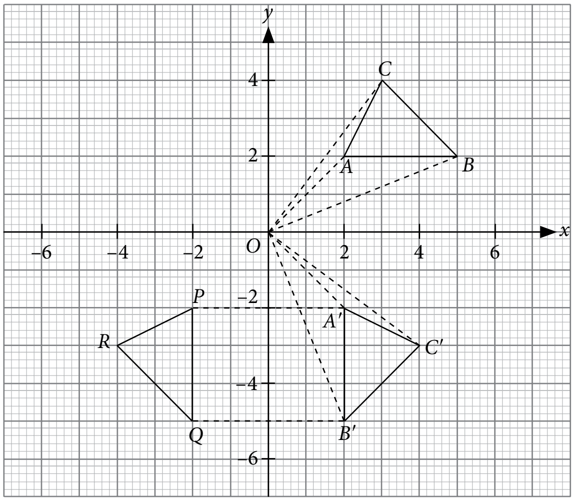
(iii) Midpoint of $AA_2 = \left(\dfrac{5+5}{2}, \dfrac{1+11}{2}\right)$

$= (5, 6)$

$\therefore k = \mathbf{6}$

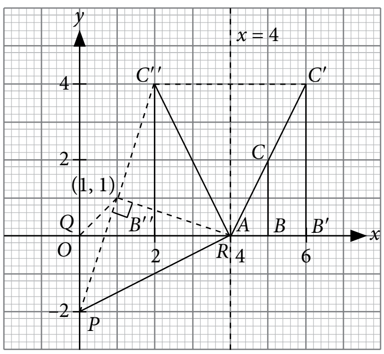
7. Let the coordinates of P be (a, b).
Under a reflection in the line $y = 0$, coordinates of $P_1 = (a, -b)$.
Under a reflection in the line $x = 0$, coordinates of $P_2 = (-a, -b)$.
$\therefore$ P will be mapped directly onto P_2 by a **rotation of 180° about the origin**.

8.

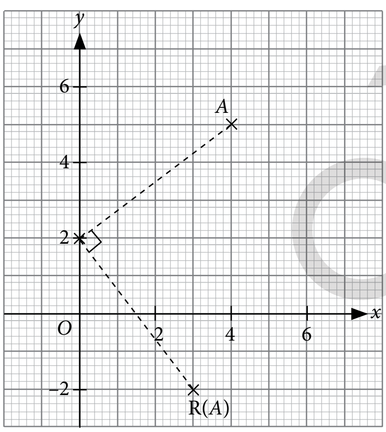


A single transformation that will map $\triangle ABC$ onto $\triangle PQR$ is a **reflection in the line $y = -x$**.

9.



10.

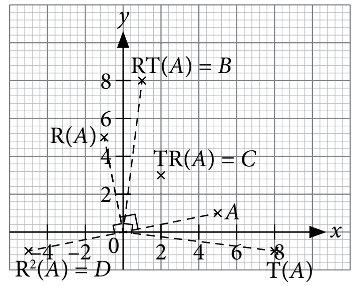


From the graph, the coordinates of R(A) are **(3, –2)**.

$$\text{Image of TR}(A) = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

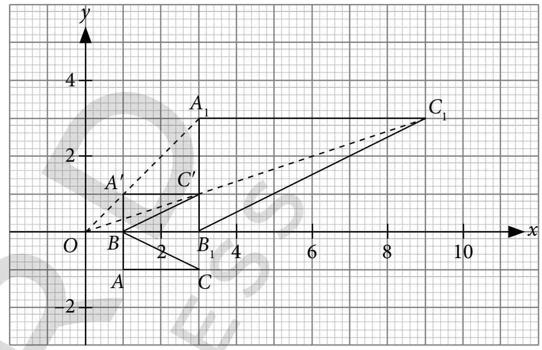
$\therefore$ the coordinates of TR(A) are **(6, –1)**.

11.



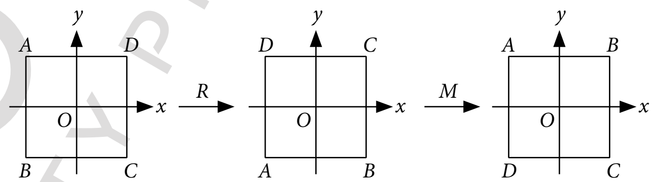
From the graph, the coordinates are **B(1, 8)**, **C(2, 3)** and **D(–5, –1)**.

12.



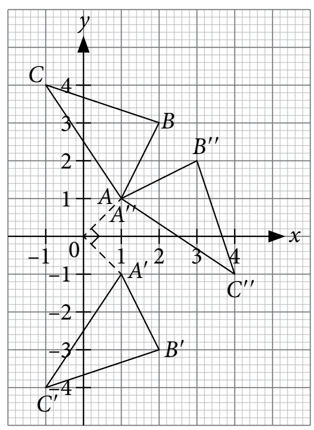
From the graph, the coordinates of A_1 are **(3, 3)**.

13. (i)



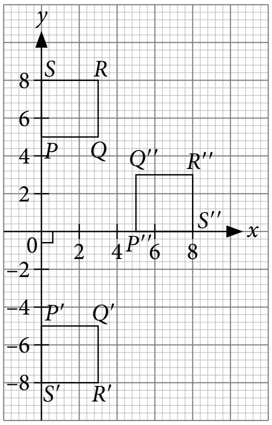
(ii) A single transformation which is equivalent to MR is a **reflection in the line AOC**.

14. (i)



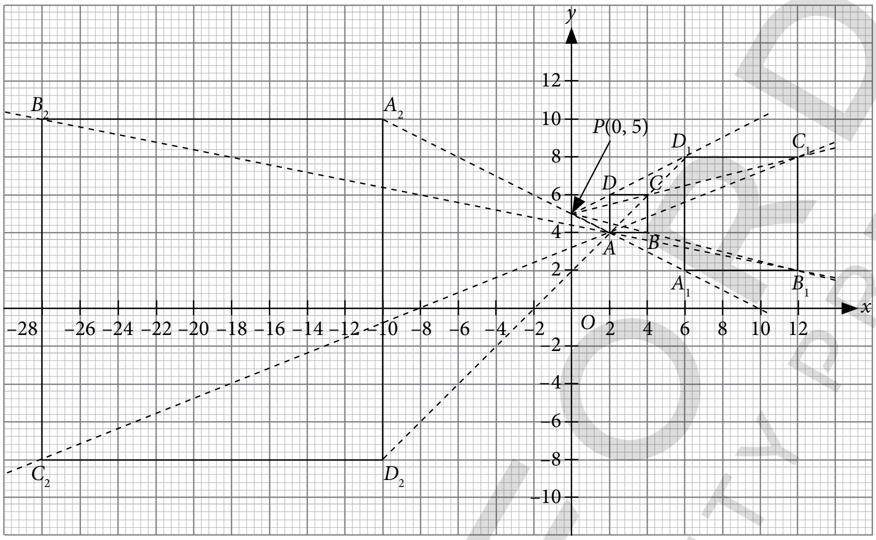
$\therefore$ the new coordinates are **A''(1, 1)**, **B''(3, 2)** and **C''(4, –1)**.

(ii)



$\therefore$ the new coordinates are $\boldsymbol{P''(5, 0)}$, $\boldsymbol{Q''(5, 3)}$, $\boldsymbol{R''(8, 3)}$ and $\boldsymbol{S''(8, 0)}$.

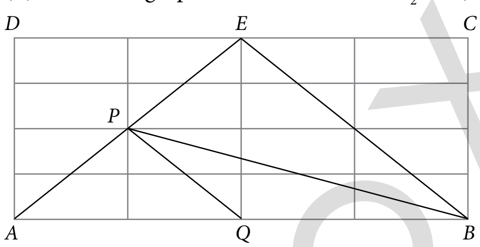
15.



(i) From the graph, the coordinates of C_1 are **(12, 8)**.

(ii) From the graph, the coordinates of C_2 are **(–28, –8)**.

16.



Chapter 11 Area and Volume of Similar Figures and Solids

TEACHING NOTES

Suggested Approach

Teachers may bring some real-life samples of similar solids, i.e. shampoo and body bath bottles in original sizes and travel sizes, and ask students whether these samples are similar. Students should be guided to draw a relationship between the lengths, heights, areas, volumes and masses of these solids. Teachers may also highlight that scale drawings of structures and models are representations of the actual size of the structures. With this, teachers may ask students to give real-life examples of such scale drawings and models.

Section 11.1: Area of similar figures

Teachers may wish to recap with students what they have learnt in Book 2 on similar figures before guiding them on how to find the relationship between the length of a figure and its area (see Investigation: Areas of similar figures). For problems involving area of similar figures, teachers can highlight to students that it might be easier to write the unknown in the numerator to help in subsequent algebraic manipulation. For example, in Worked Example 1 on page 342 of the textbook, it would be less easy had the equation been written as: $\frac{4}{A_2} = \left(\frac{3}{4.5}\right)^2$.

Section 11.2: Volume of similar solids

Now that students have learnt how to find the area of similar figures, teachers may guide students to extend what they have learnt to find the relationship among the volumes, lengths, heights and masses of similar solids (see Investigation: Volume and mass of similar solids).

Section 11.3: Solving problems involving similar solids

In this section, students will learn to apply what they have learnt in the previous sections to solve more problems, including those in real-world contexts, involving similar solids. These will require solid foundations in topics such as ratios and algebraic equations. As such, in this section, teachers should anticipate some difficulties faced by students and guide them accordingly. In the process, teachers may also revise certain topics at their own discretion.

Introductory Problem

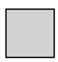
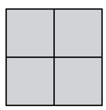
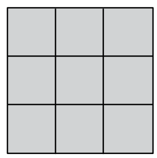
The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 5).

11.1 Area of similar figures

Investigation (Areas of similar figures)

1. **Yes**. They are similar because all the corresponding angles in the three squares are 90° and all the ratios of the lengths of the corresponding sides of the three squares are equal.

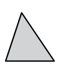
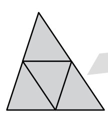
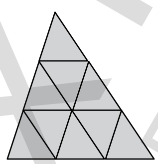
2.

Square			
Length of square	1 unit	2 units	3 units
Area of square	**1 unit²**	**4 units²**	**9 units²**

3. **(a)** The area of the second square is 4 times that of the first square.
   (b) The area of the third square is 9 times that of the first square.

4. $\frac{A_2}{A_1} = \frac{\boldsymbol{l_2^2}}{\boldsymbol{l_1^2}}$

5.

Triangle			
Length of corresponding side of triangle	1 unit	2 units	3 units
Area of triangle	1 square unit	**4 square units**	**9 square units**

6. **(a)** The area of the second triangle is 4 times that of the first triangle.
   (b) The area of the third triangle is 9 times that of the first triangle.

7. $\frac{A_2}{A_1} = \frac{\boldsymbol{l_2^2}}{\boldsymbol{l_1^2}}$

Practise Now 1

(a) $\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$

$\frac{A_2}{32} = \left(\frac{7}{4}\right)^2$

$= \frac{49}{16}$

$A_2 = \frac{49}{16} \times 32$

$= \mathbf{98\ cm^2}$

(b) Let the lengths of the corresponding sides of the smaller and larger pentagons be l_1 and l_2 respectively.

$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$

$\frac{A_1}{61.9} = \left(\frac{2.5}{6}\right)^2$

$= \frac{25}{144}$

$A_1 = \frac{25}{144} \times 61.9$

$= \mathbf{10.7\ m^2}$ (to 3 s.f.)

Practise Now 2

$\angle BAC = \angle DAE$ (common $\angle$)

$\angle ABC = \angle ADE$ (corr. $\angle$s, BC // DE)

$\therefore$ $\angle ACB = \angle AED$ and hence $\triangle ABC$ is similar to $\triangle ADE$.

$\left(\frac{AD}{AB}\right)^2 = \frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle ABC}$

$\left(\frac{AD}{8.4}\right)^2 = \frac{100}{49}$

$\frac{AD^2}{70.56} = \frac{100}{49}$

$AD^2 = \frac{100}{49} \times 70.56$

$= 144$

$AD = \sqrt{144}$ (since $AD > 0$)

$= 12$ m

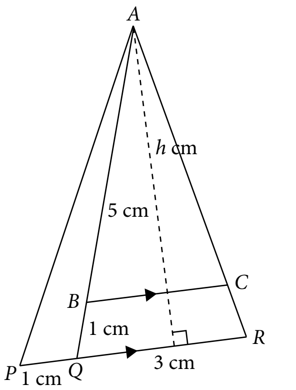
$BD = AD - AB$

$= 12 - 8.4$

$= \mathbf{3.6\ m}$

Practise Now 3

(i)



Notice that $\triangle APQ$ and $\triangle AQR$ have a common height corresponding to the bases PQ and QR respectively.
Let the common height be h cm.

$$\frac{\text{Area of }\triangle APQ}{\text{Area of }\triangle AQR} = \frac{\frac{1}{2}\times PQ\times h}{\frac{1}{2}\times QR\times h} = \frac{PQ}{QR}$$

$$\frac{\text{Area of }\triangle APQ}{8} = \frac{1}{3}$$

$$\text{Area of }\triangle APQ = \frac{1}{3}\times 8 = \mathbf{2\frac{2}{3}\ cm^2} \text{ or } \mathbf{2.67\ cm^2} \text{ (to 3 s.f.)}$$

(ii) $\angle BAC = \angle QAR$ (common $\angle$)
$\angle ABC = \angle AQR$ (corr. $\angle$s, $BC // QR$)
$\therefore \angle ACB = \angle ARQ$ and hence $\triangle ABC$ is similar to $\triangle AQR$.

$$\frac{\text{Area of }\triangle ABC}{\text{Area of }\triangle AQR} = \left(\frac{AB}{AQ}\right)^2$$

$$\frac{\text{Area of }\triangle ABC}{8} = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

$$\text{Area of }\triangle ABC = \frac{25}{36}\times 8 = \mathbf{5\frac{5}{9}\ cm^2} \text{ or } \mathbf{5.56\ cm^2} \text{ (to 3 s.f.)}$$

Exercise 11A

1. **(a)** Let the lengths of the corresponding sides of the larger and smaller figures be l_1 and l_2 respectively.

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$$

$$\frac{A_2}{64} = \left(\frac{2}{8}\right)^2 = \frac{1}{16}$$

$$A_2 = \frac{1}{16}\times 64 = \mathbf{4\ cm^2}$$

(b) Let the lengths of the corresponding sides of the larger and smaller figures be l_1 and l_2 respectively.

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{A_1}{0.06} = \left(\frac{0.4}{0.2}\right)^2 = 4$$

$$A_1 = 4\times 0.06 = \mathbf{0.24\ m^2}$$

(c) Let the lengths of the corresponding sides of the larger and smaller figures be l_1 and l_2 respectively.

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$$

$$\frac{A_2}{125} = \left(\frac{6}{15}\right)^2 = \frac{4}{25}$$

$$A_2 = \frac{4}{25}\times 125 = \mathbf{20\ cm^2}$$

(d) Let the lengths of the corresponding sides of the larger and smaller figures be l_1 and l_2 respectively.

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{A_1}{48} = \left(\frac{12}{8}\right)^2 = \frac{9}{4}$$

$$A_1 = \frac{9}{4}\times 48 = \mathbf{108\ cm^2}$$

(e) Let the corresponding arc lengths of the smaller and larger figures be l_1 and l_2 respectively.

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$$

$$\frac{A_2}{12} = \left(\frac{6}{4}\right)^2 = \frac{9}{4}$$

$$A_2 = \frac{9}{4}\times 12 = \mathbf{27\ cm^2}$$

(f) Let the lengths of the corresponding sides of the larger and smaller figures be l_1 and l_2 respectively.

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$$

$$\frac{A_2}{24} = \left(\frac{3p}{6p}\right)^2 = \frac{1}{4}$$

$$A_2 = \frac{1}{4}\times 24 = \mathbf{6\ cm^2}$$

2. Let r_1 and A_1 be the radius and the area of the smaller circle respectively, and r_2 and A_2 be the radius and the area of the larger circle respectively.

$$\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{4}{7}\right)^2 = \mathbf{\frac{16}{49}}$$

3. (i) $\angle SPT = \angle QPR$ (common $\angle$)
$\angle PST = \angle PQR$ (corr. $\angle$s, $ST // QR$)
$\therefore \angle PTS = \angle PRQ$ and hence $\triangle PST$ is similar to $\triangle PQR$.

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle PST} = \left(\frac{PR}{PT}\right)^2$$
$$\frac{\text{Area of } \triangle PQR}{24} = \left(\frac{10}{6}\right)^2 = \frac{25}{9}$$
$$\text{Area of } \triangle PQR = \frac{25}{9} \times 24 = \mathbf{66\frac{2}{3}\ m^2 \text{ or } 66.7\ m^2} \text{ (to 3 s.f.)}$$

(ii) Area of $SQRT$ = area of $\triangle PQR$ – area of $\triangle PST$
$$= 66\frac{2}{3} - 24 = \mathbf{42\frac{2}{3}\ m^2 \text{ or } 42.7\ m^2} \text{ (to 3 s.f.)}$$

4. (a) $\left(\frac{a}{3}\right)^2 = \frac{24}{6}$
$\frac{a^2}{9} = 4$
$a^2 = 9 \times 4 = 36$
$a = \sqrt{36}$ (since $a > 0$)
$= \mathbf{6}$

(b) $\left(\frac{b}{5}\right)^2 = \frac{90}{10}$
$\frac{b^2}{25} = 9$
$b^2 = 9 \times 25 = 225$
$b = \sqrt{225}$ (since $b > 0$)
$= \mathbf{15}$

(c) $\left(\frac{c}{5}\right)^2 = \frac{240}{15}$
$\frac{c^2}{25} = 16$
$c^2 = 16 \times 25 = 400$
$c = \sqrt{400}$ (since $c > 0$)
$= \mathbf{20}$

(d) $\left(\frac{d}{6}\right)^2 = \frac{12}{27}$
$\frac{d^2}{36} = \frac{12}{27}$
$d^2 = \frac{12}{27} \times 36 = 16$
$d = \sqrt{16}$ (since $d > 0$)
$= \mathbf{4}$

5. $$\frac{\text{Area of smaller hexagon}}{\text{Area of larger hexagon}} = \left(\frac{8}{10}\right)^2$$
$$\frac{\text{Area of smaller hexagon}}{200} = \frac{16}{25}$$
$$\text{Area of smaller hexagon} = \frac{16}{25} \times 200 = \mathbf{128\ m^2}$$

6. $$\frac{\text{Area of } \triangle CAE}{\text{Area of } \triangle CBD} = \left(\frac{4}{3}\right)^2$$
$$\frac{\text{Area of } \triangle CAE}{9} = \frac{16}{9}$$
$$\text{Area of } \triangle CAE = \frac{16}{9} \times 9 = 16\ cm^2$$
Area of $ABDE$ = area of $\triangle CAE$ – area of $\triangle CBD$
$= 16 - 9$
$= \mathbf{7\ cm^2}$

7. $30\text{ mm} = \frac{30}{10}\text{ cm}$ (1 cm = 10 mm)
$= 3\text{ cm}$
$$\frac{\text{Actual land area}}{\text{Area on plan}} = \left(\frac{150}{3}\right)^2$$
$$\frac{\text{Actual land area}}{3250} = 2500$$
$$\text{Actual land area} = 2500 \times 3250 = 8\,125\,000\ cm^2 = \frac{8\,125\,000}{10\,000}\ m^2 \ (1\ m^2 = 10\,000\ cm^2) = \mathbf{812.5\ m^2}$$

8. $\angle HPG = \angle QPR$ (common $\angle$)
$\angle PHG = \angle PQR$ (corr. $\angle$s, $HG // QR$)
$\therefore \angle HGP = \angle QRP$ and hence $\triangle PHG$ is similar to $\triangle PQR$.
Since $HGFQ$ is a parallelogram, $HG = QF$.
$$\frac{\text{Area of } \triangle PHG}{\text{Area of } \triangle PQR} = \left(\frac{HG}{QR}\right)^2 = \left(\frac{p}{p+q}\right)^2 = \mathbf{\frac{p^2}{p^2 + 2pq + q^2}}$$

9. (i) $$\frac{\text{Height of larger cone}}{\text{Height of smaller cone}} = 1.5$$
$$\frac{\text{Height of larger cone}}{12} = 1.5$$
$$\text{Height of larger cone} = 1.5 \times 12 = \mathbf{18\ cm}$$

(ii) $$\frac{\text{Surface area of larger cone}}{\text{Surface area of smaller cone}} = 1.5^2$$
$$\frac{\text{Surface area of larger cone}}{124} = 2.25$$
$$\text{Surface area of larger cone} = 2.25 \times 124 = \mathbf{279\ cm^2}$$

10. Area of $\triangle XYZ$ = area of $\triangle XMN$ + area of $MYZN$

$$= 14 + 22$$
$$= 36 \text{ cm}^2$$

$$\left(\frac{XY}{XM}\right)^2 = \frac{\text{Area of } \triangle XYZ}{\text{Area of } \triangle XMN}$$

$$\left(\frac{XY}{6}\right)^2 = \frac{36}{14}$$

$$\frac{XY^2}{36} = \frac{18}{7}$$

$$XY^2 = \frac{18}{7} \times 36$$
$$= \frac{648}{7}$$

$$XY = \sqrt{\frac{648}{7}} \quad (\text{since } XY > 0)$$
$$= 9.6214 \text{ cm (to 5 s.f.)}$$

$MY = XY - XM$
$= 9.6214 - 6$
$=$ **3.62 cm** (to 3 s.f.)

11. $\angle MPN = \angle QPR$ (common $\angle$)
$\angle PMN = \angle PQR$ (corr. $\angle$s, MN // QR)
$\therefore$ $\angle MNP = \angle QRP$ and hence $\triangle PMN$ is similar to $\triangle PQR$.

$$\left(\frac{MN}{QR}\right)^2 = \frac{\text{Area of } \triangle PMN}{\text{Area of } \triangle PQR}$$
$$= \frac{9}{9+16}$$
$$= \frac{9}{25}$$

$$\frac{MN}{QR} = \sqrt{\frac{9}{25}} \quad \left(\text{since } \frac{MN}{QR} > 0\right)$$
$$= \frac{3}{5}$$

$\therefore$ the ratio $MN : QR$ is **3 : 5**.

12. $C\hat{D}E = 180° - 90°$ (int. $\angle$s, AB // ED)
$= 90°$
$A\hat{B}C = E\hat{D}C = °90$
$A\hat{C}B = E\hat{C}D$ (given)
$\therefore$ $B\hat{A}C = D\hat{E}C$ and hence $\triangle ABC$ is similar to $\triangle EDC$.

$$\left(\frac{BC}{DC}\right)^2 = \frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle EDC}$$

$$\left(\frac{BC}{BC + 4.5}\right)^2 = \frac{25}{64}$$

$$\frac{BC}{BC + 4.5} = \sqrt{\frac{25}{64}} \quad \left(\text{since } \frac{BC}{BC + 4.5} > 0\right)$$
$$= \frac{5}{8}$$

$$8BC = 5BC + 22.5$$
$$3BC = 22.5$$
$BC =$ **7.5 cm**

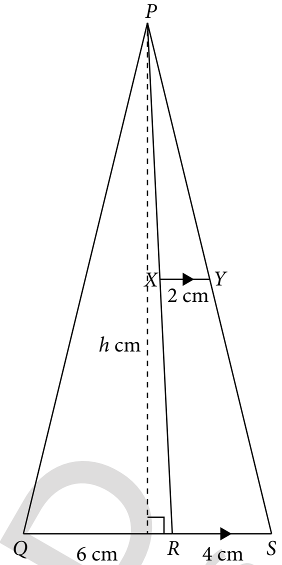
13. **(i)** $\angle XPY = \angle RPS$ (common $\angle$)
$\angle PXY = \angle PRS$ (corr. $\angle$s, XY // RS)
$\therefore$ $\angle PXY = \angle PSR$ and hence $\triangle PXY$ is similar to $\triangle PRS$.

$$\frac{\text{Area of } \triangle PRS}{\text{Area of } \triangle PXY} = \left(\frac{RS}{XY}\right)^2$$

$$\frac{\text{Area of } \triangle PRS}{10} = \left(\frac{4}{2}\right)^2$$
$$= 4$$

Area of $\triangle PRS = 4 \times 10$
$=$ **40 cm²**

(ii)



Notice that $\triangle PQR$ and $\triangle PRS$ have a common height corresponding to the bases QR and RS respectively.
Let the common height be h cm.

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle PRS} = \frac{\frac{1}{2} \times QR \times h}{\frac{1}{2} \times RS \times h}$$

$$\frac{\text{Area of } \triangle PQR}{40} = \frac{QR}{RS}$$
$$= \frac{6}{4}$$
$$= \frac{3}{2}$$

Area of $\triangle PQR = \frac{3}{2} \times 40$
$=$ **60 cm²**

14. Let the area of the smaller triangle be x cm².
Since the sum of the areas of the two triangles is 105 cm²,
Area of larger triangle $= (105 - x)$ cm²

$$\frac{\text{Area of smaller triangle}}{\text{Area of larger triangle}} = \left(\frac{3}{4}\right)^2$$

$$\frac{x}{105 - x} = \frac{9}{16}$$
$$16x = 945 - 9x$$
$$25x = 945$$
$$x = 37.8$$

Area of smaller triangle = **37.8 cm²**
Area of larger triangle = 105 − 37.8
= **67.2 cm²**

15. (i) $\angle ARB = \angle QRD$ (common $\angle$)

$\angle BAR = \angle DQR$ (corr. $\angle$s, AC // QP)

$\therefore \angle ABR = \angle QDR$ and hence $\triangle ABR$ is similar to $\triangle QDR$.

Since $\triangle ABR$ is similar to $\triangle QDR$,

$$\frac{AB}{QD} = \frac{AR}{QR}$$

$$\frac{AB}{7} = \frac{4}{7}$$

$$AB = \frac{4}{7} \times 7$$

$$= 4 \text{ cm}$$

$\angle ARC = \angle QRP$ (common $\angle$)

$\angle CAR = \angle PQR$ (corr. $\angle$s, AC // QP)

$\therefore \angle ACR = \angle QPR$ and hence $\triangle ACR$ is similar to $\triangle QPR$.

Since $\triangle ACR$ is similar to $\triangle QPR$,

$$\frac{AC}{QP} = \frac{AR}{QR}$$

$$\frac{AC}{11} = \frac{4}{7}$$

$$AC = \frac{4}{7} \times 11$$

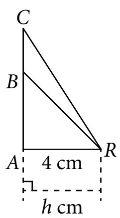
$$= 6\frac{2}{7} \text{ cm}$$

$$BC = AC - AB$$

$$= 6\frac{2}{7} - 4$$

$$= \mathbf{2\frac{2}{7}\ cm\ or\ 2.29\ cm} \text{ (to 3 s.f.)}$$

(ii)



Notice that $\triangle ARB$ and $\triangle BRC$ have a common height corresponding to the bases AB and BC respectively.

Let the common height be h cm.

$$\frac{\text{Area of } \triangle ARB}{\text{Area of } \triangle BRC} = \frac{\frac{1}{2} \times AB \times h}{\frac{1}{2} \times BC \times h}$$

$$= \frac{AB}{BC}$$

$$= \frac{4}{2\frac{2}{7}}$$

$$= \mathbf{\frac{7}{4}}$$

(iii) $\dfrac{\text{Area of } \triangle BRC}{\text{Area of } \triangle ARB} = \dfrac{4}{7}$

$$\frac{\text{Area of } \triangle QRD}{\text{Area of } \triangle ARB} = \left(\frac{QR}{AR}\right)^2$$

$$= \left(\frac{7}{4}\right)^2$$

$$= \frac{49}{16}$$

$$\frac{\text{Area of } ABDQ + \text{Area of } \triangle ARB}{\text{Area of } \triangle ARB} = \frac{49}{16}$$

$$\frac{\text{Area of } ABDQ}{\text{Area of } \triangle ARB} + 1 = \frac{49}{16}$$

$$\frac{\text{Area of } ABDQ}{\text{Area of } \triangle ARB} = \frac{33}{16}$$

$$\frac{\text{Area of } \triangle ARB}{\text{Area of } ABDQ} = \frac{16}{33}$$

$$\frac{\text{Area of } \triangle BRC}{\text{Area of } ABDQ} = \frac{\text{Area of } \triangle BRC}{\text{Area of } \triangle ARB} \times \frac{\text{Area of } \triangle ARB}{\text{Area of } ABDQ}$$

$$= \frac{4}{7} \times \frac{16}{33}$$

$$= \mathbf{\frac{64}{231}}$$

16. (i) $\angle ABC = \angle PBQ$ (common $\angle$)

$\angle BAC = \angle BPQ$ (corr. $\angle$s, AC // PQ)

$\therefore \angle ACB = \angle PQB$ and hence $\triangle BAC$ is similar to $\triangle BPQ$.

$$\frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle PBQ} = \left(\frac{BC}{BQ}\right)^2$$

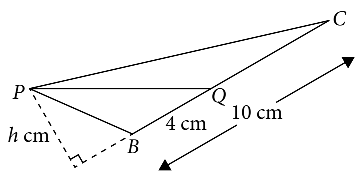
$$\frac{\text{Area of } \triangle ABC}{8} = \left(\frac{10}{4}\right)^2$$

$$= \frac{25}{4}$$

$$\text{Area of } \triangle ABC = \frac{25}{4} \times 8$$

$$= \mathbf{50\ cm^2}$$

(ii)



Notice that $\triangle PQC$ and $\triangle PBQ$ have a common height corresponding to the bases QC and BQ respectively.

Let the common height be h cm.

$$\frac{\text{Area of } \triangle PQC}{\text{Area of } \triangle PBQ} = \frac{\frac{1}{2} \times CQ \times h}{\frac{1}{2} \times BQ \times h}$$

$$\frac{\text{Area of } \triangle PQC}{8} = \frac{CQ}{BQ}$$

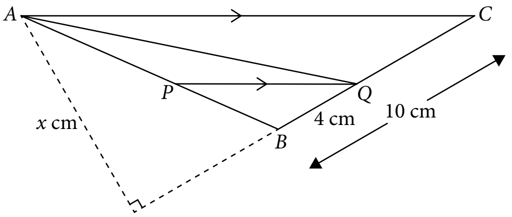
$$= \frac{10 - 4}{4}$$

$$= \frac{3}{2}$$

$$\text{Area of } \triangle PQC = \frac{3}{2} \times 8$$

$$= \mathbf{12\ cm^2}$$

(iii)



Notice that $\triangle AQC$ and $\triangle ABC$ have a common height corresponding to the bases QC and BC respectively.
Let the common height be x cm.

$$\frac{\text{Area of }\triangle AQC}{\text{Area of }\triangle ABC} = \frac{\frac{1}{2}\times CQ\times x}{\frac{1}{2}\times BC\times x}$$

$$\frac{\text{Area of }\triangle AQC}{50} = \frac{CQ}{BC} = \frac{10-4}{10} = \frac{3}{5}$$

$$\text{Area of }\triangle AQC = \frac{3}{5}\times 50 = \mathbf{30\ cm^2}$$

17. $$\left(\frac{\text{Perimeter of larger triangle}}{\text{Perimeter of smaller triangle}}\right)^2 = \frac{96}{54}$$

OPEN

$$\left(\frac{\text{Perimeter of larger triangle}}{x+1}\right)^2 = \frac{16}{9}$$

$$\frac{\text{Perimeter of larger triangle}}{x+1} = \sqrt{\frac{16}{9}}\ \left(\text{since } \frac{\text{Perimeter of larger triangle}}{x+1} > 0\right) = \frac{4}{3}$$

$$\text{Perimeter of larger triangle} = \frac{4}{3}(x+1)\text{ cm}$$

$$\text{Sum of the perimeters} = \frac{4}{3}(x+1)+(x+1) = \mathbf{\frac{7}{3}(x+1)\ cm}$$

Since the sum is $\frac{7}{3}(x+1)$ cm, it is a whole number when $(x+1)$ is a multiple of 3, e.g. $x = 35$.

$$\text{Sum of the perimeters} = \frac{7}{3}(35+1) = 84\text{ cm}$$

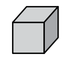
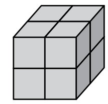
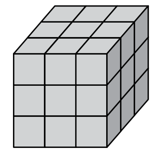
$\therefore$ a possible value of x is **35**.

11.2 Volume of similar solids

Investigation (Volume and mass of similar solids)

1. Since all the corresponding angles in the three cubes are equal, and all the ratios of the lengths of the corresponding sides of the three cubes are equal, the three cubes are similar.

2.

Cube			
Length of cube	1 unit	2 units	3 units
Volume of cube	**1 unit^3**	**8 units^3**	**27 units^3**

3. **(a)** The volume of the second cube is 8 times that of the first cube.
(b) The volume of the third cube is 27 times that of the first cube.

4. $\frac{V_2}{V_1} = \frac{\boldsymbol{l_2}^3}{\boldsymbol{l_1}^3}$

5. $\frac{r_2}{r_1} = \frac{h_2}{h_1} = \boldsymbol{k}$

6. **(a)** $V_2 = \pi r_2^2 h_2$
$= \pi\times(kr_1)^2\times(kh_1)$
$= \pi\times k^2 r_1^2\times kh_1$
$= \boldsymbol{\pi r_1^2 h_1 k^3}$

(b) $V_2 = \boldsymbol{k^3 V_1}$

7. $\frac{V_2}{V_1} = \boldsymbol{k^3} = \left(\frac{\boldsymbol{h_2}}{\boldsymbol{h_1}}\right)^3 = \left(\frac{\boldsymbol{r_2}}{\boldsymbol{r_1}}\right)^3$

8. $\frac{m_2}{m_1} = \frac{\boldsymbol{V_2}}{\boldsymbol{V_1}}$

Practise Now 4

1. Let the heights of the smaller and larger cones be l_1 and l_2 respectively.

$$\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{V_2}{16.2} = \left(\frac{10}{6}\right)^3 = \left(\frac{5}{3}\right)^3 = \frac{125}{27}$$

$$V_2 = \frac{125}{27}\times 16.2 = \mathbf{75\ cm^3}$$

2. Let r_2 be the radius of the larger cylinder.

$$\left(\frac{r_1}{r_2}\right)^3 = \frac{V_1}{V_2}$$

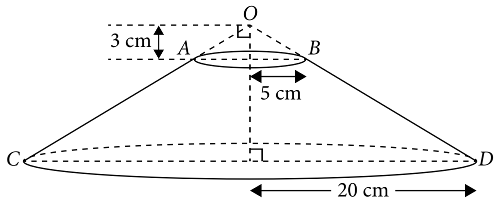
$$\left(\frac{r_1}{1}\right)^3 = \frac{2}{16}$$

$$r_1^3 = \frac{1}{8}$$

$$r_1 = \sqrt[3]{\frac{1}{8}} = \mathbf{0.5\ m}$$

Practise Now 5

Method 1:



$\angle AOB = \angle COD$ (common $\angle$)
$\angle OAB = \angle OCD$ (corr. $\angle$s, AB // CD)
$\therefore \angle ABO = \angle CDO$
Hence, $\triangle OAB$ is similar to $\triangle OCD$ and cone OAB is similar to cone OCD.

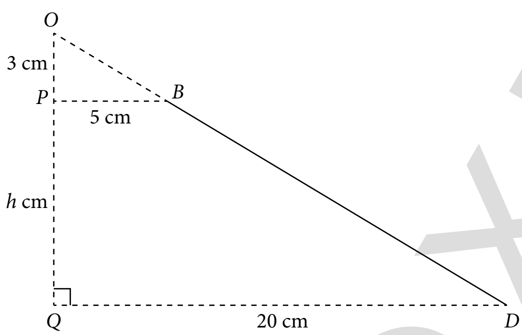
$$\frac{\text{Volume of cone } OCD}{\text{Volume of cone } OAB} = \left(\frac{20}{5}\right)^3 = 64$$

$$\begin{aligned}\text{Volume of cone } OCD &= 64 \times \text{volume of cone } OAB \\ &= 64 \times \frac{1}{3}\pi\left(5^2\right)(3) \\ &= 1600\pi \text{ cm}^3\end{aligned}$$

$$\begin{aligned}\therefore \text{volume of frustum} &= \text{volume of cone } OCD - \text{volume of cone } OAB \\ &= 1600\pi - \frac{1}{3}\pi\left(5^2\right)(3) \\ &= 1600\pi - 25\pi \\ &= 1575\pi \\ &= \mathbf{4950\ cm^3} \text{ (to 3 s.f.)}\end{aligned}$$

Method 2:

Let the height of the frustum be h cm.



$\angle POB = \angle QOD$ (common $\angle$)
$\angle OPB = \angle OQD$ (corr. $\angle$s, PB // QD)
$\therefore \angle OBP = \angle ODQ$ and hence $\triangle OPB$ is similar to $\triangle OQD$.

$$\begin{aligned}\frac{OQ}{OP} &= \frac{QD}{PB} \\ \frac{3+h}{3} &= \frac{20}{5} \\ &= 4 \\ 3 + h &= 12 \\ h &= 9\end{aligned}$$

$$\begin{aligned}\text{The height of the larger cone} &= 9 + 3 \\ &= 12 \text{ cm}\end{aligned}$$

$$\begin{aligned}\therefore \text{volume of frustum} &= \text{volume of cone } OCD - \text{volume of cone } OAB \\ &= \frac{1}{3}\pi\left(20^2\right)(12) - \frac{1}{3}\pi\left(5^2\right)(3) \\ &= \frac{1}{3}\pi\left(20^2 \times 12 - 5^2 \times 3\right) \\ &= \frac{1}{3}\pi(4725) \\ &= 1575\pi \\ &= \mathbf{4950\ cm^3} \text{ (to 3 s.f.)}\end{aligned}$$

Introductory Problem Revisited

Let V_1 and l_1 be the volume and length of the smaller cuboid respectively.
Let V_2 and l_2 be the volume and length of the larger cuboid respectively.

$$\begin{aligned}\left(\frac{l_2}{l_1}\right)^3 &= \frac{V_2}{V_1} \\ \left(\frac{l_2}{15}\right)^3 &= 60 \\ \frac{l_2^3}{15^3} &= 60 \\ \frac{l_2^3}{3375} &= 60 \\ l_2^3 &= 202\,500 \\ l_2 &= \sqrt[3]{202\,500} \\ &= 58.7 \text{ cm (to 3 s.f.)}\end{aligned}$$

$\therefore$ the length of the larger cuboid is **58.7 cm**.

Practise Now 6

1. Let m_1, V_1 and l_1 be the mass, volume and height of the smaller prism respectively.
Let m_2, V_2 and l_2 be the mass, volume and height of the larger prism respectively.

$$\begin{aligned}\frac{m_2}{m_1} &= \frac{V_2}{V_1} \\ &= \left(\frac{l_2}{l_1}\right)^3 \\ \frac{m_2}{80} &= \left(\frac{8}{5}\right)^3 \\ m_2 &= \left(\frac{8}{5}\right)^3 \times 80 \\ &= 328 \text{ g (to the nearest integer)}\end{aligned}$$

$\therefore$ the mass of the larger prism is **328 g**.

2. Let m_1, V_1 and l_1 be the mass, volume and height of the figurine respectively.
Let m_2, V_2 and l_2 be the mass, volume and height of the statue respectively.

$$\begin{aligned}l_2 &= 2 \text{ m} \\ &= 200 \text{ cm (1 m = 100 cm)}\end{aligned}$$

$$\begin{aligned}\frac{m_2}{m_1} &= \frac{V_2}{V_1} \\ &= \left(\frac{l_2}{l_1}\right)^3 \\ \frac{m_2}{3} &= \left(\frac{200}{20}\right)^3 \\ m_2 &= \left(\frac{200}{20}\right)^3 \times 3 \\ &= 3000 \text{ kg}\end{aligned}$$

$\therefore$ the mass of the statue is **3000 kg**.

11.3 Solving problems involving similar solids

Practise Now 7

(i) Let V_1 and h_1 be the volume and height of the smaller pyramid respectively.
Let V_2 and h_2 be the volume and height of the larger pyramid respectively.

$$\left(\frac{h_1}{h_2}\right)^3 = \frac{V_1}{V_2}$$

$$\left(\frac{h_1}{27}\right)^3 = \frac{1}{6}$$

$$\frac{h_1}{27} = \sqrt[3]{\frac{1}{6}}$$

$$h_1 = \sqrt[3]{\frac{1}{6}} \times 27$$

$$= 14.859 \text{ cm (to 5 s.f.)}$$

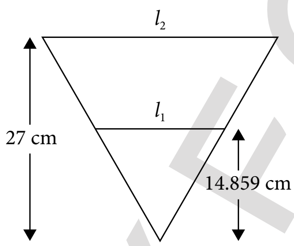
$$= 14.9 \text{ cm (to 3 s.f.)}$$

∴ the depth of the vegetable oil is **14.9 cm**.

(ii) The top surface of the vegetable oil and that of the container are rectangles.
Let l_1 and l_2 be the lengths of the smaller rectangle and the larger rectangle respectively.
Using similar triangles,

$$\frac{l_1}{l_2} = \frac{h_1}{h_2}$$

$$= \frac{14.859}{27}$$



Let A_1 be the area of the top surface of the vegetable oil and A_2 be the area of the top surface of the container.

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$= \left(\frac{14.859}{27}\right)^2$$

$$\frac{A_2}{A_1} = \frac{1}{\left(\frac{14.859}{27}\right)^2}$$

$$= 3.30 \text{ (to 3 s.f.)}$$

∴ the ratio of the area of the top surface of the vegetable oil to the area of the top surface of the container is **1 : 3.30**.

Thinking Time (Page 355)

Since the two cones are similar,

$$\frac{\text{Total surface area of smaller cone}}{\text{Total surface area of larger cone}} = \left(\frac{l_1}{l_2}\right)^2.$$

Exercise 11B

1. (a) Let the lengths of the corresponding sides of the larger and smaller solids be l_1 and l_2 respectively.

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{V_1}{72} = \left(\frac{12}{6}\right)^3$$

$$V_1 = \left(\frac{12}{6}\right)^3 \times 72$$

$$= \mathbf{576\ cm^3}$$

(b) Let the corresponding heights of the smaller and larger solids be l_1 and l_2 respectively.

$$\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{V_2}{48} = \left(\frac{6}{4}\right)^3$$

$$V_2 = \left(\frac{6}{4}\right)^3 \times 48$$

$$= \mathbf{162\ cm^3}$$

(c) Let the lengths of the corresponding sides of the larger and smaller solids be l_1 and l_2 respectively.

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{V_1}{12} = \left(\frac{7.5}{2.5}\right)^3$$

$$V_1 = \left(\frac{7.5}{2.5}\right)^3 \times 12$$

$$= \mathbf{324\ cm^3}$$

(d) Let the lengths of the corresponding sides of the smaller and larger solids be l_1 and l_2 respectively.

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{V_1}{2464} = \left(\frac{4}{16}\right)^3$$

$$V_1 = \left(\frac{4}{16}\right)^3 \times 2464$$

$$= \mathbf{38.5\ m^3}$$

(e) Let the lengths of the corresponding sides of the larger and smaller solids be l_1 and l_2 respectively.

$$\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{V_2}{3.2} = \left(\frac{32}{64}\right)^3$$

$$V_2 = \left(\frac{32}{64}\right)^3 \times 3.2$$

$$= \mathbf{0.4\ m^3}$$

2. (a) Let V_1 and l_1 be the volume and circumference of the smaller cylinder respectively.
Let V_2 and l_2 be the volume and circumference of the larger cylinder respectively.

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$= \left(\frac{8}{10}\right)^3$$

$$= \mathbf{\frac{64}{125}}$$

(b) Let V_1 and l_1 be the volume and height of the smaller cone respectively.
Let V_2 and l_2 be the volume and height of the larger cone respectively.

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$
$$= \left(\frac{9}{12}\right)^3$$
$$= \mathbf{\frac{27}{64}}$$

(c) Let V_1 and l_1 be the volume and radius of the smaller sphere respectively.
Let V_2 and l_2 be the volume and radius of the larger sphere respectively.

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$
$$= \left(\frac{4}{6}\right)^3$$
$$= \mathbf{\frac{8}{27}}$$

3. Let V_1 and l_1 be the volume and height of the Junior glass respectively.
Let V_2 and l_2 be the volume and height of the Senior glass respectively.

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$
$$\frac{V_1}{540} = \left(\frac{6}{9}\right)^3$$
$$V_1 = \left(\frac{6}{9}\right)^3 \times 540$$
$$= \mathbf{160\ cm^3}$$

4. (a) Let V_1 and l_1 be the volume and length of the larger solid respectively.
Let V_2 and l_2 be the volume and length of the smaller solid respectively.

$$\left(\frac{l_1}{l_2}\right)^3 = \frac{V_1}{V_2}$$
$$\left(\frac{a}{2}\right)^3 = \frac{48}{6}$$
$$= 8$$
$$\frac{a}{2} = \sqrt[3]{8}$$
$$= 2$$
$$a = 2 \times 2$$
$$= \mathbf{4}$$

(b) Let V_1 and l_1 be the volume and length of the smaller solid respectively.
Let V_2 and l_2 be the volume and length of the larger solid respectively.

$$\left(\frac{l_2}{l_1}\right)^3 = \frac{V_2}{V_1}$$
$$\left(\frac{b}{6}\right)^3 = \frac{54}{16}$$
$$= \frac{27}{8}$$
$$\frac{b}{6} = \sqrt[3]{\frac{27}{8}}$$
$$= \frac{3}{2}$$
$$b = \frac{3}{2} \times 6$$
$$= \mathbf{9}$$

(c) Let V_1 and l_1 be the volume and length of the larger solid respectively.
Let V_2 and l_2 be the volume and length of the smaller solid respectively.

$$\left(\frac{l_1}{l_2}\right)^3 = \frac{V_1}{V_2}$$
$$\left(\frac{c}{7}\right)^3 = \frac{80}{10}$$
$$= 8$$
$$\frac{c}{7} = \sqrt[3]{8}$$
$$= 2$$
$$c = 2 \times 7$$
$$= \mathbf{14}$$

(d) Let V_1 and l_1 be the volume and length of the smaller solid respectively.
Let V_2 and l_2 be the volume and length of the larger solid respectively.

$$\left(\frac{l_1}{l_2}\right)^3 = \frac{V_1}{V_2}$$
$$\left(\frac{d}{15}\right)^3 = \frac{20}{540}$$
$$= \frac{1}{27}$$
$$\frac{d}{15} = \sqrt[3]{\frac{1}{27}}$$
$$= \frac{1}{3}$$
$$d = \frac{1}{3} \times 15$$
$$= \mathbf{5}$$

5. **(i)** Let A_1, V_1 and l_1 be the area of the base, volume and height of the smaller cone respectively.
Let A_2, V_2 and l_2 be the area of the base, volume and height of the larger cone respectively.

$$\left(\frac{l_1}{l_2}\right)^2 = \frac{A_1}{A_2} = \frac{9}{16}$$

$$\frac{l_1}{l_2} = \sqrt{\frac{9}{16}} \text{ (since } \frac{l_1}{l_2} > 0) = \frac{3}{4}$$

∴ the ratio is **3 : 4**.

(ii) $$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{V_1}{448} = \left(\frac{3}{4}\right)^3$$

$$V_1 = \left(\frac{3}{4}\right)^3 \times 448 = \mathbf{189\ cm^3}$$

6. Let m_1, V_1 and l_1 be the mass, volume and diameter of the smaller sphere respectively.
Let m_2, V_2 and l_2 be the mass, volume and diameter of the larger sphere respectively.

$$\left(\frac{l_2}{l_1}\right)^3 = \frac{V_2}{V_1} = \frac{m_2}{m_1} = \frac{640}{270} = \frac{64}{27}$$

$$\frac{l_2}{l_1} = \sqrt[3]{\frac{64}{27}} = \mathbf{\frac{4}{3}}$$

7. Let m_1, V_1 and l_1 be the mass, volume and height of the 'mini' bottle respectively.
Let m_2, V_2 and l_2 be the mass, volume and height of the 'regular' bottle respectively.

$$\left(\frac{l_2}{l_1}\right)^3 = \frac{V_2}{V_1} = \frac{m_2}{m_1}$$

$$\left(\frac{l_2}{15}\right)^3 = \frac{750}{280}$$

$$\frac{l_2^3}{15^3} = \frac{75}{28}$$

$$\frac{l_2^3}{3375} = \frac{75}{28}$$

$$l_2^3 = \frac{75}{28} \times 3375 = \frac{253\ 125}{28}$$

$$l_2 = \sqrt[3]{\frac{253\ 125}{28}} = \mathbf{20.8\ cm} \text{ (to 3 s.f.)}$$

8. **(i)** Let m_1, A_1, V_1 and l_1 be the mass, total surface area, volume and height of the smaller candy cane respectively.
Let m_2, A_2, V_2 and l_2 be the mass, total surface area, volume and height of the larger candy cane respectively.

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2 = \left(\frac{4}{7}\right)^2 = \mathbf{\frac{16}{49}}$$

(ii) $$\frac{m_2}{m_1} = \frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{m_2}{10} = \left(\frac{7}{4}\right)^3$$

$$m_2 = \left(\frac{7}{4}\right)^3 \times 10 = \mathbf{53.6\ g} \text{ (to 3 s.f.)}$$

9. Let V_1 and l_1 be the volume and radius of the smaller sphere respectively.
Let V_2 and l_2 be the volume and radius of the larger sphere respectively.

$$\left(\frac{l_2}{l_1}\right)^3 = \frac{V_2}{V_1}$$

$$\left(\frac{l_2}{3}\right)^3 = 4$$

$$\frac{l_2^3}{3^3} = 4$$

$$\frac{l_2^3}{27} = 4$$

$$l_2^3 = 4 \times 27 = 108$$

$$l_2 = \sqrt[3]{108} = \mathbf{4.76\ cm} \text{ (to 3 s.f.)}$$

10. Let m_1, V_1 and l_1 be the mass, volume and height of the smaller glass figurine respectively.
Let m_2, V_2 and l_2 be the mass, volume and height of the larger glass figurine respectively.

$$\frac{m_1}{m_2} = \frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{m_1}{500} = \left(\frac{4}{6}\right)^3$$

$$m_1 = \left(\frac{4}{6}\right)^3 \times 500 = \mathbf{148\ g} \text{ (to 3 s.f.)}$$

11. **(i)** Let m_1, V_1 and l_1 be the mass, volume and length of the model train respectively.

Let m_2, V_2 and l_2 be the mass, volume and length of the train respectively.

$l_2 = 10$ m
$= 10 \times 100$ cm (1 m = 100 cm)
$= 1000$ cm

$m_2 = 72$ tonnes
$= 72 \times 1000$ kg (1 tonne = 1000 kg)
$= 72\ 000$ kg

$$\frac{m_1}{m_2} = \frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{m_1}{72\ 000} = \left(\frac{40}{1000}\right)^3$$

$$m_1 = \left(\frac{40}{1000}\right)^3 \times 72\ 000$$

$$= \mathbf{4.608\ kg}$$

(ii) $$\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{V_2}{0.85} = \left(\frac{1000}{40}\right)^3$$

$$V_2 = \left(\frac{1000}{40}\right)^3 \times 0.85$$

$= \mathbf{13\ 281}$ ***l*** (to the nearest integer)

12. Let m_1, A_1, V_1 and l_1 be the mass, base area, volume and height of the smaller tank respectively.

Let m_2, A_2, V_2 and l_2 be the mass, base area, volume and height of the larger tank respectively.

$$\left(\frac{l_1}{l_2}\right)^3 = \frac{V_1}{V_2} = \frac{m_1}{m_2} = \frac{4.29}{8.58} = \frac{1}{2}$$

$$\frac{l_1}{l_2} = \sqrt[3]{\frac{1}{2}}$$

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{A_1}{12.94} = \left(\sqrt[3]{\frac{1}{2}}\right)^2$$

$$A_1 = \left(\sqrt[3]{\frac{1}{2}}\right)^2 \times 12.94$$

$= \mathbf{8.15\ m^2}$ (to 3 s.f.)

13. **(i)** Let A_1, V_1 and l_1 be the area of the top surface, volume and depth of the water respectively.

Let A_2, V_2 and l_2 be the area of the top surface, volume and height of the container respectively.

Volume of water $= 336$ cm^3

$$\frac{1}{3}A_1l_1 = 336$$

$$\frac{1}{3} \times 28 \times l_1 = 336$$

$$l_1 = \frac{336 \times 3}{28}$$

$$= \mathbf{36\ cm}$$

(ii) $$\left(\frac{l_1}{l_2}\right)^2 = \frac{A_1}{A_2} = \frac{28}{63} = \frac{4}{9}$$

$$\frac{l_1}{l_2} = \sqrt{\frac{4}{9}} \text{ (since } \frac{l_1}{l_2} > 0)$$

$$= \mathbf{\frac{2}{3}}$$

(iii) $$\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{V_2}{336} = \left(\frac{3}{2}\right)^3$$

$$V_2 = \left(\frac{3}{2}\right)^3 \times 336$$

$$= \mathbf{1134\ cm^3}$$

14. **(i)** Let A_1, V_1 and l_1 be the area of the top surface, volume and depth of the mercury respectively.

Let A_2, V_2 and l_2 be the area of the top surface, volume and height of the container respectively.

$$\left(\frac{l_1}{l_2}\right)^3 = \frac{V_1}{V_2}$$

$$\left(\frac{l_1}{15}\right)^3 = \frac{8}{27}$$

$$\frac{l_1}{15} = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

$$l_1 = \frac{2}{3} \times 15$$

$$= \mathbf{10\ cm}$$

(ii) $$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{A_1}{45} = \left(\frac{10}{15}\right)^2$$

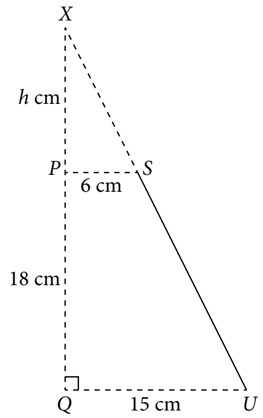
$$A_1 = \left(\frac{10}{15}\right)^2 \times 45$$

$$= \mathbf{20\ cm^2}$$

(iii) $$V_2 = \frac{1}{3}A_2l_2 = \frac{1}{3} \times 45 \times 15$$

$$= \mathbf{225\ cm^3}$$

15. Let the height of the cone XRS be h cm.



$\angle PXS = \angle QXU$ (common $\angle$)

$\angle XPS = \angle XQU$ (corr. $\angle$s, PS // QU)

$\therefore \angle PSX = \angle QUX$ and hence $\triangle XPS$ is similar to $\triangle XQU$.

$$\frac{QX}{PX} = \frac{QU}{PS}$$

$$\frac{h+18}{h} = \frac{15}{6}$$

$$6h + 108 = 15h$$

$$9h = 108$$

$$h = 12$$

The height of the larger cone $= 12 + 18$

$= 30$ cm

$\therefore$ volume of frustum

$=$ volume of cone XTU – volume of cone XRS

$= \frac{1}{3}\pi(15^2)(30) - \frac{1}{3}\pi(6^2)(12)$

$= \frac{1}{3}\pi(15^2 \times 30 - 6^2 \times 12)$

$= \frac{1}{3}\pi(6318)$

$= 2106\pi$

$=$ **6620 cm³** (to 3 s.f.)

16. Let m_1, V_1 and l_1 be the mass, volume and height of the smaller clay model respectively.
Let m_2, V_2 and l_2 be the mass, volume and height of the larger clay model respectively.

$$\frac{m_2}{m_1} = \frac{V_2}{V_1}$$

$$= \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{x^2}{x+0.3} = \left(\frac{30}{20}\right)^3$$

$$= \frac{27}{8}$$

$$8x^2 = 27x + 8.1$$

$$8x^2 - 27x - 8.1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-27) \pm \sqrt{(-27)^2 - 4(8)(-8.1)}}{2(8)}$$

$$= \frac{27 \pm \sqrt{988.2}}{16}$$

$= 3.65$ or -0.277 (to 3 s.f.)

If $x = -0.277$, $(x + 0.3)$ kg $= (-0.277 + 0.3)$ kg

$= 0.023$ kg

$\therefore x = -0.277$ is an acceptable value.

$\therefore x =$ **3.65 or −0.277**

17. **(a)** Let A_1, V_1 and l_1 be the area, volume and diameter of C_A respectively.
Let A_2, V_2 and l_2 be the area, volume and diameter of C_B respectively.

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$= \left(\frac{20.4}{26.4}\right)^2$$

$$= \mathbf{\frac{289}{484}}$$

(b) **Yes**. Since circles are similar, cones made from $\frac{3}{4}$-circles are also similar.

(c) **No**, because $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$ but $\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$.

Notes