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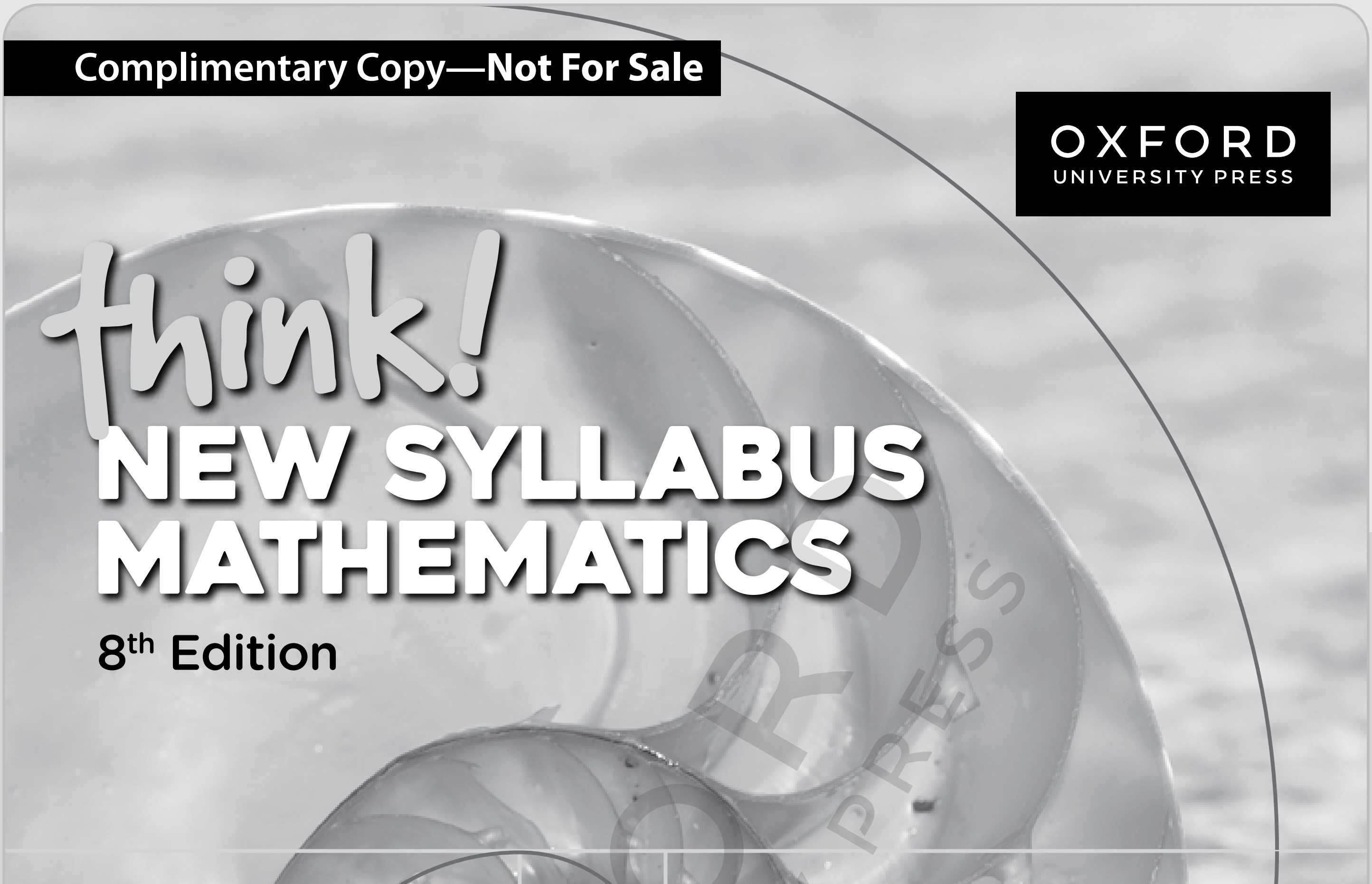

think!
NEW SYLLABUS MATHEMATICS

8th Edition

Teacher's Resource Book 3

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CONTENTS

Syllabus Matching Grid

Cambridge O Level Mathematics (Syllabus D) 4024. Syllabus for examination in 2025, 2026 and 2027.

Theme or Topic	Subject Content	Reference
1. Number	Identify and use: • Natural numbers • Integers (positive, negative and zero) • Prime numbers • Square numbers • Cube numbers • Common factors and common multiples • Rational and irrational numbers (e.g. π, $\sqrt{2}$) • Reciprocals.	Book 1: Chapter 1 Chapter 2 Chapter 4
2. Sets	• Use set language, set notation and Venn diagrams to describe sets and represent relationships between sets	Book 2: Chapter 12 Book 4: Chapter 1
2. Powers and roots	Calculate with the following: • Squares • Square roots • Cubes and cube roots of numbers • Cube roots • other powers and roots of numbers	Book 1: Chapter 1 Chapter 2 Chapter 3 Chapter 4
4. Fractions, decimals and percentages	• Use the language and notation of the following in appropriate contexts: (a) proper fractions (b) improper fractions (c) mixed numbers (d) decimals (e) percentages. • Recognise equivalence and convert between these forms.	Book 1: Chapter 2 Chapter 3 Chapter 8
5. Ordering	• Order quantities by magnitude and demonstrate familiarity with the symbols $=, \neq, <, >, \leqslant, \geqslant$.	Book 1: Chapter 2 Chapter 3 Chapter 4 Book 2: Chapter 3
6. The four operations	• Use the four operations for calculations with integers, fractions and decimals, including correct ordering of operations and use of brackets.	Book 1: Chapter 2 Chapter 3 Chapter 4
7. Indices I	• Understand and use indices (positive, zero, negative and fractional). • Understand and use the rules of indices.	Book 3: Chapter 4
8. Standard form	• Use the standard form $A \times 10^n$, where n is a positive or negative integer, and $1 \leqslant A < 10$. • Convert numbers into and out of standard form. • Calculate with values in standard form.	Book 3: Chapter 4

9. Estimation	• Round values to a specified degree of accuracy. • Make estimates for calculations involving numbers, quantities and measurements. • Round answers to a reasonable degree of accuracy in the context of a given problem.	Book 1: Chapter 5
10. Limits of accuracy	• Give upper and lower bounds for data rounded to a specified accuracy. • Find upper and lower bounds of the results of calculations which have used data rounded to a specified accuracy.	Book 1: Chapter 5
11. Ratio and proportion	Understand and use ratio and proportion to: • give ratios in their simplest form • divide a quantity in a given ratio • use proportional reasoning and ratios in context.	Book 1: Chapter 9
12. Rates	• Use common measures of rate. • Apply other measures of rate. • Solve problems involving average speed.	Book 1: Chapter 9
13. Percentages	• Calculate a given percentage of a quantity. • Express one quantity as a percentage of another. • Calculate percentage increase or decrease. • Calculate with simple and compound interest. • Calculate using reverse percentages.	Book 1: Chapter 8
14. Using a calculator	• Use a calculator efficiently. • Enter values appropriately on a calculator. • Interpret the calculator display appropriately.	Book 1: Chapter 1 Chapter 4 Chapter 12
15. Time	• Calculate with time: seconds (s), minutes (min), hours (h), days, weeks, months, years, including the relationship between units. • Calculate times in terms of the 24-hour and 12-hour clock. • Read clocks and timetables.	Book 1: Chapter 9
16. Money	• Calculate with money. • Convert from one currency to another.	Book 1: Chapter 9 Book 2: Chapter 6
17. Exponential growth and decay	• Use exponential growth and decay.	Book 3: Chapter 4
18. Surds	• Understand and use surds, including simplifying expressions. • Rationalise the denominator.	Book 3: Chapter 4
19. Introduction to algebra	• Know that letters can be used to represent generalised numbers. • Substitute numbers into expressions and formulas.	Book 1: Chapter 6
20. Algebraic manipulation	• Simplify expressions by collecting like terms. • Expand products of algebraic expressions. • Factorise by extracting common factors. • Factorise expressions of the form: (a) $ax + bx + kay + kby$ (b) $a^2x^2 - b^2y^2$ (c) $a^2 + 2ab + b^2$ (d) $ax^2 + bx + c$ (e) $ax^2 + bx^2 + cx$ • Complete the square for expressions in the form $ax^2 + bx + c$.	Book 1: Chapter 6 Book 2: Chapter 4

21. Algebraic fractions	• Manipulate algebraic fractions. • Factorise and simplify rational expressions.	Book 3: Chapter 1
22. Indices II	• Understand and use indices (positive, zero, negative and fractional). • Understand and use the rules of indices.	Book 3: Chapter 4
23. Equations	• Construct expressions, equations and formulas. • Solve linear equations in one unknown. • Solve fractional equations with numerical and linear algebraic denominators. • Solve simultaneous linear equations in two unknowns. • Solve quadratic equations by factorisation, completing the square and by use of the quadratic formula. • Change the subject of formulas.	Book 1: Chapter 7 Book 2: Chapter 2 Book 3: Chapter 1 Chapter 2 Chapter 3
24. Inequalities	• Represent and interpret inequalities, including on a number line. • Construct, solve and interpret linear inequalities. • Represent and interpret linear inequalities in two variables graphically. • List inequalities that define a given region.	Book 2: Chapter 3
25. Sequences	• Continue a given number sequence or pattern. • Recognise patterns in sequences, including the term-to-term rule, and relationships between different sequences. • Find and use the *n*th term of sequences.	Book 2: Chapter 5
26. Proportion	• Express direct and inverse proportion in algebraic terms and use this form of expression to find unknown quantities.	Book 2: Chapter 7
27. Graphs in practical situations	• Use and interpret graphs in practical situations including travel graphs and conversion graphs. • Draw graphs from given data. • Apply the idea of rate of change to simple kinematics involving distance-time and speed-time graphs, acceleration and deceleration. • Calculate distance travelled as area under a speed-time graph.	Book 2: Chapter 1 Book 3: Chapter 6
28. Graphs of functions	• Construct tables of values, and draw, recognise and interpret graphs for functions of the following forms: (a) ax^n (includes sums of no more than three of these) (b) $ab^x + c$ where $n = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2, 3$; a and c are rational numbers; and b is a positive integer. • Solve associated equations graphically, including finding and interpreting roots by graphical methods. • Draw and interpret graphs representing exponential growth and decay problems. • Estimate gradients of curves by drawing tangents.	Book 3: Chapter 6
29. Sketching curves	• Recognise, sketch and interpret graphs of the following functions: (a) linear (b) quadratic (c) cubic (d) reciprocal (e) exponential.	Book 2: Chapter 1 Book 3: Chapter 2 Chapter 3 Chapter 6
30. Functions	• Understand functions, domain and range, and use function notation. • Understand and find inverse functions $f^{-1}(x)$. • Form composite functions as defined by $gf(x) = g(f(x))$.	Book 4: Chapter 5

31. Coordinates	• Use and interpret Cartesian coordinates in two dimensions.	Book 2: Chapter 1
32. Drawing linear graphs	• Draw straight-line graphs for linear equations.	Book 2: Chapter 1
33. Gradient of linear graphs	• Find the gradient of a straight line. • Calculate the gradient of a straight line from the coordinates of two points on it.	Book 2: Chapter 1 Book 3: Chapter 5
34. Length and midpoint	• Calculate the length of a line segment. • Find the coordinates of the midpoint of a line segment.	Book 3: Chapter 5
35. Equations of linear graphs	• Interpret and obtain the equation of a straight-line graph.	Book 2: Chapter 1 Book 3: Chapter 5
36. Parallel lines	• Find the gradient and equation of a straight line parallel to a given line.	Book 3: Chapter 5
37. Perpendicular lines	• Find the gradient and equation of a straight line perpendicular to a given line.	Book 3: Chapter 5
38. Geometrical terms	• Use and interpret the following geometrical terms: (a) point (b) vertex (c) line (d) plane (e) parallel (f) perpendicular (g) perpendicular bisector (h) bearing (i) right angle (j) acute, obtuse and reflex angles (k) interior and exterior angles (l) similar (m) congruent (n) scale factor. • Use and interpret the vocabulary of: (a) triangles (b) special quadrilaterals (c) polygons (d) nets (e) solids. • Use and interpret the vocabulary of a circle.	Book 1: Chapter 10 Chapter 11 Book 2: Chapter 8 Chapter 11
39. Geometrical constructions	• Measure and draw lines and angles. • Construct a triangle, given the lengths of all sides, using a ruler and pair of compasses only. • Draw, use and interpret nets.	Book 1: Chapter 11

40. Scale drawings	• Draw and interpret scale drawings. • Use and interpret three-figure bearings.	Book 2: Chapter 8 Book 4: Chapter 7
41. Similarity	• Calculate lengths of similar shapes. • Use the relationships between lengths and areas of similar shapes and lengths, surface areas and volumes of similar solids. • Solve problems and give simple explanations involving similarity.	Book 2: Chapter 8 Book 4: Chapter 11
42. Symmetry	• Recognise line symmetry and order of rotational symmetry in two dimensions. • Recognise symmetry properties of prisms, cylinders, pyramids and cones.	Book 1: Chapter 11 Book 2: Chapter 11 Book 3: Chapter 7
43. Angles	• Calculate unknown angles and give simple explanations using the following geometrical properties: (a) sum of angles at a point = 360° (b) sum of angles at a point on a straight line = 180° (c) vertically opposite angles are equal (d) angle sum of a triangle = 180° and angle sum of a quadrilateral = 360°. • Calculate unknown angles and give geometric explanations for angles formed within parallel lines: (a) corresponding angles are equal (b) alternate angles are equal (c) co-interior (supplementary) angles sum to 180°. • Know and use angle properties of regular and irregular polygons.	Book 1: Chapter 10 Chapter 11
44. Circle theorems I	• Calculate unknown angles and give explanations using the following geometrical properties of circles: • angle in a semicircle = 90° • angle between tangent and radius = 90° • angle at the centre is twice the angle at the circumference • angles in the same segment are equal • opposite angles of a cyclic quadrilateral sum to 180° (supplementary) • alternate segment theorem.	Book 4: Chapter 9
45. Circle theorems II	• Use the following symmetry properties of circles: • equal chords are equidistant from the centre • the perpendicular bisector of a chord passes through the centre • tangents from an external point are equal in length.	Book 4: Chapter 9
46. Units of measure	• Use metric units of mass, length, area, volume and capacity in practical situations and convert quantities into larger or smaller units.	Book 1: Chapter 3 Chapter 12 Book 2: Chapter 11
47. Area and perimeter	• Carry out calculations involving the perimeter and area of a rectangle, triangle, parallelogram and trapezium.	Book 1: Chapter 12

48. Circles, arcs and sectors	• Carry out calculations involving the circumference and area of a circle. • Carry out calculations involving arc length and sector area as fractions of the circumference and area of a circle.	Book 1: Chapter 12 Book 4: Chapter 8
49. Surface area and volume	Carry out calculations and solve problems involving the surface area and volume of a: • cuboid • prism • cylinder • sphere • pyramid • cone.	Book 2: Chapter 11 Book 3: Chapter 7
50. Compound shapes and parts of shapes	• Carry out calculations and solve problems involving perimeters and areas of: (a) compound shapes (b) parts of shapes. • Carry out calculations and solve problems involving surface areas and volumes of: (a) compound solids (b) parts of solids.	Book 1: Chapter 12 Book 2: Chapter 11 Book 3: Chapter 7
51. Pythagoras' theorem	• Know and use Pythagoras' theorem.	Book 2: Chapter 9
52. Right-angled triangles	• Know and use the sine, cosine and tangent ratios for acute angles in calculations involving sides and angles of a right-angled triangle. • Solve problems in two dimensions using Pythagoras' theorem and trigonometry. • Know that the perpendicular distance from a point to a line is the shortest distance to the line. • Carry out calculations involving angles of elevation and depression.	Book 2: Chapter 9 Chapter 10 Book 4: Chapter 6 Chapter 7
53. Non-right-angled triangles	• Use the sine and cosine rules in calculations involving lengths and angles for any triangle. • Use the formula area of triangle = $\frac{1}{2}ab\sin C$	Book 4: Chapter 6
54. Pythagoras' theorem and trigonometry in 3D	• Carry out calculations and solve problems in three dimensions using Pythagoras' theorem and trigonometry, including calculating the angle between a line and a plane.	Book 4: Chapter 7
55. Transformations	Recognise, describe and draw the following transformations: • Reflection of a shape in a straight line. • Rotation of a shape about a centre through multiples of 90°. • Enlargement of a shape from a centre by a scale factor. • Translation of a shape by a vector $\begin{pmatrix} x \\ y \end{pmatrix}$.	Book 4: Chapter 10
56. Vectors in two dimensions	• Describe a translation using a vector represented by $\begin{pmatrix} x \\ y \end{pmatrix}$, $\overrightarrow{AB}$ or $\mathbf{a}$ • Add and subtract vectors. • Multiply a vector by a scalar.	Book 4: Chapter 7
57. Magnitude of a vector	• Calculate the magnitude of a vectorvector $\begin{pmatrix} x \\ y \end{pmatrix}$ as $\sqrt{x^2 + y^2}$	Book 4: Chapter 4

58. Vector geometry	• Represent vectors by directed line segments. • Use position vectors. • Use the sum and difference of two or more vectors to express given vectors in terms of two coplanar vectors. • Use vectors to reason and to solve geometric problems.	Book 4: Chapter 4
59. Introduction to probability	• Understand and use the probability scale from 0 to 1. • Understand and use probability notation. • Calculate the probability of a single event. • Understand that the probability of an event not occurring = 1 – the probability of the event occurring.	Book 2: Chapter 12
60. Relative and expected frequencies	• Understand relative frequency as an estimate of probability. • Calculate expected frequencies.	Book 2: Chapter 12
61. Probability of combined events	Calculate the probability of combined events using, where appropriate: • sample space diagrams • Venn diagrams • tree diagrams.	Book 4: Chapter 2
62. Classifying statistical data	• Classify and tabulate statistical data.	Book 1: Chapter 13
63. Interpreting statistical data	• Read, interpret and draw inferences from tables and statistical diagrams. • Compare sets of data using tables, graphs and statistical measures. • Appreciate restrictions on drawing conclusions from given data.	Book 1: Chapter 13
64. Averages and measures of spread	• Calculate the mean, median, mode and range for individual data and distinguish between the purposes for which these are used. • Calculate an estimate of the mean for grouped discrete or grouped continuous data. • Identify the modal class from a grouped frequency distribution.	Book 3: Chapter 8
65. Statistical charts and diagrams	• Draw and interpret: (a) bar charts (b) pie charts (c) pictograms (d) simple frequency distributions.	Book 1: Chapter 13
66. Scatter diagrams	• Draw and interpret scatter diagrams. • Understand what is meant by positive, negative and zero correlation. • Draw by eye, interpret and use a straight line of best fit.	Book 4: Chapter 3
67. Cumulative frequency diagrams	• Draw and interpret cumulative frequency tables and diagrams. • Estimate and interpret the median, percentiles, quartiles and interquartile range from cumulative frequency diagrams.	Book 4: Chapter 3
68. Histograms	• Draw and interpret histograms. • Calculate with frequency density.	Book 2: Chapter 13

Secondary 3 Mathematics Scheme of Work

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
1	**1 Algebraic Fractions and Formulae**	**1.1** Algebraic fractions (pp. 2 – 5)	• Simplify algebraic fractions	Manipulate algebraic fractions	Thinking Time (pp. 4 – 5)		
1		**1.2** Multiplication and division of algebraic fractions (pp. 5 – 7)	• Multiply and divide algebraic fractions				
1		**1.3** Addition and subtraction of algebraic fractions (pp. 7 – 10)	• Add and subtract algebraic fractions • Express algebraic fractions with linear or quadratic denominators into a single fraction				Worked Example 7 (pp. 9 – 10) Practise Now 7 (p. 10)
2		**1.4** Solving equations involving algebraic fractions (pp. 10 – 13)	• Solve equations involving algebraic fractions which are reducible to linear equations				Ex 1B Q 9 (p. 13)
2		**1.5** Manipulating algebraic formulae (pp. 14 – 22)	• Change the subject of formula • Find value of an unknown in a formula		Journal Writing (p. 15)		Worked Example 12(b) (p. 17) Practise Now 12 Q 3(b) (p. 18) Ex 1C Q 15(ii) (p. 20)
2		**Miscellaneous**					
3	**2 Quadratic Equations and Graphs**	**2.1** Solving quadratic equations by factorisation (pp. 24 – 31)	• Solve quadratic equations by factorisation	Solve quadratic equations by factorisation	Class Discussion – Linear equations (p. 24) Class Discussion – Mathematical fallacy (p. 26) Class Discussion – Mathematical fallacy: 2 = 1? (p. 28)		Class Discussion – Mathematical fallacy (p. 26) Worked Example 3(b) (p. 27) Practise Now 3 Q 2 (p. 28) Class Discussion – Mathematical fallacy: 2 = 1? (p. 28) Ex 2A Q 14(b) (p. 31)

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
4		**2.2** Quadratic functions and graphs (pp. 32 – 44)	• Know what quadratic functions and the properties of their graphs are • Draw graphs of quadratic functions • Solve problems involving quadratic functions in real-world contexts	Interpret graphs of quadratic functions	Investigation – Relationship between area of square and its length (p. 32) Investigation – Graphs of $y = x^2$ and $y = -x^2$ (p. 33) Investigation – Graphs of quadratic functions $y = ax^2 + bx + c$ (p. 34)	Investigation – Relationship between area of square and its length (p. 32) Investigation – Graphs of $y = x^2$ and $y = -x^2$ (p. 33) Investigation – Graphs of quadratic functions $y = ax^2 + bx + c$ (p. 34)	Practise Now 10 Q 2(ii) (p. 40) Ex 2B Q 11(ii), 13(iii) (p. 43)
4		**Miscellaneous**					
5	**3 Quadratic and Fractional Equations**	**3.1** Solving quadratic equations by completing the square (pp. 46 – 52)	• Solve quadratic equations by completing the square	Solve quadratic equations by completing the square	Investigation – Completing the square for quadratic expressions of the form $x^2 + bx$ (pp. 48 – 49)		
5		**3.2** Solving quadratic equations using formula (pp. 53 – 55)	• Solve quadratic equations using formula	Solve quadratic equations by use of the quadratic formula	Class Discussion – Number of real solutions to quadratic equations (p. 54)		
5		**3.3** Solving fractional equations reducible to quadratic equations (pp. 55 – 57)	• Solve fractional equations that can be reduced to quadratic equations				
6		**3.4** Solving quadratic equations by graphical method (pp. 57 – 61)	• Solve quadratic equations by graphical method	Solve quadratic equations graphically, including finding and interpreting roots by graphical methods	Thinking Time (p. 60) Journal Writing (p. 60)		

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
6		**3.5** Sketching graphs of quadratic functions (pp. 61 – 71)	• Sketch graphs of quadratic functions	Sketch and interpret graphs of quadratic functions	Investigation – Characteristics of graphs of $y = \pm(x - h)(x - k)$ (pp. 62 – 63) Investigation – Characteristics of graphs of $y = \pm(x - p)^2 + q$ (pp. 65 – 66) Thinking Time (p. 69) Class Discussion – Matching quadratic graphs with the corresponding functions (p. 70)	Investigation – Characteristics of graphs of $y = \pm(x - h)(x - k)$ (pp. 62 – 63) Investigation – Characteristics of graphs of $y = \pm(x - p)^2 + q$ (pp. 65 – 66)	
7		**3.6** Applications of quadratic equations and functions in real-world contexts (pp. 72 – 80)	• Formulate and solve quadratic equations in one variable to solve real-world problems	Solve quadratic equations by factorisation, completing the square and by use of the quadratic formula			Worked Example 14(iii) (p. 72) Practise Now 14(iv) (p. 73) Ex 3E Q 5(iv), 6(iii) (p. 77)
7		**Miscellaneous**					
8	**4 Indices, Surds, Standard Form and Exponential Growth and Decay**	**4.1** Indices (p. 82)	• Use index notation to represent a number multiplied by itself	Understand and use indices (positive, zero, negative and fractional)			
8		**4.2** Laws of Indices (pp. 82 – 91)	• State and apply the five Laws of Indices	Understand and use the rules of indices	Investigation – Discovering Law 1 of Indices (p. 83) Investigation – Discovering Law 2 of Indices (pp. 84 – 85) Investigation – Discovering Law 3 of Indices (p. 86) Investigation – Discovering Law 4 of Indices (p. 87) Investigation – Discovering Law 5 of Indices (pp. 88 – 89)		

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
9		**4.3** Zero and negative indices (pp. 91 – 95)	• State and use the definitions of zero and negative indices		Investigation – Making sense of the zero index (pp. 91 – 92) Investigation – Making sense of negative indices (p. 93) Thinking Time (p. 95)		
9		**4.4** Rational indices (pp. 96 – 103)	• State and use the definition of rational indices		Investigation – Making sense of rational indices (p. 97) Thinking Time (p. 97) Investigation – Discovering a result involving rational indices (p. 98) Thinking Time (pp. 99 – 100)		
10		**4.5** Surds (pp. 104 – 110)	• Simplify expressions involving surds • Use conjugate surd to rationalise the denominator of an expression containing a surd	Understand and use surds, including simplifying expressions Rationalise the denominator	Investigation – Simplifying surds (p. 104) Class Discussion – Mathematical fallacy: 1 = –1? (p. 105) Class Discussion – Product of two irrational numbers (p. 106) Thinking Time (p. 107) Investigation – Rational and irrational roots of quadratic equations (p. 108)		Ex 4C Q 15 (p. 110)

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
10		**4.6** Exponential growth and decay (pp. 110 – 112)	• Formulate expressions to solve problems involving exponential growth and decay				
11		**4.7** Standard form (pp. 112 – 120)	• Represent very large or very small numbers using standard form • Solve problems involving the use of standard form	Use the standard form $A \times 10^n$ where n is a positive or negative integer and $1 \leqslant A < 10$ Convert numbers into and out of standard form Calculate with values in standard form	Class Discussion – Exploring standard form (p. 113) Performance Task (p. 116)		
11		**Miscellaneous**					
12	**5 Coordinate Geometry**	**5.1** Length of a line segment (pp. 122 – 128)	• Find the length of a line segment given the coordinates of its endpoints	Calculate the length of a line segment	Class Discussion – Determining the length of a line segment in a Cartesian plane (pp. 122 – 123) Thinking Time (p. 124)		
12		**5.2** Gradient of a straight line (pp. 128 – 133)	• Find the gradient of a straight line given the coordinates of two points on it	Calculate the gradient of a straight line from the coordinates of two points on it	Class Discussion – Finding the gradient of a straight line (pp. 129 – 130) Thinking Time (p. 130) Investigation – Gradient of a straight line (pp. 130 – 131)	Investigation – Gradient of a straight line (pp. 130 – 131)	
13		**5.3** Equation of a straight line (pp. 134 – 139)	• Interpret and find the equation of a straight line	Interpret and obtain the equation of a straight-line graph	Journal Writing (p. 136)		
13		**5.4** Midpoint of a line segment (pp. 139 – 144)	• Find the midpoint of a line segment given the coordinates of its endpoints	Find the coordinates of the midpoint of a line segment	Thinking Time (p. 142) Class Discussion – Finding 4th vertex of quadrilateral (p. 143)		

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
14		**5.5** Parallel and perpendicular lines (pp. 144 – 150)	• Find the gradients of parallel lines and perpendicular lines	Find the gradient of a straight line parallel to a given line Find the gradient of a straight line perpendicular to a given line	Investigation – Angle of inclination (pp. 144 – 145) Class Discussion – Parallel lines (pp. 145 – 146) Thinking Time (p. 146) Class Discussion – Perpendicular lines (p. 147)		Worked Example 12 (p. 148) Practise Now 12 Q 1 (p. 148) Ex 5E Q 9 (p. 149)
14		**5.6** Equation of a straight line involving parallel and perpendicular lines (pp. 150 – 156)	• Solve geometry problems involving parallel and perpendicular lines	Find the equation of a straight line parallel to a given line Find the equation of a straight line perpendicular to a given line			
14		**Miscellaneous**					
15	**6 Graphs of Functions and Graphical Solution**	**6.1** Graphs of cubic functions (pp. 158 – 165)	• Draw and sketch graphs of cubic functions	Construct tables of values, and draw, recognise and interpret graphs for functions of the following forms: • ax^n (includes sums of no more than three of these) • $ab^x + c$ where $n = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}$, 1, 2, 3; a and c are rational numbers; and b is a positive integer Solve associated equations graphically, including finding and interpreting roots by graphical methods Draw and interpret graphs representing exponential growth and decay problems	Investigation – Graphs of cubic functions (p. 159) Investigation – Graphical solution to equations (p. 161)	Investigation – Graphs of cubic functions (p. 159) Investigation – Graphical solution to equations (p. 161)	
15		**6.2** Graphs of reciprocal functions (pp. 165 – 172)	• Draw and sketch graphs of reciprocal functions		Investigation – Graphs of $y = \frac{a}{x}$ (p. 166) Thinking Time (p. 167) Investigation – Graphs of $y = \frac{a}{x^2}$ (p. 170)	Investigation – Graphs of $y = \frac{a}{x}$ (p. 166) Investigation – Graphs of $y = \frac{a}{x^2}$ (p. 170)	

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
16		**6.3** Graphs of functions involving $\sqrt{x}$ (pp. 172 – 177)	• Draw the graphs of functions involving $\sqrt{x}$		Investigation – Graphs of $y = a\sqrt{x}$ (p. 172) Investigation – Graphs of $y = \frac{a}{\sqrt{x}}$ (p. 174)	Investigation – Graphs of $y = a\sqrt{x}$ (p. 172) Investigation – Graphs of $y = \frac{a}{\sqrt{x}}$ (p. 174)	
16		**6.4** Graphs of exponential functions (pp. 178 – 182)	• Draw and sketch graphs of exponential functions		Investigation – Graph of $y = a^x$ and $y = ka^x$ (p. 178) Journal Writing (p. 179) Class Discussion – Matching graphs of power functions with the corresponding functions (p. 182)	Investigation – Graph of $y = a^x$ and $y = ka^x$ (p. 178)	
17		**6.5** Graphs of rational functions (pp. 183 – 186)	• Draw graphs of rational functions		Investigation – Graph of a rational function involving polynomials and its asymptotes (p. 183)	Investigation – Graph of a rational function involving polynomials and its asymptotes (p. 183)	
17		**6.6** Gradient of a curve (pp. 186 – 192)	• Estimate the gradient of a curve by drawing a tangent	Estimate gradients of curves by drawing tangents			Ex 6B Q 14(b) (p. 192)

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
18		**6.7** Applications of graphs in real-world contexts (pp. 193 – 208)	• Solve problems in real-world contexts, including distance-time and speed-time graphs	Use and interpret graphs in practical situations including travel graphs and conversion graphs Draw graphs from given data Apply the idea of rate of change to simple kinematics involving distance-time and speed-time graphs, acceleration and deceleration Calculate distance travelled as area under a speed-time graph	Class Discussion – Linear distance-time graphs (p. 193)		
18		**Miscellaneous**					
19	**7 Volume, Surface Area and Symmetry of Pyramids, Cones and Spheres**	**7.1** Volume, surface area and symmetry of pyramids (pp. 210 – 224)	• Identify pyramids • Find volume and surface area of pyramids, including the use of nets • Recognise the symmetrical properties of pyramids	Carry out calculations and solve problems involving the surface area and volume of a pyramid	Class Discussion – What are pyramids? (p. 210) Thinking Time (p. 212) Journal Writing (p. 212) Investigation – Volume of pyramid (pp. 213 – 214) Investigation – Symmetry in pyramids (pp. 220 – 221)		Ex 7A Q 18 (p. 224)

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
20		**7.2** Volume, surface area and symmetry of cones (pp. 225 – 233)	• Identify cones • Find volume and surface area of cones • Recognise the symmetrical properties of cones	Carry out calculations and solve problems involving the surface area and volume of a cone	Class Discussion – What are cones? (p. 225) Thinking Time (p. 226) Investigation – Comparison between a cone and a pyramid (p. 227) Thinking Time (p. 228) Investigation – Curved surface area of cone (p. 229) Thinking Time (p. 230) Investigation – Symmetry in cones (p. 231)		
20		**7.3** Volume and surface area of spheres (pp. 234 – 240)	• Identify spheres • Find volume and surface area of spheres	Carry out calculations and solve problems involving the surface area and volume of a sphere	Thinking Time (p. 234) Class Discussion – Is the King's crown made of pure gold? (pp. 234 – 235) Investigation – Volume of sphere (p. 235) Investigation – Surface area of sphere (pp. 237 – 238) Thinking Time (p. 238) Journal Writing (p. 238)		

Week (5 classes × 45 min)	Chapter	Section	Specific Instructional Objectives (SIOs)	Syllabus Subject Content	Activity	ICT	Reasoning, Communication and Connection
21		**7.4** Volume and surface area of composite solids (pp. 240 – 246)	• Solve problems involving the volume and surface area of composite solids	Carry out calculations and solve problems involving surface areas and volumes of: • compound solids • parts of solids.			
21		**Miscellaneous**					
22	**8 Averages of Statistical Data**	**8.1** Mean (pp. 248 – 258)	• Find the mean of a set of data • Calculate an estimate for the mean	Calculate the mean for individual data Calculate an estimate of the mean for grouped discrete or grouped continuous data	Class Discussion – Understanding the concept of mean (p. 251)		
23		**8.2** Median (pp. 258 – 262)	• Find the median of a set of data • Find the class interval where the median lies	Calculate the median for individual data	Class Discussion – Understanding the concept of median (p. 261)		
23		**8.3** Mode (pp. 263 – 265)	• Find the mode of a set of data • State the modal class of a set of grouped data	Calculate the mode for individual data Identify the modal class from a grouped frequency distribution	Class Discussion – Mode as an average (p. 263) Class Discussion – Understanding the concept of mode (p. 264)		
24		**8.4** Measures of central tendency (pp. 265 – 276)	• Evaluate the purposes and appropriateness of the use of the mean, median and mode in various contexts	Distinguish between the purposes for which the mean, median and mode are used	Class Discussion – Comparing different measures of central tendency (p. 266) Journal Writing (p. 271)		Ex 8B Q 8(v), 9(iv) (p. 273)
24		**Miscellaneous**					

Chapter 1 Algebraic Fractions and Formulae

TEACHING NOTES

Suggested Approach

Teachers may want to relate algebraic fractions with numerical fractions by simply replacing numerals with algebraic expressions. Students are expected to know how to manipulate algebraic fractions, perform the operations on algebraic fractions and express one variable in terms of the other variables at the end of this chapter.

Section 1.1: Algebraic Fractions

Students are to recall that the value of a fraction does not change when multiplied or divided by a non-zero number or expression. The same applies to algebraic fractions. The usual rules of expansion and factorisation can be used as well.

Section 1.2: Multiplication and Division of Algebraic Fractions

Teachers can let students observe that the multiplication and division of algebraic fractions are very much similar to the multiplication and division of numerical fractions. The notion of reciprocals should be illustrated. Students may need reminders to check that their final expression are in its simplest form.

Section 1.3: Addition and Subtraction of Algebraic Fractions

In Book 1, students have learnt to manipulate linear expressions with fractional coefficients. Here, the scope is extended to algebraic fractions. Teachers should highlight the important step of converting algebraic fractions to like fractions, and after this step, the fractions can be combined to a single fraction and then simplified.

Section 1.4: Manipulation of Algebraic Formulae

Teachers can work through the worked examples together with the students so as to illustrate how one variable can be made the subject of a formula.

Teachers should emphasise that when making a single variable the subject of a formula, the single variable must be on the left-hand side of the equation while all other variables must be on the right-hand side of the equation. Often, students may make mistakes whereby the subject of the formula is also found on the right-hand side of the equation.

Another error students may make is to substitute the wrong values when evaluating the expression, particularly when there are multiple variables in the equation.

Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 12).

1.1 Algebraic fractions

Practise Now 1

(a) $\dfrac{{}^{2}\cancel{8}x^{\cancel{5}\,2}\cancel{y}^{1}}{{}_{3}\cancel{12}\cancel{x^3}_{1}y^{\cancel{4}\,3}} = \dfrac{\mathbf{2x^2}}{\mathbf{3y^3}}$

(b) $\dfrac{{}^{1}\cancel{9}x^{\cancel{4}\,2}(x-y)^{\cancel{3}\,2}}{{}_{3}\cancel{27}\cancel{x^2}_{1}y^3\cancel{(x-y)}_{1}} = \dfrac{\mathbf{x^2(x-y)^2}}{\mathbf{3y^3}}$

Thinking Time (Page 3)

(a) $\dfrac{ab}{ab} = 1$

When ab in the numerator and the denominator are divided by each other, the result is 1.

(b) $\dfrac{b+4b}{ab} = \dfrac{1+4}{a}$

Both terms in the numerator must be divided by b.

(c) $\dfrac{1+4}{4} = \dfrac{5}{4}$

The terms in the numerator must be added before dividing the numerator and denominator by their common factor. In this case, the resultant fraction is already in its simplest form.

Practise Now 2

(a) $\dfrac{h^2+7hk}{5hk} = \dfrac{{}^{1}\cancel{h}(h+7k)}{5\cancel{h}_{1}k}$

$= \dfrac{\mathbf{h+7k}}{\mathbf{5k}}$

(b) $\dfrac{15p}{10p^2-5p} = \dfrac{{}^{3}\cancel{15p}^{1}}{{}_{1}\cancel{5p}_{1}(2p-1)}$

$= \dfrac{\mathbf{3}}{\mathbf{2p-1}}$

(c) $\dfrac{z^2-4z}{4z-16} = \dfrac{z\cancel{(z-4)}_{1}}{4\cancel{(z-4)}_{1}}$

$= \dfrac{\mathbf{z}}{\mathbf{4}}$

Practise Now 3

1. (a) $\dfrac{3v^2-9v}{v^2-9} = \dfrac{3v\cancel{(v-3)}_{1}}{(v+3)\cancel{(v-3)}_{1}}$

$= \dfrac{\mathbf{3v}}{\mathbf{v+3}}$

(b) $\dfrac{p^2-7pq+12q^2}{5p^2-20pq} = \dfrac{\cancel{(p-4q)}^{1}(p-3q)}{5p\cancel{(p-4q)}_{1}}$

$= \dfrac{\mathbf{p-3q}}{\mathbf{5p}}$

(c) $\dfrac{x^2-3xy+2xz-6yz}{xy-2xz-3y^2+6yz} = \dfrac{x(x-3y)+2z(x-3y)}{x(y-2z)-3y(y-2z)}$

$= \dfrac{\cancel{(x-3y)}^{1}(x+2z)}{(y-2z)\cancel{(x-3y)}_{1}}$

$= \dfrac{\mathbf{x+2z}}{\mathbf{y-2z}}$

2. $\dfrac{n^4-5n^2+6}{n^4-9} = \dfrac{\cancel{(n^2-3)}^{1}(n^2-2)}{(n^2+3)\cancel{(n^2-3)}_{1}}$

$= \dfrac{\mathbf{n^2-2}}{\mathbf{n^2+3}}$

Thinking Time (Page 4)

For $\dfrac{28x^3(x+y)^2}{63xy^2(x+y)^4}$, $x \neq 0$, $y \neq 0$ and $x \neq -y$.

For $\dfrac{x^2-3x}{3x-9}$, $x \neq 3$.

For $\dfrac{2m^2-4m}{m^2-4}$, $m \neq \pm 2$.

1.2 Multiplication and division of algebraic fractions

Practise Now 4

1. (a) $\dfrac{{}^{1}\cancel{2a}^{\cancel{2}\,1}}{{}_{1}\cancel{5c}^{\cancel{3}\,2}} \times \dfrac{{}^{3}\cancel{15c}^{1}}{{}_{4}\cancel{8a}^{\cancel{4}\,2}} = \dfrac{\mathbf{3}}{\mathbf{4a^2c^2}}$

(b) $\dfrac{p^2}{5q^3} \times \dfrac{35qr}{12p} \div \dfrac{5r^4}{6p^3q^2} = \dfrac{\cancel{p^2}^{1}}{{}_{1}\cancel{5q^3}_{1}} \times \dfrac{{}^{7}\cancel{35qr}^{1\,1}}{{}_{2}\cancel{12p}_{1}} \times \dfrac{{}^{1}\cancel{6}p^3\cancel{q^2}^{1}}{5r^{\cancel{4}\,3}}$

$= \dfrac{\mathbf{7p^4}}{\mathbf{10r^3}}$

(c) $\dfrac{2x-6}{5x+5y} \div \dfrac{3}{7y+7x} = \dfrac{2(x-3)}{5\cancel{(x+y)}_{1}} \times \dfrac{7\cancel{(x+y)}^{1}}{3}$

$= \dfrac{\mathbf{14(x-3)}}{\mathbf{15}}$

(d) $\dfrac{h^2-6h+9}{h^2-2h} \times \dfrac{h-2}{h-3} = \dfrac{(h-3)^{\cancel{2}\,1}}{h\cancel{(h-2)}_{1}} \times \dfrac{\cancel{h-2}^{1}}{\cancel{h-3}_{1}}$

$= \dfrac{\mathbf{h-3}}{\mathbf{h}}$

2. $\dfrac{3m-n}{n+m} \div \dfrac{2n-6m}{m+n} = \dfrac{3m-n}{n+m} \times \dfrac{m+n}{2(n-3m)}$

$= \dfrac{-\cancel{(n-3m)}^{1}}{\cancel{m+n}_{1}} \times \dfrac{\cancel{m+n}^{1}}{2\cancel{(n-3m)}_{1}}$

$= -\dfrac{\mathbf{1}}{\mathbf{2}}$

Exercise 1A

1. (a) $\dfrac{\cancel{4x^4}}{\cancel{12}x^5} = \mathbf{\dfrac{1}{3x}}$

(b) $\dfrac{\cancel{16}a^3b^4}{\cancel{24}a^5b^2} = \mathbf{\dfrac{2b^2}{3a^2}}$

(c) $\dfrac{\cancel{23}q^3r}{\cancel{69}qr^3s} = \mathbf{\dfrac{q^2}{3r^2s}}$

(d) $\dfrac{\cancel{3}mn^2\cancel{p}}{\cancel{18}m^3np^4} = \mathbf{\dfrac{n}{6m^2p^3}}$

(e) $\dfrac{\cancel{15}ac^3}{\cancel{75}a^2b^4\cancel{c}} = \mathbf{\dfrac{c^2}{5ab^4}}$

(f) $\dfrac{\cancel{16}xy^3\cancel{z}}{\cancel{64}x^2y^4z^3} = \mathbf{\dfrac{1}{4xyz^2}}$

2. (a) $\dfrac{xy+3y}{4x+12} = \dfrac{y\cancel{(x+3)}}{4\cancel{(x+3)}} = \mathbf{\dfrac{y}{4}}$

(b) $\dfrac{8a+4b}{bc+2ac} = \dfrac{4\cancel{(2a+b)}}{c\cancel{(b+2a)}} = \mathbf{\dfrac{4}{c}}$

(c) $\dfrac{a^2+2ab}{6a} = \dfrac{\cancel{a}(a+2b)}{6\cancel{a}} = \mathbf{\dfrac{a+2b}{6}}$

(d) $\dfrac{c^2}{c^2-cd} = \dfrac{c^{\cancel{2}}}{\cancel{c}(c-d)} = \mathbf{\dfrac{c}{c-d}}$

(e) $\dfrac{(m-n)^2}{m^2-mn} = \dfrac{(m-n)^{\cancel{2}}}{m\cancel{(m-n)}} = \mathbf{\dfrac{m-n}{m}}$

(f) $\dfrac{5pq}{15p-10pq} = \dfrac{\cancel{5p}q}{\cancel{5p}(3-2q)} = \mathbf{\dfrac{q}{3-2q}}$

3. (a) $\dfrac{2a+b}{4a^2-b^2} = \dfrac{\cancel{2a+b}}{\cancel{(2a+b)}(2a-b)} = \mathbf{\dfrac{1}{2a-b}}$

(b) $\dfrac{c^2+2cd-15d^2}{4c^2+20cd} = \dfrac{\cancel{(c+5d)}(c-3d)}{4c\cancel{(c+5d)}} = \mathbf{\dfrac{c-3d}{4c}}$

(c) $\dfrac{3a-6}{a^2+a-6} = \dfrac{3\cancel{(a-2)}}{(a+3)\cancel{(a-2)}} = \mathbf{\dfrac{3}{a+3}}$

(d) $\dfrac{x^2+6x-7}{x^2-x} = \dfrac{(x+7)\cancel{(x-1)}}{x\cancel{(x-1)}} = \mathbf{\dfrac{x+7}{x}}$

(e) $\dfrac{k^2-9}{k^2-7k+12} = \dfrac{(k+3)\cancel{(k-3)}}{\cancel{(k-3)}(k-4)} = \mathbf{\dfrac{k+3}{k-4}}$

(f) $\dfrac{km+8k}{m^2+4m-32} = \dfrac{k\cancel{(m+8)}}{\cancel{(m+8)}(m-4)} = \mathbf{\dfrac{k}{m-4}}$

4. (a) $\dfrac{\cancel{15a}}{\cancel{8}b^3\cancel{c}} \times \dfrac{\cancel{4c}}{\cancel{5a}b} = \mathbf{\dfrac{3}{2b^4}}$

(b) $\dfrac{3(c+d)}{c-d} \times \dfrac{2c-2d}{8c+8d} = \dfrac{3\cancel{(c+d)}}{\cancel{c-d}} \times \dfrac{\cancel{2}\cancel{(c-d)}}{\cancel{8}\cancel{(c+d)}} = \mathbf{\dfrac{3}{4}}$

(c) $\dfrac{a-2b}{16} \div \dfrac{4a-8b}{24} = \dfrac{\cancel{a-2b}}{\cancel{16}} \times \dfrac{\cancel{24}}{4\cancel{(a-2b)}} = \mathbf{\dfrac{3}{8}}$

(d) $\dfrac{8c^3}{6(c+d)} \div \dfrac{2c^2}{(3c+3d)} = \dfrac{\cancel{8}c^{\cancel{3}}}{\cancel{6}\cancel{(c+d)}} \times \dfrac{\cancel{3}\cancel{(c+d)}}{\cancel{2}\cancel{c^2}} = \mathbf{2c}$

5. (a) $\dfrac{\cancel{9}\cancel{x}\cancel{(a-b)^2}}{\cancel{27}x^3(a-b)^3} = \mathbf{\dfrac{1}{3x^2(a-b)}}$

(b) $\dfrac{\cancel{7}a^3(a-3b)^4}{\cancel{21}a^2b\cancel{(a-3b)^2}} = \mathbf{\dfrac{a(a-3b)^2}{3b}}$

(c) $\dfrac{\cancel{8}ab^3(2a+3b)^{\cancel{2}}}{\cancel{32}a^2b\cancel{(3b+2a)}} = \mathbf{\dfrac{b^2(2a+3b)}{4a}}$

(d) $\dfrac{\cancel{8}an^3\cancel{(b+c)}}{\cancel{96}a^2n(c+b)^{\cancel{2}}} = \mathbf{\dfrac{n^2}{12a(b+c)}}$

(e) $\dfrac{y^2-2y-15}{y^2-3y-10} = \dfrac{\cancel{(y-5)}(y+3)}{\cancel{(y-5)}(y+2)} = \mathbf{\dfrac{y+3}{y+2}}$

(f) $\dfrac{8-2m-m^2}{2m^2-3m-2} = \dfrac{-(m^2+2m-8)}{2m^2-3m-2} = \dfrac{-(m+4)\cancel{(m-2)}}{(2m+1)\cancel{(m-2)}} = \mathbf{\dfrac{-m-4}{2m+1}}$

(g) $\dfrac{9x^2-y^2}{y^2-2xy-3x^2} = \dfrac{(3x+y)(3x-y)}{(y-3x)(y+x)} = \dfrac{-(3x+y)\cancel{(y-3x)}}{\cancel{(y-3x)}(y+x)} = \mathbf{\dfrac{-y-3x}{y+x}}$

(h) $\dfrac{3x^2+5xy-2y^2}{4x^2+7xy-2y^2} = \dfrac{(3x-y)\cancel{(x+2y)}}{(4x-y)\cancel{(x+2y)}} = \mathbf{\dfrac{3x-y}{4x-y}}$

(i) $\dfrac{b^2-a^2}{2a^2+ab-3b^2} = \dfrac{(b+a)(b-a)}{(2a+3b)(a-b)}$

$= \dfrac{-(b+a)\cancel{(a-b)}^{1}}{(2a+3b)\cancel{(a-b)}_{1}}$

$= \mathbf{\dfrac{-a-b}{2a+3b}}$

(j) $\dfrac{y^2-6y-7}{2y^2-17y+21} = \dfrac{{}^{1}\cancel{(y-7)}(y+1)}{(2y-3)\cancel{(y-7)}_{1}}$

$= \mathbf{\dfrac{y+1}{2y-3}}$

(k) $\dfrac{3x-3y}{ax-ay-x+y} = \dfrac{3(x-y)}{x(a-1)-y(a-1)}$

$= \dfrac{3\cancel{(x-y)}^{1}}{(a-1)\cancel{(x-y)}_{1}}$

$= \mathbf{\dfrac{3}{a-1}}$

(l) $\dfrac{a^2-ab-ac+bc}{a^2+ab-ac-bc} = \dfrac{a(a-b)-c(a-b)}{a(a+b)-c(a+b)}$

$= \dfrac{(a-b)\cancel{(a-c)}^{1}}{(a+b)\cancel{(a-c)}_{1}}$

$= \mathbf{\dfrac{a-b}{a+b}}$

(m) $\dfrac{a^2+am-an-mn}{a^2+am+an+mn} = \dfrac{a(a+m)-n(a+m)}{a(a+m)+n(a+m)}$

$= \dfrac{\overset{1}{\cancel{(a+m)}}(a-n)}{\underset{1}{\cancel{(a+m)}}(a+n)}$

$= \mathbf{\dfrac{a-n}{a+n}}$

6. (a) $\dfrac{5a}{9b}\times\dfrac{ac^2}{2b}\times\dfrac{c^3}{8b^4} = \mathbf{\dfrac{5a^2c^5}{144b^6}}$

(b) $\dfrac{6d}{9f^3}\times\dfrac{3f^2}{16d^5}\div\dfrac{8f^2}{27d} = \dfrac{{}^{3}\cancel{6d}^{1}}{{}_{1}\cancel{9}f^3}\times\dfrac{3\cancel{f^2}^{1}}{{}_{8}\cancel{16d^5}^{3}}\times\dfrac{{}^{3}\cancel{27d}^{1}}{8\cancel{f^2}_{1}}$

$= \mathbf{\dfrac{27}{64d^3f^3}}$

(c) $\dfrac{2y^3}{x^3}\div\dfrac{4y}{5x}\times\dfrac{64x}{10y^4} = \dfrac{{}^{1}\cancel{2y^3}^{1}}{\cancel{x^3}_{1}}\times\dfrac{{}^{1}\cancel{5x}^{1}}{{}_{1}\cancel{4y}}\times\dfrac{{}^{16}\cancel{64x}^{1}}{{}_{1}\cancel{10y^4}_{1}}$

$= \mathbf{\dfrac{16}{xy^2}}$

(d) $\dfrac{3sq}{4p^2r}\div\dfrac{3q^3}{12p^3s^2}\times\dfrac{14ps}{7qr} = \dfrac{{}^{1}\cancel{3sq}^{1}}{{}_{1}\cancel{4p^2}r}\times\dfrac{{}^{3}\cancel{12}p^3s^2}{{}_{1}\cancel{3q^3}^{2}}\times\dfrac{{}^{2}\cancel{14}ps}{{}_{1}\cancel{7}qr}$

$= \mathbf{\dfrac{6p^2s^4}{r^2q^3}}$

(e) $\dfrac{3w-7}{5w^3}\div\dfrac{21-9w}{27w} = \dfrac{3w-7}{5w^3}\times\dfrac{27w}{3(7-3w)}$

$= \dfrac{-\cancel{(7-3w)}^{1}}{5\cancel{w^3}^{2}}\times\dfrac{{}^{9}\cancel{27w}^{1}}{{}_{1}\cancel{3(7-3w)}_{1}}$

$= -\mathbf{\dfrac{9}{5w^2}}$

(f) $\dfrac{6x^2y}{16y-8x}\times\dfrac{12x-24y}{4xy^2} = \dfrac{6x^2y}{8(2y-x)}\times\dfrac{12(x-2y)}{4xy^2}$

$= \dfrac{{}^{3}\cancel{6x^2y}^{1}}{-{}_{4}\cancel{8(x-2y)}_{1}}\times\dfrac{{}^{3}\cancel{12(x-2y)}^{1}}{{}_{1}\cancel{4xy^2}^{1}}$

$= -\mathbf{\dfrac{9x}{4y}}$

(g) $\dfrac{h^2-h-6}{h^2-9}\times\dfrac{h^2}{h^2+2h} = \dfrac{{}^{1}\cancel{(h-3)}\cancel{(h+2)}^{1}}{(h+3)\cancel{(h-3)}_{1}}\times\dfrac{\cancel{h^2}^{1}}{\cancel{h}\cancel{(h+2)}_{1}}$

$= \mathbf{\dfrac{h}{h+3}}$

(h) $\dfrac{c^2-d^2}{c^2-2cd+d^2}\div\dfrac{1}{cd+d^2} = \dfrac{(c+d)\cancel{(c-d)}^{1}}{(c-d)^{\cancel{2}1}}\times\dfrac{d(c+d)}{1}$

$= \mathbf{\dfrac{d(c+d)^2}{c-d}}$

(i) $\dfrac{m^2-4}{m^2-3m+2}\div\dfrac{m}{m-1} = \dfrac{(m+2)\cancel{(m-2)}^{1}}{{}_{1}\cancel{(m-2)}\cancel{(m-1)}_{1}}\times\dfrac{\cancel{m-1}^{1}}{m}$

$= \mathbf{\dfrac{m+2}{m}}$

(j) $\dfrac{z^2}{z^2-4}\div\dfrac{3z-z^2}{z^2-5z+6} = \dfrac{\cancel{z^2}^{1}}{(z+2)\cancel{(z-2)}_{1}}\times\dfrac{(z-3)\cancel{(z-2)}^{1}}{{}_{1}\cancel{z}(3-z)}$

$= \dfrac{z}{z+2}\times\dfrac{-\cancel{(3-z)}^{1}}{\cancel{3-z}_{1}}$

$= -\mathbf{\dfrac{z}{z+2}}$

(k) $(a^2-4b^2)\div\dfrac{a^2+2ab}{ab} = \overset{1}{\cancel{(a+2b)}}(a-2b)\times\dfrac{{}^{1}\cancel{a}b}{{}_{1}\cancel{a(a+2b)}_{1}}$

$= \mathbf{b(a-2b)}$

(l) $\dfrac{y^2-4y+4}{2-6y}\times\dfrac{2y+4}{3y^2-12} = \dfrac{(y-2)^2}{2(1-3y)}\times\dfrac{2(y+2)}{3(y^2-4)}$

$= \dfrac{(y-2)^{\cancel{2}1}}{{}_{1}\cancel{2}(1-3y)}\times\dfrac{{}^{1}\cancel{2}\cancel{(y+2)}^{1}}{{}_{1}3\cancel{(y+2)}\cancel{(y-2)}_{1}}$

$= \mathbf{\dfrac{y-2}{3(1-3y)}}$

7. $\dfrac{x^2+y^2-z^2+2xy}{x^2-y^2-z^2-2yz} = \dfrac{(x^2+2xy+y^2)-z^2}{x^2-(y^2+2yz+z^2)}$

$= \dfrac{(x+y)^2-z^2}{x^2-(y+z)^2}$

$= \dfrac{\overset{1}{\cancel{[(x+y)+z]}}[(x+y)-z]}{\underset{1}{\cancel{[x+(y+z)]}}[x-(y+z)]}$

$= \mathbf{\dfrac{x+y-z}{x-y-z}}$

1.3 Addition and subtraction of algebraic fractions

Practise Now 5

1. (a) $\dfrac{6}{5a}+\dfrac{3}{8a} = \dfrac{48}{40a}+\dfrac{15}{40a}$

$= \mathbf{\dfrac{63}{40a}}$

(b) $\dfrac{4}{2b+3c}-\dfrac{7}{6b+9c} = \dfrac{4}{2b+3c}-\dfrac{7}{3(2b+3c)}$

$= \dfrac{12}{3(2b+3c)}-\dfrac{7}{3(2b+3c)}$

$= \mathbf{\dfrac{5}{3(2b+3c)}}$

(c) $\frac{h}{2-3k}-\frac{3h}{3k-2} = \frac{h}{2-3k}-\frac{3h}{-(2-3k)}$

$= \frac{h}{2-3k}+\frac{3h}{2-3k}$

$= \boldsymbol{\frac{4h}{2-3k}}$

2. (a) $\frac{2m+3n}{3m}-\frac{m-n}{6n} = \frac{2n(2m+3n)}{6mn}-\frac{m(m-n)}{6mn}$

$= \frac{4mn+6n^2-m^2+mn}{6mn}$

$= \boldsymbol{\frac{6n^2+5mn-m^2}{6mn}}$

(b) $\frac{3p}{4p-4q}-\frac{5p-2q}{3p-3q} = \frac{3p}{4(p-q)}-\frac{5p-2q}{3(p-q)}$

$= \frac{9p}{12(p-q)}-\frac{4(5p-2q)}{12(p-q)}$

$= \frac{9p-20p+8q}{12(p-q)}$

$= \boldsymbol{\frac{8q-11p}{12(p-q)}}$

(c) $\frac{5x}{4x-3y}-\frac{7y}{6y-8x} = \frac{5x}{4x-3y}-\frac{7y}{2(3y-4x)}$

$= \frac{10x}{2(4x-3y)}-\frac{7y}{-2(4x-3y)}$

$= \frac{10x}{2(4x-3y)}+\frac{7y}{2(4x-3y)}$

$= \boldsymbol{\frac{10x+7y}{2(4x-3y)}}$

Practise Now 6

(a) $\frac{2}{x+1}-\frac{3}{2x-5} = \frac{2(2x-5)}{(x+1)(2x-5)}-\frac{3(x+1)}{(x+1)(2x-5)}$

$= \frac{4x-10-3x-3}{(x+1)(2x-5)}$

$= \boldsymbol{\frac{x-13}{(x+1)(2x-5)}}$

(b) $\frac{2}{y^2-9}-\frac{y}{y-3} = \frac{2}{(y+3)(y-3)}-\frac{y}{y-3}$

$= \frac{2}{(y+3)(y-3)}-\frac{y(y+3)}{(y+3)(y-3)}$

$= \boldsymbol{\frac{2-y^2-3y}{(y+3)(y-3)}}$

(c) $\frac{1}{z+5}-\frac{1}{z-5}+\frac{2z}{z^2-25}$

$= \frac{z-5}{(z+5)(z-5)}-\frac{z+5}{(z+5)(z-5)}+\frac{2z}{(z+5)(z-5)}$

$= \frac{z-5-z-5+2z}{(z+5)(z-5)}$

$= \frac{2z-10}{(z+5)(z-5)}$

$= \frac{2\cancel{(z-5)}^{1}}{(z+5)\cancel{(z-5)}_{1}}$

$= \boldsymbol{\frac{2}{z+5}}$

(d) $\frac{4}{w+5}-\frac{7}{w^2+8w+15} = \frac{4}{w+5}-\frac{7}{(w+5)(w+3)}$

$= \frac{4(w+3)}{(w+5)(w+3)}-\frac{7}{(w+5)(w+3)}$

$= \frac{4w+12-7}{(w+5)(w+3)}$

$= \boldsymbol{\frac{4w+5}{(w+5)(w+3)}}$

Practise Now 7

Yes, I agree.

$\frac{2(x+3)}{x^2-9}$ can be simplified to $\frac{2}{x-3}$. The LCM of the denominators, $x - 3$ and $x + 3$, is $(x - 3)(x + 3) = x^2 - 9$.

$\frac{2(x+3)}{x^2-9}+\frac{1}{x+3} = \frac{2\cancel{(x+3)}^{1}}{{}_{1}\cancel{(x+3)}(x-3)}+\frac{1}{x+3}$

$= \frac{2}{x-3}+\frac{1}{x+3}$

$= \frac{2(x+3)}{(x+3)(x-3)}+\frac{x-3}{(x+3)(x-3)}$

$= \frac{2x+6+x-3}{(x+3)(x-3)}$

$= \boldsymbol{\frac{3x+3}{(x-3)(x+3)}}$

1.4 Solving equations involving algebraic fractions

Practise Now 8

(a) **Method 1:**

$\frac{a-3}{2}+\frac{2a-1}{7} = 4$

$\frac{7(a-3)}{14}+\frac{2(2a-1)}{14} = 4$

$\frac{7a-21+4a-2}{14} = 4$

$\frac{11a-23}{14} = 4$

$11a - 23 = 56$

$11a = 79$

$a = \boldsymbol{\frac{79}{11}}$

Method 2:

$\frac{a-3}{2}+\frac{2a-1}{7} = 4$

$7(a - 3) + 2(2a - 1) = 4(14)$

$7a - 21 + 4a - 2 = 56$

$11a - 23 = 56$

$11a = 79$

$a = \boldsymbol{\frac{79}{11}}$

(b) Method 1:

$$\frac{3}{2b+3}-\frac{5}{3b-4}=0$$
$$3(3b-4)-5(2b+3)=0$$
$$9b-12-10b-15=0$$
$$-b-27=0$$
$$b=\mathbf{-27}$$

Method 2:

$$\frac{3}{2b+3}-\frac{5}{3b-4}=0$$
$$\frac{3}{2b+3}=\frac{5}{3b-4}$$
$$3(3b-4)=5(2b+3)$$
$$9b-12=10b+15$$
$$b=\mathbf{-27}$$

Exercise 1B

1. (a) $\frac{7}{6a}+\frac{4}{9a}=\frac{21}{18a}+\frac{8}{18a}$

$=\boldsymbol{\frac{29}{18a}}$

(b) $\frac{3}{2b}+\frac{1}{3b}-\frac{5}{6b}=\frac{9}{6b}+\frac{2}{6b}-\frac{5}{6b}$

$=\frac{6}{6b}$

$=\boldsymbol{\frac{1}{b}}$

(c) $\frac{1}{3c}-\frac{1}{3d}=\frac{d}{3cd}-\frac{c}{3cd}$

$=\boldsymbol{\frac{d-c}{3cd}}$

(d) $\frac{f-4h}{3k}-\frac{2f-5h}{8k}=\frac{8(f-4h)}{24k}-\frac{3(2f-5h)}{24k}$

$=\frac{8f-32h-6f+15h}{24k}$

$=\boldsymbol{\frac{2f-17h}{24k}}$

(e) $\frac{4a}{x-3y}+\frac{3a}{3x-9y}=\frac{4a}{x-3y}+\frac{3a}{3(x-3y)}$

$=\frac{12a}{3(x-3y)}+\frac{3a}{3(x-3y)}$

$=\frac{\overset{5}{\cancel{15}}a}{\underset{1}{\cancel{3}}(x-3y)}$

$=\boldsymbol{\frac{5a}{x-3y}}$

(f) $\frac{p+3}{2z}+\frac{p-1}{6z}-\frac{2p+1}{3z}=\frac{3(p+3)}{6z}+\frac{p-1}{6z}-\frac{2(2p+1)}{6z}$

$=\frac{3p+9+p-1-4p-2}{6z}$

$=\frac{6}{6z}$

$=\boldsymbol{\frac{1}{z}}$

2. (a) $\frac{5}{a}+\frac{3}{a+4}=\frac{5(a+4)}{a(a+4)}+\frac{3a}{a(a+4)}$

$=\frac{5a+20+3a}{a(a+4)}$

$=\boldsymbol{\frac{8a+20}{a(a+4)}}$

(b) $\frac{1}{2b}-\frac{3}{b+c}=\frac{b+c}{2b(b+c)}-\frac{3(2b)}{2b(b+c)}$

$=\frac{b+c-6b}{2b(b+c)}$

$=\boldsymbol{\frac{c-5b}{2b(b+c)}}$

(c) $\frac{4}{d-5}+\frac{2}{2d+3}=\frac{4(2d+3)}{(d-5)(2d+3)}+\frac{2(d-5)}{(d-5)(2d+3)}$

$=\frac{8d+12+2d-10}{(d-5)(2d+3)}$

$=\boldsymbol{\frac{10d+2}{(d-5)(2d+3)}}$

(d) $\frac{2}{f+5}-\frac{3}{f-1}=\frac{2(f-1)}{(f+5)(f-1)}-\frac{3(f+5)}{(f+5)(f-1)}$

$=\frac{2f-2-3f-15}{(f+5)(f-1)}$

$=\boldsymbol{\frac{-f-17}{(f+5)(f-1)}}$

(e) $\frac{11}{3h-7}+\frac{2}{6-5h}=\frac{11(6-5h)}{(3h-7)(6-5h)}+\frac{2(3h-7)}{(3h-7)(6-5h)}$

$=\frac{66-55h+6h-14}{(3h-7)(6-5h)}$

$=\boldsymbol{\frac{52-49h}{(3h-7)(6-5h)}}$

(f) $\frac{3}{k^2-1}+\frac{2}{k-1}=\frac{3}{(k+1)(k-1)}+\frac{2(k+1)}{(k+1)(k-1)}$

$=\frac{3+2k+2}{(k+1)(k-1)}$

$=\boldsymbol{\frac{2k+5}{(k+1)(k-1)}}$

(g) $\frac{3}{4m^2-1}-\frac{5}{2m+1}=\frac{3}{(2m+1)(2m-1)}-\frac{5}{2m+1}$

$=\frac{3}{(2m+1)(2m-1)}-\frac{5(2m-1)}{(2m+1)(2m-1)}$

$=\frac{3-10m+5}{(2m+1)(2m-1)}$

$=\boldsymbol{\frac{8-10m}{(2m+1)(2m-1)}}$

(h) $\frac{2}{n-2}+\frac{3}{(n-2)^2}=\frac{2(n-2)}{(n-2)^2}+\frac{3}{(n-2)^2}$

$=\frac{2n-4+3}{(n-2)^2}$

$=\boldsymbol{\frac{2n-1}{(n-2)^2}}$

3. (a) $\frac{a}{a+2}=\frac{3}{5}$

$5a=3a+6$

$2a=6$

$a=\mathbf{3}$

(b) $\frac{1}{b-2}=\frac{2}{b-1}$

$b-1=2(b-2)$

$b-1=2b-4$

$b=\mathbf{3}$

(c) $\dfrac{4}{c+3}-\dfrac{3}{c+2}=0$

$$\frac{4}{c+3}=\frac{3}{c+2}$$
$$4c+8=3c+9$$
$$c=\mathbf{1}$$

(d) $\dfrac{5}{d+4}-\dfrac{2}{d-2}=0$

$$\frac{5}{d+4}=\frac{2}{d-2}$$
$$5(d-2)=2(d+4)$$
$$5d-10=2d+8$$
$$3d=18$$
$$d=\mathbf{6}$$

(e) $\dfrac{6}{f}-\dfrac{10}{3f}=2$

$$\frac{18}{3f}-\frac{10}{3f}=2$$
$$\frac{8}{3f}=2$$
$$8=6f$$
$$f=\frac{8}{6}$$
$$=\mathbf{\frac{4}{3}}$$

(f) $\dfrac{5}{6h}+\dfrac{6}{7h}-\dfrac{9}{14h}=4$

$$\frac{35}{42h}+\frac{36}{42h}-\frac{27}{42h}=4$$
$$\frac{44}{42h}=4$$
$$\frac{22}{21h}=4$$
$$22=84h$$
$$h=\frac{22}{84}$$
$$=\mathbf{\frac{11}{42}}$$

(g) $\dfrac{3}{k+1}-\dfrac{1}{2k+2}=5$

$$\frac{3}{k+1}-\frac{1}{2(k+1)}=5$$
$$\frac{6}{2(k+1)}-\frac{1}{2(k+1)}=5$$
$$\frac{5}{2(k+1)}=5$$
$$5=10k+10$$
$$10k=-5$$
$$k=-\frac{5}{10}$$
$$=-\mathbf{\frac{1}{2}}$$

4. **(a)** $\dfrac{5}{2(a-b)}+\dfrac{4}{3(b-a)}=\dfrac{5}{2(a-b)}+\dfrac{4}{-3(a-b)}$

$$=\frac{15}{6(a-b)}-\frac{8}{6(a-b)}$$
$$=\frac{15-8}{6(a-b)}$$
$$=\mathbf{\frac{7}{6(a-b)}}$$

(b) $\dfrac{c-1}{3c-7}-\dfrac{1}{14-6c}=\dfrac{c-1}{3c-7}-\dfrac{1}{2(7-3c)}$

$$=\frac{c-1}{3c-7}-\frac{1}{-2(3c-7)}$$
$$=\frac{2(c-1)}{2(3c-7)}+\frac{1}{2(3c-7)}$$
$$=\frac{2c-2+1}{2(3c-7)}$$
$$=\mathbf{\frac{2c-1}{2(3c-7)}}$$

(c) $\dfrac{4f}{10f-5d}+\dfrac{2d}{6f-3d}=\dfrac{4f}{5(2f-d)}+\dfrac{2d}{3(2f-d)}$

$$=\frac{12f}{15(2f-d)}+\frac{10d}{15(2f-d)}$$
$$=\mathbf{\frac{12f+10d}{15(2f-d)}}$$

(d) $\dfrac{u+1}{2u-8}-\dfrac{u+2}{12-3u}=\dfrac{u+1}{2(u-4)}-\dfrac{u+2}{-3(u-4)}$

$$=\frac{3(u+1)}{6(u-4)}+\frac{2(u+2)}{6(u-4)}$$
$$=\frac{3u+3+2u+4}{6(u-4)}$$
$$=\mathbf{\frac{5u+7}{6(u-4)}}$$

(e) $\dfrac{2m-5}{9n-6}-\dfrac{m+3}{4-6n}=\dfrac{2m-5}{3(3n-2)}-\dfrac{m+3}{-2(3n-2)}$

$$=\frac{2(2m-5)}{6(3n-2)}+\frac{3(m+3)}{6(3n-2)}$$
$$=\frac{4m-10+3m+9}{6(3n-2)}$$
$$=\mathbf{\frac{7m-1}{6(3n-2)}}$$

(f) $\dfrac{h+k}{p-q}+\dfrac{3h+k}{8q-8p}=\dfrac{h+k}{p-q}+\dfrac{3h+k}{-8(p-q)}$

$$=\frac{8(h+k)}{8(p-q)}-\frac{3h+k}{8(p-q)}$$
$$=\frac{8h+8k-3h-k}{8(p-q)}$$
$$=\mathbf{\frac{5h+7k}{8(p-q)}}$$

(g) $\dfrac{5x^2}{6x-6y}-\dfrac{2x^2}{3y-3x}=\dfrac{5x^2}{6(x-y)}-\dfrac{2x^2}{-3(x-y)}$

$$=\frac{5x^2}{6(x-y)}+\frac{4x^2}{6(x-y)}$$
$$=\frac{9x^2}{6(x-y)}$$
$$=\mathbf{\frac{3x^2}{2(x-y)}}$$

(h) $\dfrac{3x}{4y-2z}-\dfrac{2x}{z-2y}+\dfrac{5}{3z-6y}=\dfrac{3x}{-2(z-2y)}-\dfrac{2x}{z-2y}+\dfrac{5}{3(z-2y)}$

$$=\frac{-9x}{6(z-2y)}-\frac{12x}{6(z-2y)}+\frac{10}{6(z-2y)}$$
$$=\mathbf{\frac{10-21x}{6(z-2y)}}$$

5. **(a)** $\frac{3a}{3a-5}+\frac{4a}{4a-1} = \frac{3a(4a-1)}{(3a-5)(4a-1)}+\frac{4a(3a-5)}{(3a-5)(4a-1)}$

$= \frac{12a^2-3a+12a^2-20a}{(3a-5)(4a-1)}$

$= \mathbf{\frac{24a^2-23a}{(3a-5)(4a-1)}}$

(b) $\frac{5}{2b+1}-\frac{2b}{(2b+1)^2} = \frac{5(2b+1)}{(2b+1)^2}-\frac{2b}{(2b+1)^2}$

$= \frac{10b+5-2b}{(2b+1)^2}$

$= \mathbf{\frac{8b+5}{(2b+1)^2}}$

(c) $\frac{h+5}{h^2-6h}-\frac{3}{h-6} = \frac{h+5}{h(h-6)}-\frac{3}{h-6}$

$= \frac{h+5}{h(h-6)}-\frac{3h}{h(h-6)}$

$= \frac{h+5-3h}{h(h-6)}$

$= \mathbf{\frac{5-2h}{h(h-6)}}$

(d) $\frac{1}{m}+\frac{2}{m-4}+\frac{3}{m-3}$

$= \frac{(m-4)(m-3)}{m(m-4)(m-3)}+\frac{2m(m-3)}{m(m-4)(m-3)}+\frac{3m(m-4)}{m(m-4)(m-3)}$

$= \frac{m^2-7m+12+2m^2-6m+3m^2-12m}{m(m-4)(m-3)}$

$= \mathbf{\frac{6m^2-25m+12}{m(m-4)(m-3)}}$

(e) $\frac{x+y}{x-y}+\frac{x^2-4y^2}{x^2-y^2}-\frac{x-3y}{x+y}$

$= \frac{(x+y)^2}{(x+y)(x-y)}+\frac{x^2-4y^2}{(x+y)(x-y)}-\frac{(x-3y)(x-y)}{(x+y)(x-y)}$

$= \frac{x^2+2xy+y^2+x^2-4y^2-(x^2-4xy+3y^2)}{(x+y)(x-y)}$

$= \mathbf{\frac{x^2+6xy-6y^2}{(x-y)(x+y)}}$

(f) $\frac{1}{2z-3}-\frac{2}{3-2z}+\frac{18}{9-4z^2}$

$= \frac{1}{-(3-2z)}-\frac{2}{3-2z}+\frac{18}{(3+2z)(3-2z)}$

$= \frac{-(3+2z)}{(3+2z)(3-2z)}-\frac{2(3+2z)}{(3+2z)(3-2z)}+\frac{18}{(3+2z)(3-2z)}$

$= \frac{-3-2z-6-4z+18}{(3+2z)(3-2z)}$

$= \frac{9-6z}{(3+2z)(3-2z)}$

$= \frac{3\cancel{(3-2z)}^{1}}{(3+2z)\cancel{(3-2z)}_{1}}$

$= \mathbf{\frac{3}{2z+3}}$

6. **(a)** $\frac{2}{a+3}+\frac{3}{a^2+4a+3} = \frac{2}{a+3}+\frac{3}{(a+3)(a+1)}$

$= \frac{2(a+1)}{(a+3)(a+1)}+\frac{3}{(a+3)(a+1)}$

$= \frac{2a+2+3}{(a+3)(a+1)}$

$= \mathbf{\frac{2a+5}{(a+3)(a+1)}}$

(b) $\frac{1}{b^2-5b-6}-\frac{b}{b-6} = \frac{1}{(b-6)(b+1)}-\frac{b}{b-6}$

$= \frac{1}{(b-6)(b+1)}-\frac{b(b+1)}{(b-6)(b+1)}$

$= \mathbf{\frac{1-b^2-b}{(b+1)(b-6)}}$

(c) $\frac{1}{2p^2-8p-10}+\frac{2p}{p-5} = \frac{1}{2(p^2-4p-5)}+\frac{2p}{p-5}$

$= \frac{1}{2(p+1)(p-5)}+\frac{4p(p+1)}{2(p+1)(p-5)}$

$= \mathbf{\frac{4p^2+4p+1}{2(p+1)(p-5)}}$

(d) $\frac{x}{x+y}+\frac{4}{x^2+3xy+2y^2}-\frac{3x}{x+2y}$

$= \frac{x}{x+y}+\frac{4}{(x+y)(x+2y)}-\frac{3x}{x+2y}$

$= \frac{x(x+2y)}{(x+y)(x+2y)}+\frac{4}{(x+y)(x+2y)}-\frac{3x(x+y)}{(x+y)(x+2y)}$

$= \frac{x^2+2xy+4-3x^2-3xy}{(x+y)(x+2y)}$

$= \mathbf{\frac{4-2x^2-xy}{(x+y)(x+2y)}}$

7. **(a)** $2-\frac{5}{x+2} = 1\frac{3}{5}$

$\frac{5}{x+2} = 2-1\frac{3}{5}$

$= \frac{2}{5}$

$25 = 2(x+2)$

$2x = 21$

$x = \mathbf{\frac{21}{2}}$

(b) $\frac{7x-4}{15}+\frac{x-1}{3} = \frac{3x-1}{5}-\frac{7+x}{10}$

$\frac{2(7x-4)}{30}+\frac{10(x-1)}{30} = \frac{6(3x-1)}{30}-\frac{3(7+x)}{30}$

$14x-8+10x-10 = 18x-6-21-3x$

$24x-18 = 15x-27$

$9x = -9$

$x = \mathbf{-1}$

(c) $\frac{x+1}{5x-1}+\frac{1}{2(1-5x)} = \frac{1}{4}$

$\frac{2(x+1)}{-2(1-5x)}+\frac{1}{2(1-5x)} = \frac{1}{4}$

$\frac{-2x-2+1}{2(1-5x)} = \frac{1}{4}$

$4(-2x-1) = 2(1-5x)$

$-8x-4 = 2-10x$

$2x = 6$

$x = \mathbf{3}$

(d)

$$\frac{x+1}{2x-1}-\frac{4}{4x-2}-\frac{3}{6x-3}=1$$

$$\frac{x+1}{2x-1}-\frac{4}{2(2x-1)}-\frac{3}{3(2x-1)}=1$$

$$\frac{6(x+1)}{6(2x-1)}-\frac{12}{6(2x-1)}-\frac{6}{6(2x-1)}=1$$

$$\frac{6x+6-12-6}{6(2x-1)}=1$$

$$\frac{6x-12}{6(2x-1)}=1$$

$$6x-12=12x-6$$

$$6x=-6$$

$$x=\mathbf{-1}$$

(e)

$$\frac{3}{2-x}+\frac{5}{4-2x}-\frac{1}{x-2}=4$$

$$\frac{3}{2-x}+\frac{5}{2(2-x)}-\frac{1}{-(2-x)}=4$$

$$\frac{6}{2(2-x)}+\frac{5}{2(2-x)}+\frac{2}{2(2-x)}=4$$

$$\frac{13}{2(2-x)}=4$$

$$13=16-8x$$

$$8x=3$$

$$x=\mathbf{\frac{3}{8}}$$

8.

$$\frac{\frac{1}{3x}+\frac{2}{y}}{\frac{2}{x}}=\left(\frac{1}{3x}+\frac{2}{y}\right)\div\frac{2}{x}$$

$$=\left(\frac{y}{3xy}+\frac{6x}{3xy}\right)\times\frac{x}{2}$$

$$=\frac{y+6x}{3\cancel{x}_{1}y}\times\frac{\cancel{x}^{1}}{2}$$

$$=\mathbf{\frac{y+6x}{6y}}$$

9. **Albert** will be able to simplify the expression in a fewer number of steps, as Joyce would have to deal with a cubic equation in b.

Albert's working:

$$\frac{b-2}{b^2-5b+6}+\frac{b}{b+3}=\frac{b-2}{(b-2)(b-3)}+\frac{b}{b+3}$$

$$=\frac{1}{b-3}+\frac{b}{b+3}$$

$$=\frac{b+3}{(b+3)(b-3)}+\frac{b(b-3)}{(b+3)(b-3)}$$

$$=\frac{b+3+b^2-3b}{(b+3)(b-3)}$$

$$=\frac{b^2-2b+3}{(b+3)(b-3)}$$

10. If $\boldsymbol{Q=a^2+3a+2}$, then $P+R=2a+7$.

$\therefore$ if $\boldsymbol{P=2a+4}$, then $\boldsymbol{R=3}$.

11.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{2}{x}}=\frac{4}{3}$$

$$\left(\frac{1}{x}+\frac{1}{y}\right)\div\frac{2}{x}=\frac{4}{3}$$

$$\left(\frac{y}{xy}+\frac{x}{xy}\right)\times\frac{x}{2}=\frac{4}{3}$$

$$\frac{x+y}{xy}\times\frac{x}{2}=\frac{4}{3}$$

$$\frac{x+y}{2y}=\frac{4}{3}$$

$$3x+3y=8y$$

$$3x=5y$$

$$\frac{y}{x}=\mathbf{\frac{3}{5}}$$

1.5 Manipulating algebraic formulae

Practise Now 9

1. **(i)** $v=u+at$

$at=v-u$

$\boldsymbol{a=\frac{v-u}{t}}$

(ii) When $t=4$, $u=10$ and $v=50$,

$a=\frac{50-10}{4}$

$=\frac{40}{4}$

$=\mathbf{10}$

2. **(i)** $I=\frac{PRT}{100}$

$100I=PRT$

$\boldsymbol{T=\frac{100I}{PR}}$

(ii) When $P=50\ 000$, $R=2$ and $I=4000$,

$T=\frac{100(4000)}{50\ 000(2)}$

$=4$

$\therefore$ the investment must be held in the bank for **4 years**.

Journal Writing (Page 15)

1. Changing the subject of a formula to the unknown variable makes it easier to find its value when the other known values are substituted into the formula.
2. A formula is an equation with more than one variable. For example, $x+y=7$ is a formula because it shows the relationship between the two variables x and y. However, $2x-8=0$ is not a formula because it only has one variable x.

Practise Now 10

1. **(i)**

$$\begin{aligned} y &= \frac{2x+5}{3x-7} \\ y(3x-7) &= 2x+5 \\ 3xy - 7y &= 2x+5 \\ 3xy - 2x &= 7y+5 \\ x(3y-2) &= 7y+5 \\ \boldsymbol{x} &= \boldsymbol{\frac{7y+5}{3y-2}} \end{aligned}$$

(ii) When $y = -3$,

$$\begin{aligned} x &= \frac{7(-3)+5}{3(-3)-2} \\ &= \frac{-21+5}{-9-2} \\ &= \frac{-16}{-11} \\ &= \boldsymbol{\frac{16}{11}} \end{aligned}$$

2. **(i)**

$$\begin{aligned} p &= a+\frac{bx^2}{3k} \\ p-a &= \frac{bx^2}{3k} \\ 3k(p-a) &= bx^2 \\ \boldsymbol{k} &= \boldsymbol{\frac{bx^2}{3(p-a)}} \end{aligned}$$

(ii) When $a = 1$, $b = -2$, $p = 3$ and $x = 9$,

$$\begin{aligned} k &= \frac{-2(9^2)}{3(3-1)} \\ &= \frac{-162}{6} \\ &= \boldsymbol{-27} \end{aligned}$$

Practise Now 11

1. **(i)**

$$\begin{aligned} 3y &= \sqrt{b^2-4ax} \\ 9y^2 &= b^2 - 4ax \\ 4ax &= b^2 - 9y^2 \\ \boldsymbol{x} &= \boldsymbol{\frac{b^2-9y^2}{4a}} \end{aligned}$$

(ii) When $a = -5$, $b = 4$ and $y = 2$,

$$\begin{aligned} x &= \frac{4^2-9(2^2)}{4(-5)} \\ &= \frac{16-36}{-20} \\ &= \boldsymbol{1} \end{aligned}$$

(iii) For x to be defined, $4a \neq 0$.

$\therefore$ $a \neq 0$ and $t = \boldsymbol{0}$.

2. **(i)**

$$\begin{aligned} p &= a+\frac{bx^2}{3k} \\ \frac{bx^2}{3k} &= p-a \\ bx^2 &= 3k(p-a) \\ x^2 &= \frac{3k(p-a)}{b} \\ \boldsymbol{x} &= \boldsymbol{\pm\sqrt{\frac{3k(p-a)}{b}}} \end{aligned}$$

(ii) When $a = -1$, $b = 2$, $k = 1$ and $p = 5$,

$$\begin{aligned} x &= \pm\sqrt{\frac{3(1)[5-(-1)]}{2}} \\ &= \pm\sqrt{\frac{18}{2}} \\ &= \pm\sqrt{9} \\ &= 3\pm \end{aligned}$$

(iii) For x to be defined, $\frac{3k(p-a)}{b} > 0$. Since $3k(p-a) < 0$, then $b < 0$.

$\therefore$ a possible value of b is $\boldsymbol{-1}$.

Practise Now 12

1. **(a)** When $x = 5$,

$$\begin{aligned} y &= \sqrt{\frac{5+7}{5-2}} \\ &= \sqrt{\frac{12}{3}} \\ &= \sqrt{4} \\ &= \boldsymbol{2} \end{aligned}$$

(b) When $y = 4$,

$$\begin{aligned} 4 &= \sqrt{\frac{x+7}{x-2}} \\ 4^2 &= \frac{x+7}{x-2} \\ 16(x-2) &= x+7 \\ 16x - 32 &= x+7 \\ 15x &= 39 \\ x &= \frac{39}{15} \\ &= \boldsymbol{\frac{13}{5}} \end{aligned}$$

(c) For $\sqrt{\frac{x+7}{x-2}}$ to be defined, $x - 2 \neq 0$.

$\therefore$ $x \neq 2$ and $k = \boldsymbol{2}$.

2. When $a = 3$,

$$\begin{aligned} 3 &= \sqrt[3]{\frac{5b+16}{2b-23}} \\ 3^3 &= \frac{5b+16}{2b-23} \\ 27(2b-23) &= 5b+16 \\ 54b - 621 &= 5b+16 \\ 49b &= 637 \\ b &= \boldsymbol{13} \end{aligned}$$

3. **(a)** When $y = 4$ and $z = 2$,

$$\begin{aligned} \sqrt[3]{\frac{x+4}{x-4}} &= 2 \\ \frac{x+4}{x-4} &= 2^3 \\ x+4 &= 8(x-4) \\ x+4 &= 8x-32 \\ 7x &= 36 \\ x &= \boldsymbol{\frac{36}{7}} \end{aligned}$$

(b) If $x = y$, $\sqrt[3]{\frac{x+y}{x-y}}$ will be undefined.

$\therefore$ it is **not possible** to have a value of z.

Introductory Problem Revisited

(i) $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

$\frac{1}{20} = \frac{1}{1000} + \frac{1}{v}$

$\frac{1}{v} = \frac{1}{20} - \frac{1}{1000}$

$= \frac{49}{1000}$

$49v = 1000$

$v = \mathbf{\frac{1000}{49}}$

(ii) The corresponding values of v can be found by first making v the subject of the formula.

$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$

$= \frac{u-f}{fu}$

$v = \frac{fu}{u-f}$

f	u	v
20	1500	$v = \frac{20(1500)}{1500-20}$ $= \mathbf{\frac{750}{37}}$
10	1200	$v = \frac{10(1200)}{1200-10}$ $= \mathbf{\frac{1200}{119}}$
4	30	$v = \frac{4(30)}{30-4}$ $= \mathbf{\frac{60}{13}}$
7	160	$v = \frac{7(160)}{160-7}$ $= \mathbf{\frac{1120}{153}}$

Exercise 1C

1. **(a)** $ax + by = k$

$by = k - ax$

$\boldsymbol{y = \frac{k-ax}{b}}$

(b) $PV = nRT$

$\boldsymbol{n = \frac{PV}{RT}}$

(c) $5b - 2d = 3c$

$2d = 5b - 3c$

$\boldsymbol{d = \frac{5b-3c}{2}}$

(d) $R = m(a + g)$

$R = am + gm$

$am = R - gm$

$a = \frac{R-gm}{m}$

$\boldsymbol{a = \frac{R}{m} - g}$

2. **(a)** $\frac{a}{m} = b + c$

$\boldsymbol{a = m(b + c)}$

(b) $5q - r = \frac{2p}{3}$

$3(5q - r) = 2p$

$\boldsymbol{p = \frac{3}{2}(5q-r)}$

(c) $\frac{k+a}{5} = 3k$

$k + a = 15k$

$14k = a$

$\boldsymbol{k = \frac{a}{14}}$

(d) $A = \frac{1}{2}(a+b)h$

$\frac{2A}{h} = a + b$

$\boldsymbol{b = \frac{2A}{h} - a}$

3. **(a)** $\sqrt[3]{h-k} = m$

$h - k = m^3$

$\boldsymbol{h = m^3 + k}$

(b) $b = \sqrt{D+4ac}$

$b^2 = D + 4ac$

$\boldsymbol{D = b^2 - 4ac}$

(c) $P = \frac{V^2}{R}$

$PR = V^2$

$\boldsymbol{V = \pm\sqrt{PR}}$

(d) $A = \frac{\theta}{360} \times \pi r^2$

$\frac{\theta}{360} = \frac{A}{\pi r^2}$

$\boldsymbol{\theta = \frac{360A}{\pi r^2}}$

4. When $a = 2$, $b = 7$ and $c = 5$,

$\sqrt{2x^2-7} = 5$

$2x^2 - 7 = 5^2$

$2x^2 = 25 + 7$

$x^2 = 16$

$x = \mathbf{\pm 4}$

5. **(a)** When $b = 7$ and $c = 2$,

$a^2(7 + 2) = 2(7) - 2$

$9a^2 = 12$

$a^2 = \frac{12}{9}$

$= \frac{4}{3}$

$a = \pm\sqrt{\frac{4}{3}}$

$= \mathbf{\pm 1.15}$ (to 3 s.f.)

(b) When $a = 4$ and $b = -1$,

$4^2(-1 + c) = 2(-1) - c$

$16(-1 + c) = -2 - c$

$-16 + 16c = -2 - c$

$17c = 14$

$c = \mathbf{\frac{14}{17}}$

6. (a) $F = \frac{9}{5}C + 32$

$\frac{9}{5}C = F - 32$

$\boldsymbol{C = \frac{5}{9}(F-32)}$

(b) $A = 2\pi r^2 + \pi rl$

$\pi rl = A - 2\pi r^2$

$\boldsymbol{l = \frac{A - 2\pi r^2}{\pi r}}$

(c) $s = ut + \frac{1}{2}at^2$

$ut = s - \frac{1}{2}at^2$

$\boldsymbol{u = \frac{s}{t} - \frac{1}{2}at}$

(d) $S = \frac{n}{2}[2a + (n-1)d]$

$\frac{2S}{n} = 2a + (n-1)d$

$(n-1)d = \frac{2S}{n} - 2a$

$(n-1)d = \frac{2S - 2an}{n}$

$\boldsymbol{d = \frac{2S - 2an}{n(n-1)}}$

7. (a) $\frac{1}{h+1} + 2 = k$

$\frac{1}{h+1} = k - 2$

$h + 1 = \frac{1}{k-2}$

$h = \frac{1}{k-2} - 1$

$h = \frac{1-(k-2)}{k-2}$

$\boldsymbol{h = \frac{3-k}{k-2}}$

(b) $z = \frac{y(z-y)}{x}$

$zx = yz - y^2$

$yz - zx = y^2$

$z(y - x) = y^2$

$\boldsymbol{z = \frac{y^2}{y-x}}$

(c) $\frac{px}{q} = p + q$

$px = pq + q^2$

$px - pq = q^2$

$p(x - q) = q^2$

$\boldsymbol{p = \frac{q^2}{x-q}}$

(d) $\frac{1}{a} + \frac{1}{b} = 1$

$\frac{1}{b} = 1 - \frac{1}{a}$

$= \frac{a-1}{a}$

$\boldsymbol{b = \frac{a}{a-1}}$

8. (i) $V = \pi r^2 h + \frac{2}{3}\pi r^3$

$\pi r^2 h = V - \frac{2}{3}\pi r^3$

$\boldsymbol{h = \frac{V}{\pi r^2} - \frac{2}{3}r}$

(ii) When $V = 1000$ and $r = 7$,

$h = \frac{1000}{\pi(7^2)} - \frac{2}{3}(7)$

$= \frac{1000}{49\pi} - \frac{14}{3}$

$=$ **1.83** (to 3 s.f.)

9. (a) $V = \frac{4}{3}\pi r^3$

$r^3 = \frac{3V}{4\pi}$

$\boldsymbol{r = \sqrt[3]{\frac{3V}{4\pi}}}$

(b) $v^2 = u^2 + 2as$

$u^2 = v^2 - 2as$

$\boldsymbol{u = \pm\sqrt{v^2 - 2as}}$

(c) $y = (x - p)^2 + q$

$(x - p)^2 = y - q$

$x - p = \pm\sqrt{y-q}$

$\boldsymbol{x = p \pm \sqrt{y-q}}$

(d) $t = \sqrt{\frac{4z}{m-3}}$

$t^2 = \frac{4z}{m-3}$

$t^2(m - 3) = 4z$

$\boldsymbol{z = \frac{t^2(m-3)}{4}}$

10. (i) $a = \sqrt{\frac{3b+c}{b-c}}$

$a^2 = \frac{3b+c}{b-c}$

$a^2b - a^2c = 3b + c$

$a^2b - 3b = c + a^2c$

$b(a^2 - 3) = c + a^2c$

$\boldsymbol{b = \frac{c+a^2c}{a^2-3}}$

(ii) When $a = 2$ and $c = 5$,

$b = \frac{5+(2^2)(5)}{(2^2)-3}$

$= \frac{5+20}{4-3}$

$=$ **25**

(iii) For b to be defined, $a^2 - 3 \neq 0$. $\therefore$ $a^2 \neq 3$ and $t =$ **3**.

11. (i) $T = 2\pi\sqrt{\frac{l}{g}}$

$\sqrt{\frac{l}{g}} = \frac{T}{2\pi}$

$\frac{l}{g} = \left(\frac{T}{2\pi}\right)^2$

$\boldsymbol{l = g\left(\frac{T}{2\pi}\right)^2}$

(ii) Time taken to complete one oscillation

$= \frac{12}{20}$

$= \frac{3}{5}$ s

When $T = \frac{3}{5}$ and $g = 10$,

$l = 10\left(\frac{\frac{3}{5}}{2\pi}\right)^2$

$= 0.0912$ (to 3 s.f.)

$\therefore$ the length of the pendulum is **0.0912 m**.

12. (i) $E = mgh + \frac{1}{2}mv^2$

$\frac{1}{2}mv^2 = E - mgh$

$v^2 = \frac{2(E-mgh)}{m}$

$\boldsymbol{v = \sqrt{\frac{2(E-mgh)}{m}}}$ (since $v \geqslant 0$)

(ii) When $m = 0.5$, $g = 10$, $h = 2$ and $E = 100$,

$v = \sqrt{\frac{2(100-0.5\times10\times2)}{0.5}}$

$= \sqrt{\frac{2(90)}{0.5}}$

$= 19.0$ (to 3 s.f.)

$\therefore$ the velocity of the object is **19.0 m s^{-1}**.

13. (a) When $m = 5$, $n = 7$, $x = 4$ and $y = -2$,

$\frac{5\left[7\times4-(-2)^2\right]}{p} = 3(7)$

$5(28 - 4) = 21p$

$21p = 120$

$p = \boldsymbol{\frac{40}{7}}$

(b) When $m = 14$, $p = 9$, $x = 2$ and $y = 3$,

$\frac{14(2n-3^2)}{9} = 3n$

$14(2n - 9) = 27n$

$28n - 126 = 27n$

$n = \mathbf{126}$

(c) When $m = 5$, $n = 4$, $p = 15$ and $x = 42$,

$\frac{5(4\times42-y^2)}{15} = 3(4)$

$5(168 - y^2) = 180$

$168 - y^2 = 36$

$y^2 = 132$

$y = \pm\sqrt{132}$

$= \mathbf{\pm11.5}$ (to 3 s.f.)

(d) For $\frac{m(nx-y^2)}{p}$ to be defined, $p \neq 0$. $\therefore$ $k = \mathbf{0}$.

14. (a) When $\pi = 3.142$, $h = 15$ and $r = 7$,

$A = \frac{1}{3}(3.142)(7^2)(15)+\frac{4}{3}(3.142)(7^3)$

$= \frac{2309.37}{3}+\frac{4310.824}{3}$

$= \mathbf{2210}$ (to 3 s.f.)

(b) When $\pi = 3.142$, $A = 15\,400$ and $r = 14$,

$15\,400 = \frac{1}{3}(3.142)(14^2)h+\frac{4}{3}(3.142)(14^3)$

$46\,200 = 615.832h + 34\,486.592$

$615.832h = 11\,713.408$

$h = \mathbf{19.0}$ (to 3 s.f.)

15. (i) (a) When $a = 13$, $b = 15$ and $x = 3.8$,

$y = 3(3.8) + \sqrt[3]{13+15^2}$

$= 11.4 + \sqrt[3]{238}$

$= \mathbf{17.6}$ (to 3 s.f.)

(b) When $b = 13$, $x = 8.5$ and $y = 35$,

$35 = 3(8.5) + \sqrt[3]{a+13^2}$

$\sqrt[3]{a+13^2} = 35 - 25.5$

$a + 13^2 = 9.5^3$

$a + 169 = 857.375$

$a = \mathbf{688}$ (to 3 s.f.)

(c) When $a = 23$, $x = 15.6$ and $y = 56$,

$56 = 3(15.6) + \sqrt[3]{23+b^2}$

$\sqrt[3]{23+b^2} = 56 - 46.8$

$23 + b^2 = 9.2^3$

$b^2 = 9.2^3 - 23$

$b = \pm\sqrt{9.2^3-23}$

$= \mathbf{\pm27.5}$ (to 3 s.f.)

(ii) **Yes**, it is possible because the cube root of a negative number will be a negative real number.

Chapter 2 Quadratic Equations and Graphs

TEACHING NOTES

Suggested Approach

To solve quadratic equations by factorisation, students should be familiar with the "zero product" property, i.e., if 2 factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$ or both P and Q are equal to 0.

Teachers should use graphing software to explore the 2 important properties of quadratic graphs - symmetry and the maximum/minimum point.

This chapter exposes students to mathematical and real-life problems that can be solved using quadratic equations and that involve the graphs of quadratic functions.

Section 2.1: Solving quadratic equations by factorisation

Teachers may need to recap the steps involved in factorising quadratic expressions using the "multiplication frame" method with students before illustrating how to solve quadratic equations by factorisation together with the "zero product" property.

Errors that students may make in solving quadratic equations should be highlighted (see Class Discussion on page 26 of the textbook).

Section 2.2: Quadratic functions and graphs

As a lesson introduction, students should go through the activity on page 32 (see Investigation: Relationship Between the Area of a Square and the Length of its side). Students are expected to deduce that the shape of the graph is non-linear. As such, for each value of x, there is exactly one corresponding value of A, then $A = x^2$ is the equation of the function.

Students should also explore the effect of the values of a, b and c in the quadratic graphs of equation $y = ax^2 + bx + c$ where $c \neq 0$ (see Investigation: Graphs of $y = x^2$ and $y = -x^2$)

In Book 2, students have learnt to plot graphs of linear functions. Using a given table of values and scales, teachers should provide students with ample practices to plot different types of quadratic graphs. Besides plotting the quadratic graphs, students should be able to recognise and write down the equation of the line of symmetry of the plotted quadratic graphs. Students are to understand that for a given y value, there could be 2 possible x values which will satisfy the quadratic equations. Teachers can emphasise this by asking students to state the values of x for a given value of y by highlighting the 2 possible answers.

Teachers should also highlight to students that quadratic graphs have either a maximum or minimum y value. Students have to observe that the maximum/minimum point is also on the line of symmetry of the graph. Teachers may pose questions about maximum/minimum points in different forms to give students added practice, e.g.

- State the maximum/minimum point;
- State the maximum/minimum value of y;
- Sate the value of x for which y is maximum/minimum.

Teachers should go through the mathematical and real-life problems that involve the graphs of quadratic functions (see Worked Example 10 on page 39 of the textbook).

In addition, teachers should remind students to check for any invalid answers (e.g. negative values for positive quantities such as length and mass etc.) and indicate the rejection of such values in their answers.

Introductory Problem

1. If $ab = 0$, then $\boldsymbol{a = 0}$ **or** $\boldsymbol{b = 0}$.
2. **No**. There are other possible combinations of values for x and for y such that $xy = 6$, e.g. $x = 3$ and $y = 2$.

2.1 Solving quadratic equations by factorisation

Class Discussion (Linear equations)

(a) $$\begin{aligned} 3x + 2 &= 5 \\ 3x &= 5 - 2 \\ &= 3 \\ x &= \mathbf{1} \end{aligned}$$

(b) $$\begin{aligned} 4y - 3 &= 7 \\ 4y &= 7 + 3 \\ &= 10 \\ y &= \frac{10}{4} \\ &= \mathbf{\frac{5}{2}} \end{aligned}$$

(c) $$\begin{aligned} \frac{7}{2}x - 8 &= 6 \\ \frac{7}{2}x &= 6 + 8 \\ &= 14 \\ x &= 14 \div \frac{7}{2} \\ &= 14 \times \frac{2}{7} \\ &= \mathbf{4} \end{aligned}$$

(d) $$\begin{aligned} -\frac{3}{4}y - 9 &= -3 \\ -\frac{3}{4}y &= -3 + 9 \\ &= 6 \\ y &= 6 \div \left(-\frac{3}{4}\right) \\ &= 6 \times \left(-\frac{4}{3}\right) \\ &= \mathbf{-8} \end{aligned}$$

Practise Now 1

(a) $x(x - 2) = 0$

$x = \mathbf{0}$ or $x - 2 = 0$

$x = \mathbf{2}$

(b) $4x(x + 1) = 0$

$4x = 0$ or $x + 1 = 0$

$x = \mathbf{0}$ or $x = \mathbf{-1}$

(c) $3x^2 + 18x = 0$

$3x(x + 6) = 0$

$3x = 0$ or $x + 6 = 0$

$x = \mathbf{0}$ or $x = \mathbf{-6}$

(d) $-\frac{2}{3}x^2 + 5x = 0$

$x\left(-\frac{2}{3}x + 5\right) = 0$

$x = \mathbf{0}$ or $-\frac{2}{3}x + 5 = 0$

$x = \mathbf{\frac{15}{2}}$

Class Discussion (Mathematical fallacy)

$4x^2 + 6x = 0$ — (1)

$2x(2x + 3) = 0$

$2x = 0$ or $2x + 3 = 0$

It is incorrect to divide both sides of equation (1) by $2x$ since $2x$ can be 0 and we cannot divide by 0.

Practise Now 2

1. **(a)** $(x + 5)(x - 7) = 0$

$x + 5 = 0$ or $x - 7 = 0$

$x = \mathbf{-5}$ or $x = \mathbf{7}$

(b) $(3y + 2)(4y - 5) = 0$

$3y + 2 = 0$ or $4y - 5 = 0$

$y = \mathbf{-\frac{2}{3}}$ or $y = \mathbf{\frac{5}{4}}$

(c) $$\begin{aligned} w^2 + 8w + 16 &= 0 \\ w^2 + 2(w)(4) + 4^2 &= 0 \\ (w + 4)^2 &= 0 \\ w + 4 &= 0 \\ w &= \mathbf{-4} \end{aligned}$$

(d) $\frac{3}{2}z^2 - 5z + 4 = 0$

$3z^2 - 10z + 8 = 0$

$(3z - 4)(z - 2) = 0$

$3z - 4 = 0$ or $z - 2 = 0$

$z = \mathbf{\frac{4}{3}}$ or $z = \mathbf{2}$

(e) $$\begin{aligned} 4h^2 - 20h + 25 &= 0 \\ (2h)^2 - 2(2h)(5) + 5^2 &= 0 \\ (2h - 5)^2 &= 0 \\ 2h - 5 &= 0 \\ h &= \mathbf{\frac{5}{2}} \end{aligned}$$

(f) $3k^2 + 30k + 48 = 0$

$k^2 + 10k + 16 = 0$

$(k + 8)(k + 2) = 0$

$k + 8 = 0$ or $k + 2 = 0$

$k = \mathbf{-8}$ or $k = \mathbf{-2}$

2. **(i)** $3x^2 - 10x + 8 = 0$

$(3x - 4)(x - 2) = 0$

$3x - 4 = 0$ or $x - 2 = 0$

$x = \mathbf{\frac{4}{3}}$ or $x = \mathbf{2}$

(ii) $3(y+1)^2 - 10(y+1) + 8 = 0$ — (1)
Let $y + 1 = x$.
Then equation (1) becomes: $3x^2 - 10x + 8 = 0$
From **(i)**,

$$x = \frac{4}{3} \quad \text{or} \quad x = 2$$
$$y + 1 = \frac{4}{3} \quad \text{or} \quad y + 1 = 2$$
$$y = \mathbf{\frac{1}{3}} \quad \text{or} \quad y = \mathbf{1}$$

3. **(i)** $x^2 + px + 8 = 0$ — (1)
Substitute $x = -2$ into (1):

$$\begin{aligned}(-2)^2 + p(-2) + 8 &= 0\\ 4 - 2p + 8 &= 0\\ 2p &= 12\\ p &= \mathbf{6}\end{aligned}$$

(ii)

$$\begin{aligned}x^2 + 6x + 8 &= 0\\ (x+4)(x+2) &= 0\end{aligned}$$
$$x + 4 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = -4 \quad \text{or} \quad x = -2$$

$\therefore$ the other solution of the equation is $x = \mathbf{-4}$.

Practise Now 3

1. **(a)**

$$\begin{aligned}9x^2 - 4 &= 0\\ (3x)^2 - 2^2 &= 0\\ (3x+2)(3x-2) &= 0\end{aligned}$$
$$3x + 2 = 0 \quad \text{or} \quad 3x - 2 = 0$$
$$x = \mathbf{-\frac{2}{3}} \quad \text{or} \quad x = \mathbf{\frac{2}{3}}$$

(b)

$$\begin{aligned}100y^2 - 25 &= 0\\ (10y)^2 - 5^2 &= 0\\ (10y+5)(10y-5) &= 0\end{aligned}$$
$$10y + 5 = 0 \quad \text{or} \quad 10y - 5 = 0$$
$$y = \mathbf{-\frac{1}{2}} \quad \text{or} \quad y = \mathbf{\frac{1}{2}}$$

(c)

$$\begin{aligned}36m^2 &= 1\\ m^2 &= \frac{1}{36}\\ m &= \pm\mathbf{\frac{1}{6}}\end{aligned}$$

(d)

$$\begin{aligned}\frac{1}{4}n^2 - 4 &= 0\\ \left(\frac{1}{2}n\right)^2 - 2^2 &= 0\\ \left(\frac{1}{2}n+2\right)\left(\frac{1}{2}n-2\right) &= 0\end{aligned}$$
$$\frac{1}{2}n + 2 = 0 \quad \text{or} \quad \frac{1}{2}n - 2 = 0$$
$$n = \mathbf{-4} \quad \text{or} \quad n = \mathbf{4}$$

2.

$$\begin{aligned}16x^2 + 49 &= 0\\ 16x^2 &= -49\end{aligned}$$

Since $x^2 \geqslant 0$ for all real values of x, then $16x^2 \geqslant 0$.
Thus $16x^2$ can never be equal to -49.
$\therefore$ $16x^2 + 49 = 0$ has no real solutions.

Practise Now 4

(a)

$$\begin{aligned}x(x+6) &= -5\\ x^2 + 6x &= -5\\ x^2 + 6x + 5 &= 0\\ (x+5)(x+1) &= 0\end{aligned}$$
$$x + 5 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = \mathbf{-5} \quad \text{or} \quad x = \mathbf{-1}$$

(b)

$$\begin{aligned}9y(1-y) &= 2\\ 9y - 9y^2 &= 2\\ 9y^2 - 9y + 2 &= 0\\ (3y-1)(3y-2) &= 0\end{aligned}$$
$$3y - 1 = 0 \quad \text{or} \quad 3y - 2 = 0$$
$$y = \mathbf{\frac{1}{3}} \quad \text{or} \quad y = \mathbf{\frac{2}{3}}$$

(c)

$$\begin{aligned}(3t+5)(t-2) &= -6\\ 3t^2 - t - 10 &= -6\\ 3t^2 - t - 4 &= 0\\ (3t-4)(t+1) &= 0\end{aligned}$$
$$3t - 4 = 0 \quad \text{or} \quad t + 1 = 0$$
$$t = \mathbf{\frac{4}{3}} \quad \text{or} \quad t = \mathbf{-1}$$

(d)

$$\begin{aligned}(2v+1)^2 &= \frac{1}{5}(v+2)\\ 5(2v+1)^2 &= v + 2\\ 5(4v^2 + 4v + 1) &= v + 2\\ 20v^2 + 20v + 5 &= v + 2\\ 20v^2 + 19v + 3 &= 0\\ (5v+1)(4v+3) &= 0\end{aligned}$$
$$5v + 1 = 0 \quad \text{or} \quad 4v + 3 = 0$$
$$v = \mathbf{-\frac{1}{5}} \quad \text{or} \quad v = \mathbf{-\frac{3}{4}}$$

Class Discussion (Mathematical fallacy: 2 = 1?)

Since it is given that $x = y$, then $x - y = 0$. Therefore, we cannot divide both sides of the equation by $(x - y)$ because we cannot divide by 0.

Practise Now 5

1. Let the smaller even number be x.
Then the next consecutive even number is $x + 2$.

$$\begin{aligned}x^2 + (x+2)^2 &= 164\\ x^2 + x^2 + 4x + 4 &= 164\\ 2x^2 + 4x - 160 &= 0\\ x^2 + 2x - 80 &= 0\\ (x-8)(x+10) &= 0\end{aligned}$$
$$x - 8 = 0 \quad \text{or} \quad x + 10 = 0$$
$$x = 8 \quad \text{or} \quad x = -10 \text{ (rejected since } x > 0)$$

When $x = 8$, $x + 2 = 8 + 2$
$= 10$

$\therefore$ the two consecutive positive even numbers are **8** and **10**.

2. Let the smaller number be x. Then the other number will be $x + 5$.

$$[x + (x + 5)]^2 = 169$$
$$(2x + 5)^2 = 169$$
$$4x^2 + 20x + 25 = 169$$
$$4x^2 + 20x - 144 = 0$$
$$x^2 + 5x - 36 = 0$$
$$(x - 4)(x + 9) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x + 9 = 0$$
$$x = 4 \quad \text{or} \quad x = -9 \text{ (rejected since } x > 0)$$

When $x = 4$, $y = x + 5$
$= 4 + 5$
$= 9$

$\therefore$ the two numbers are **4** and **9**.

Practise Now 6

Let the length of the rectangle be x cm.

Then the breadth of the rectangle is $\left(\frac{20-2x}{2}\right)$ cm = $(10 - x)$ cm.

$$x(10 - x) = 24$$
$$10x - x^2 = 24$$
$$x^2 - 10x + 24 = 0$$
$$(x - 4)(x - 6) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x - 6 = 0$$
$$x = 4 \quad \text{or} \quad x = 6$$

When $x = 4$, breadth of the rectangle $= 10 - 4$
$= 6$ cm

When $x = 6$, breadth of the rectangle $= 10 - 6$
$= 4$ cm

$\therefore$ the length of the rectangle is **6 cm** and the breadth of the rectangle is **4 cm**.

Exercise 2A

1. **(a)** $a(a - 9) = 0$
$a = \mathbf{0}$ or $a - 9 = 0$
$a = \mathbf{9}$

(b) $b(b + 7) = 0$
$b = \mathbf{0}$ or $b + 7 = 0$
$b = \mathbf{-7}$

(c) $5c^2 + 25c = 0$
$5c(c + 5) = 0$
$5c = 0$ or $c + 5 = 0$
$c = \mathbf{0}$ or $c = \mathbf{-5}$

(d) $11d^2 - d = 0$
$d(11d - 1) = 0$
$d = \mathbf{0}$ or $11d - 1 = 0$
$d = \mathbf{\frac{1}{11}}$

(e) $-4h^2 - 16h = 0$
$-4h(h + 4) = 0$
$-4h = 0$ or $h + 4 = 0$
$h = \mathbf{0}$ or $h = \mathbf{-4}$

(f) $3k - 81k^2 = 0$
$3k(1 - 27k) = 0$
$3k = 0$ or $1 - 27k = 0$
$k = \mathbf{0}$ or $k = \mathbf{\frac{1}{27}}$

(g) $-\frac{1}{2}x(2x+3) = 0$
$-\frac{1}{2}x = 0$ or $2x + 3 = 0$
$x = \mathbf{0}$ or $x = \mathbf{-\frac{3}{2}}$

(h) $\frac{4}{5}y^2 - 4y = 0$
$4y\left(\frac{1}{5}y - 1\right) = 0$
$4y = 0$ or $\frac{1}{5}y - 1 = 0$
$y = \mathbf{0}$ or $y = \mathbf{5}$

2. **(a)** $(m - 4)(m - 9) = 0$
$m = \mathbf{4}$ or $m = \mathbf{9}$

(b) $(n - 3)(n + 5) = 0$
$n = \mathbf{3}$ or $n = \mathbf{-5}$

(c) $(p + 1)(p + 2) = 0$
$p = \mathbf{-1}$ or $p = \mathbf{-2}$

(d) $(7q - 6)(4q - 5) = 0$
$7q - 6 = 0$ or $4q - 5 = 0$
$q = \mathbf{\frac{6}{7}}$ or $q = \mathbf{\frac{5}{4}}$

(e) $(5s + 3)(2 - s) = 0$
$5s + 3 = 0$ or $2 - s = 0$
$s = \mathbf{-\frac{3}{5}}$ or $s = \mathbf{2}$

(f) $(-2t - 5)(8t - 5) = 0$
$-2t - 5 = 0$ or $8t - 5 = 0$
$t = \mathbf{-\frac{5}{2}}$ or $t = \mathbf{\frac{5}{8}}$

3. **(a)** $s^2 + 10s + 21 = 0$
$(s + 3)(s + 7) = 0$
$s + 3 = 0$ or $s + 7 = 0$
$s = \mathbf{-3}$ or $s = \mathbf{-7}$

(b) $t^2 - 16t + 63 = 0$
$(t - 7)(t - 9) = 0$
$t - 7 = 0$ or $t - 9 = 0$
$t = \mathbf{7}$ or $t = \mathbf{9}$

(c) $3u^2 + 49u + 60 = 0$
$(3u + 4)(u + 15) = 0$
$3u + 4 = 0$ or $u + 15 = 0$
$u = \mathbf{-\frac{4}{3}}$ or $u = \mathbf{-15}$

(d) $6w^2 - 29w + 20 = 0$
$(6w - 5)(w - 4) = 0$
$6w - 5 = 0$ or $w - 4 = 0$
$w = \mathbf{\frac{5}{6}}$ or $w = \mathbf{4}$

(e) $\frac{1}{3}x^2 + 2x - 9 = 0$
$x^2 + 6x - 27 = 0$
$(x + 9)(x - 3) = 0$
$x + 9 = 0$ or $x - 3 = 0$
$x = \mathbf{-9}$ or $x = \mathbf{3}$

(f) $9y^2 + 21y - 18 = 0$
$3y^2 + 7y - 6 = 0$
$(3y - 2)(y + 3) = 0$
$3y - 2 = 0$ or $y + 3 = 0$
$y = \mathbf{\frac{2}{3}}$ or $y = \mathbf{-3}$

(g) $m^2 - 16m + 64 = 0$

$m^2 - 2(m)(8) + 8^2 = 0$

$(m - 8)^2 = 0$

$m - 8 = 0$

$m = \mathbf{8}$

(h) $k^2 + 12k + 36 = 0$

$k^2 + 2(k)(6) + 6^2 = 0$

$(k + 6)^2 = 0$

$k + 6 = 0$

$k = \mathbf{-6}$

(i) $25p^2 + 70p + 49 = 0$

$(5p)^2 + 2(5p)(7) + 7^2 = 0$

$(5p + 7)^2 = 0$

$5p + 7 = 0$

$p = \mathbf{-\frac{7}{5}}$

(j) $\frac{4}{9}q^2 - \frac{4}{3}q + 1 = 0$

$\left(\frac{2}{3}q\right)^2 - 2\left(\frac{2}{3}q\right)(1) + 1^2 = 0$

$\left(\frac{2}{3}q - 1\right)^2 = 0$

$\frac{2}{3}q - 1 = 0$

$q = \mathbf{\frac{3}{2}}$

4. (a) $a^2 - 16 = 0$

$a^2 - 4^2 = 0$

$(a + 4)(a - 4) = 0$

$a + 4 = 0$ or $a - 4 = 0$

$a = \mathbf{-4}$ or $a = \mathbf{4}$

(b) $4b^2 - 100 = 0$

$b^2 - 25 = 0$

$b^2 - 5^2 = 0$

$(b + 5)(b - 5) = 0$

$b + 5 = 0$ or $b - 5 = 0$

$b = \mathbf{-5}$ or $b = \mathbf{5}$

(c) $121 - c^2 = 0$

$11^2 - c^2 = 0$

$(11 + c)(11 - c) = 0$

$11 + c = 0$ or $11 - c = 0$

$c = \mathbf{-11}$ or $c = \mathbf{11}$

(d) $25d^2 - \frac{1}{4} = 0$

$(5d)^2 - \left(\frac{1}{2}\right)^2 = 0$

$\left(5d + \frac{1}{2}\right)\left(5d - \frac{1}{2}\right) = 0$

$5d + \frac{1}{2} = 0$ or $5d - \frac{1}{2} = 0$

$d = \mathbf{-\frac{1}{10}}$ or $d = \mathbf{\frac{1}{10}}$

(e) $9x^2 - 64 = 0$

$(3x)^2 - 8^2 = 0$

$(3x + 8)(3x - 8) = 0$

$3x + 8 = 0$ or $3x - 8 = 0$

$x = \mathbf{-\frac{8}{3}}$ or $x = \mathbf{\frac{8}{3}}$

(f) $\frac{1}{18} - \frac{1}{2}y^2 = 0$

$1 - 9y^2 = 0$

$1^2 - (3y)^2 = 0$

$(1 + 3y)(1 - 3y) = 0$

$1 + 3y = 0$ or $1 - 3y = 0$

$y = \mathbf{-\frac{1}{3}}$ or $y = \mathbf{\frac{1}{3}}$

5. (a) $k(2k + 5) = 3$

$2k^2 + 5k = 3$

$2k^2 + 5k - 3 = 0$

$(2k - 1)(k + 3) = 0$

$2k - 1 = 0$ or $k + 3 = 0$

$k = \mathbf{\frac{1}{2}}$ or $k = \mathbf{-3}$

(b) $2m(m - 5) = 5m - 18$

$2m^2 - 10m = 5m - 18$

$2m^2 - 15m + 18 = 0$

$(2m - 3)(m - 6) = 0$

$2m - 3 = 0$ or $m - 6 = 0$

$m = \mathbf{\frac{3}{2}}$ or $m = \mathbf{6}$

(c) $(n - 2)(n + 4) = 27$

$n^2 + 2n - 8 = 27$

$n^2 + 2n - 35 = 0$

$(n + 7)(n - 5) = 0$

$n + 7 = 0$ or $n - 5 = 0$

$n = \mathbf{-7}$ or $n = \mathbf{5}$

(d) $(p - 1)(p - 6) = 126$

$p^2 - 7p + 6 = 126$

$p^2 - 7p - 120 = 0$

$(p - 15)(p + 8) = 0$

$p - 15 = 0$ or $p + 8 = 0$

$p = \mathbf{15}$ or $p = \mathbf{-8}$

(e) $3r^2 - 5(r + 1) = 7r + 58$

$3r^2 - 5r - 5 = 7r + 58$

$3r^2 - 12r - 63 = 0$

$r^2 - 4r - 21 = 0$

$(r - 7)(r + 3) = 0$

$r - 7 = 0$ or $r + 3 = 0$

$r = \mathbf{7}$ or $r = \mathbf{-3}$

(f) $(3s + 1)(s - 4) = -5(s - 1)$

$3s^2 - 11s - 4 = -5s + 5$

$3s^2 - 6s - 9 = 0$

$s^2 - 2s - 3 = 0$

$(s - 3)(s + 1) = 0$

$s - 3 = 0$ or $s + 1 = 0$

$s = \mathbf{3}$ or $s = \mathbf{-1}$

6. Let the whole number be x.

$2x^2 + x = 10$

$2x^2 + x - 10 = 0$

$(x - 2)(2x + 5) = 0$

$x - 2 = 0$ or $2x + 5 = 0$

$x = 2$ or $x = -\frac{5}{2}$ (rejected since x is a whole number)

$\therefore$ the number is **2**.

7. Let the whole number be x.

$$3x^2 - 4x = 15$$
$$3x^2 - 4x - 15 = 0$$
$$(x - 3)(3x + 5) = 0$$
$$x - 3 = 0 \quad \text{or} \quad 3x + 5 = 0$$
$$x = 3 \quad \text{or} \quad x = -\frac{5}{3} \text{ (rejected since } x \text{ is a whole number)}$$

$\therefore$ the number is **3**.

8. Let the smaller number be x.

Then the next consecutive number is $x + 1$.

$$x^2 + (x + 1)^2 = 113$$
$$x^2 + x^2 + 2x + 1 = 113$$
$$2x^2 + 2x - 112 = 0$$
$$x^2 + x - 56 = 0$$
$$(x - 7)(x + 8) = 0$$
$$x - 7 = 0 \quad \text{or} \quad x + 8 = 0$$
$$x = 7 \quad \text{or} \quad x = -8 \text{ (rejected since } x > 0)$$

When $x = 7$, $x + 1 = 7 + 1$
$= 8$

$\therefore$ the two consecutive positive numbers are **7** and **8**.

9. Let the smaller number be x. Then the other number will be $x + 7$.

$$[x + (x + 7)]^2 = 289$$
$$(2x + 7)^2 = 289$$
$$4x^2 + 28x + 49 = 289$$
$$4x^2 + 28x - 240 = 0$$
$$x^2 + 7x - 60 = 0$$
$$(x - 5)(x + 12) = 0$$
$$x - 5 = 0 \quad \text{or} \quad x + 12 = 0$$
$$x = 5 \quad \text{or} \quad x = -12 \text{ (rejected since } x > 0)$$

When $x = 5$, $y = x + 7$
$= 5 + 7$
$= 12$

$\therefore$ the two numbers are **5** and **12**.

10. Let the smaller number be x. Then the other number will be $x + 9$.

$$x(x + 9) = 162$$
$$x^2 + 9x = 162$$
$$x^2 + 9x - 162 = 0$$
$$(x - 9)(x + 18) = 0$$
$$x - 9 = 0 \quad \text{or} \quad x + 18 = 0$$
$$x = 9 \quad \text{or} \quad x = -18$$

When $x = 9$, $y = x + 9$
$= 9 + 9$
$= 18$

When $x = -18$, $y = x + 9$
$= -18 + 9$
$= -9$

$\therefore$ the two numbers are **9** and **18** or **–9** and **–18**.

11. $7x^3 + 21x^2 = 0$

$$7x^2(x + 3) = 0$$
$$7x^2 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = \mathbf{0} \quad \text{or} \quad x = \mathbf{-3}$$

12. (a)

$$7f + f^2 = 60$$
$$f^2 + 7f - 60 = 0$$
$$(f + 12)(f - 5) = 0$$
$$f + 12 = 0 \quad \text{or} \quad f - 5 = 0$$
$$f = \mathbf{-12} \quad \text{or} \quad f = \mathbf{5}$$

(b)

$$15 = 8h^2 - 2h$$
$$8h^2 - 2h - 15 = 0$$
$$(4h + 5)(2h - 3) = 0$$
$$4h + 5 = 0 \quad \text{or} \quad 2h - 3 = 0$$
$$h = -\mathbf{\frac{5}{4}} \quad \text{or} \quad h = \mathbf{\frac{3}{2}}$$

(c)

$$\frac{1}{2}x^2 - \frac{11}{4}x + \frac{5}{4} = 0$$
$$2x^2 - 11x + 5 = 0$$
$$(2x - 1)(x - 5) = 0$$
$$2x - 1 = 0 \quad \text{or} \quad x - 5 = 0$$
$$x = \mathbf{\frac{1}{2}} \quad \text{or} \quad x = \mathbf{5}$$

(d)

$$2 - 3.5y - 9.75y^2 = 0$$
$$8 - 14y - 39y^2 = 0$$
$$(4 - 13y)(2 + 3y) = 0$$
$$4 - 13y = 0 \quad \text{or} \quad 2 + 3y = 0$$
$$y = \mathbf{\frac{4}{13}} \quad \text{or} \quad y = -\mathbf{\frac{2}{3}}$$

13. (i)

$$6x^2 - x - 15 = 0$$
$$(3x - 5)(2x + 3) = 0$$
$$3x - 5 = 0 \quad \text{or} \quad 2x + 3 = 0$$
$$x = \mathbf{\frac{5}{3}} \quad \text{or} \quad x = -\mathbf{\frac{3}{2}}$$

(ii) $6(y - 3)^2 - (y - 3) - 15 = 0 \quad$ — (1)

Let $y - 3 = x$.

Then equation (1) becomes: $6x^2 - x - 15 = 0$

From **(i)**,

$$x = \frac{5}{3} \quad \text{or} \quad x = -\frac{3}{2}$$
$$y - 3 = \frac{5}{3} \quad \text{or} \quad y - 3 = -\frac{3}{2}$$
$$y = \mathbf{\frac{14}{3}} \quad \text{or} \quad y = \mathbf{\frac{3}{2}}$$

14. (a)

$$\frac{4}{9} - \frac{d^2}{25} = 0$$
$$\left(\frac{2}{3}\right)^2 - \left(\frac{d}{5}\right)^2 = 0$$
$$\left(\frac{2}{3} + \frac{d}{5}\right)\left(\frac{2}{3} - \frac{d}{5}\right) = 0$$
$$\frac{2}{3} + \frac{d}{5} = 0 \quad \text{or} \quad \frac{2}{3} - \frac{d}{5} = 0$$
$$\boldsymbol{d} = -\mathbf{\frac{10}{3}} \quad \text{or} \quad \boldsymbol{d} = \mathbf{\frac{10}{3}}$$

(b)

$$\frac{4}{9} + \frac{d^2}{25} = 0$$
$$\frac{d^2}{25} = -\frac{4}{9}$$

Since $d^2 \geqslant 0$ for all real values of x, then $\frac{d^2}{25} \geqslant 0$.

Thus $\frac{d^2}{25}$ can never be equal to $-\frac{4}{9}$.

$\therefore \frac{4}{9} + \frac{d^2}{25} = 0$ has no real solutions.

15. (a)

$$\frac{1}{3}(2q - 3)(q - 4) = 6$$
$$(2q - 3)(q - 4) = 18$$
$$2q^2 - 11q + 12 = 18$$
$$2q^2 - 11q - 6 = 0$$
$$(2q + 1)(q - 6) = 0$$
$$2q + 1 = 0 \quad \text{or} \quad q - 6 = 0$$
$$\boldsymbol{q} = -\mathbf{\frac{1}{2}} \quad \text{or} \quad \boldsymbol{q} = \mathbf{6}$$

(b) $\frac{1}{4}(t+2)(2t-6) = \frac{t}{2}+1$

$(t + 2)(2t - 6) = 2t + 4$

$2t^2 - 2t - 12 = 2t + 4$

$2t^2 - 4t - 16 = 0$

$t^2 - 2t - 8 = 0$

$(t - 4)(t + 2) = 0$

$t - 4 = 0$ or $t + 2 = 0$

$\boldsymbol{t = 4}$ or $\boldsymbol{t = -2}$

16. Let the length of the campsite be x m.

Then the breadth of the campsite is $\left(\frac{64-2x}{2}\right)$ m = $(32 - x)$ m.

$x(32 - x) = 207$

$32x - x^2 = 207$

$x^2 - 32x + 207 = 0$

$(x - 9)(x - 23) = 0$

$x - 9 = 0$ or $x - 23 = 0$

$x = 9$ or $x = 23$

When $x = 9$, breadth of the campsite $= 32 - 9$

$= 23$ m

When $x = 23$, breadth of the campsite $= 32 - 23$

$= 9$ m

∴ the length of the campsite is **23 m** and the breadth of the campsite is **9 m**.

17. Let the width of the concrete path be x m.

Area of the rectangular field

$= 70 \times 50$

$= 3500$ m^2

Total area of concrete path and rectangular field

$= (70 + 2x)(50 + 2x)$

$= (4x^2 + 240x + 3500)$ m^2

Area of the concrete path

$= 4x^2 + 240x + 3500 - 3500$

$= (4x^2 + 240x)$ m^2

$4x^2 + 240x = 1024$

$4x^2 + 240x - 1024 = 0$

$x^2 + 60x - 256 = 0$

$(x - 4)(x + 64) = 0$

$x - 4 = 0$ or $x + 64 = 0$

$x = 4$ or $x = -64$ (rejected since $x > 0$)

∴ the width of the path is **4 m**.

18. $\frac{1}{2}(x+3)(2x-5) = 20$

$(x + 3)(2x - 5) = 40$

$2x^2 + x - 15 = 40$

$2x^2 + x - 55 = 0$

$(x - 5)(2x + 11) = 0$

$x - 5 = 0$ or $2x + 11 = 0$

$x = 5$ or $x = -\frac{11}{2}$

When $x = 5$,

$x + 3 = 5 + 3$

$= 8$

$2x - 5 = 2(5) - 5$

$= 5$

When $x = -\frac{11}{2}$,

$x + 3 = -\frac{11}{2}+3$

$= -\frac{5}{2}$

$2x - 5 = 2\left(-\frac{11}{2}\right)-5$

$= -16$

Since $x + 3 > 0$ and $2x - 5 > 0$, $x = $ **5**.

19. Let the length of one part of the wire be x cm.

Then the length of the remaining part of the wire is $(44 - x)$ cm.

$\left(\frac{x}{4}\right)^2+\left(\frac{44-x}{4}\right)^2 = 65$

$\frac{x^2}{16}+\frac{1936-88x+x^2}{16} = 65$

$x^2 + 1936 - 88x + x^2 = 1040$

$2x^2 - 88x + 896 = 0$

$x^2 - 44x - 448 = 0$

$(x - 28)(x - 16) = 0$

$x - 28 = 0$ or $x - 16 = 0$

$x = 28$ or $x = 16$

When $x = 28$, the length of the remaining part of the wire

$= 44 - 28$

$= 16$ cm

When $x = 16$, the length of the remaining part of the wire

$= 44 - 16$

$= 28$ cm

∴ the perimeters of the squares are **16 cm** and **28 cm**.

20. (i) Area of each face of the coin

$= \pi(3)^2 - x^2$

$= (9\pi - x^2)$ cm^2

$9\pi - x^2 = 7\pi$

$2\pi - x^2 = 0$ (shown)

(ii) $2\pi - x^2 = 0$

$x^2 = 2\pi$

$x = \pm\sqrt{2\pi}$

$= \boldsymbol{\pm 2.51}$ (to 3 s.f.)

(iii) Since the length of the sides of the square > 0, the perimeter of the square

$= 4x$

$= 4\sqrt{2\pi}$

$=$ **10.0 cm** (to 3 s.f.)

21. (i) Total distance covered

$= (x + 2)(x - 3) + x(3x + 5)$

$= x^2 - x - 6 + 3x^2 + 5x$

$= (4x^2 + 4x - 6)$ km

$4x^2 + 4x - 6 = 74$

$4x^2 + 4x - 80 = 0$ (shown)

(ii) $4x^2 + 4x - 80 = 0$

$x^2 + x - 20 = 0$

$(x + 5)(x - 4) = 0$

$x + 5 = 0$ or $x - 4 = 0$

$x = \boldsymbol{-5}$ or $x = \boldsymbol{4}$

(iii) Since $x > 0$, $x = 4$.

Time taken for her entire journey

$= x - 3 + x$

$= 4 - 3 + 4$

$=$ **5 hours**

22. $9x^2y^2 - 12xy + 4 = 0$
$(3xy)^2 - 2(3xy)(2) + 2^2 = 0$
$(3xy - 2)^2 = 0$
$3xy - 2 = 0$
$3xy = 2$
$\boldsymbol{y = \dfrac{2}{3x}}$

23. **(i)** $x^2 - qx + 10 = 0$ — (1)
Substitute $x = 5$ into (1):
$5^2 - q(5) + 10 = 0$
$25 - 5q + 10 = 0$
$5q = 35$
$q = \mathbf{7}$

(ii) $x^2 - 7x + 10 = 0$
$(x - 5)(x - 2) = 0$
$x - 5 = 0$ or $x - 2 = 0$
$x = 5$ or $x = 2$
$\therefore$ the other solution of the equation is $x = \mathbf{2}$.

24. $x - (2x - 3)^2 = -6(x^2 + x - 2)$
$x - (4x^2 - 12x + 9) = -6x^2 - 6x + 12$
$x - 4x^2 + 12x - 9 = -6x^2 - 6x + 12$
$2x^2 + 19x - 21 = 0$
$(2x + 21)(x - 1) = 0$
$2x + 21 = 0$ or $x - 1 = 0$
$x = \mathbf{-\dfrac{21}{2}}$ or $x = \mathbf{1}$

25. $(x + 1)^2 = 16(x - 2) + (x - 3)$
$x^2 + 2x + 1 = 16x - 32 + x - 3$
$x^2 - 15x + 36 = 0$
$(x - 12)(x - 3) = 0$
$x - 12 = 0$ or $x - 3 = 0$
$x = \mathbf{12}$ or $x = \mathbf{3}$

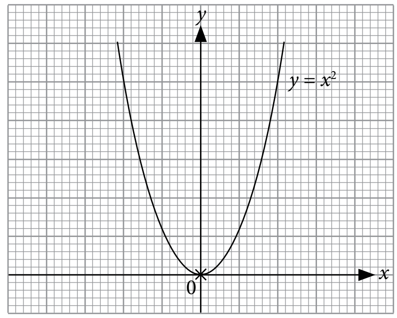
2.2 Quadratic functions and graphs

Investigation (Relationship between area of square and its length)

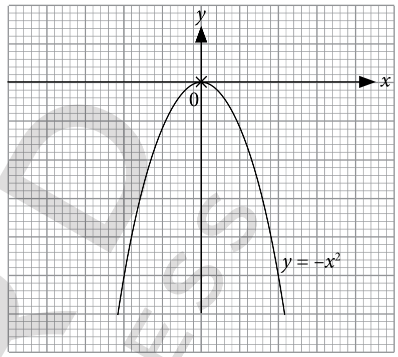
1. For each value of x, there is one corresponding value of A. Since every input x has a unique image, $A = x^2$ is the equation of a function.
2. The graph is non-linear as an increase in the value of x brings about a more than proportionate increase in the value of A.

Investigation (Graphs of $y = x^2$ and $y = -x^2$)

1. **(a)**



(b)



2. **(a)** **(0, 0)**
(b) The coordinates of the lowest point of the graph $y = x^2$ is **(0, 0)**.
The coordinates of the highest point of the graph $y = -x^2$ is **(0, 0)**.
(c) Both graphs are symmetrical about the ***y*-axis**. Therefore, the equation of the line of symmetry of the graphs is $\boldsymbol{x = 0}$.
(d) The line of symmetry of $y = x^2$ and $y = -x^2$ passes through the turning points of the graphs, **(0, 0)**.

Investigation (Graphs of quadratic functions $y = ax^2 + bx + c$)

1. As the value of a increases, the shape of the graph becomes narrower.
2. When $a > 0$, the shape of the graph becomes wider as the value of a decreases.
3. As the value of a decreases until it becomes negative, the shape of the graph changes from one that opens upwards to one that opens downwards.
4. The value of a determines whether the shape of the graph becomes narrower or wider.
When a is positive, the curve opens upwards indefinitely and when a is negative, the curve opens downwards indefinitely.
5. As the value of c increases, the position of the graph changes by shifting upwards.
6. As the value of c decreases, the position of the graph changes by shifting downwards.
7. The value of c affects the distance the graph is from the x-axis.

8.

Quadratic Graph	Coefficient of x^2	Opens upwards / downwards	Coordinates of minimum / maximum point	Equation of line of symmetry	x-intercept(s)	y-intercept
$y = x^2 - 4x + 3$	1	Opens upwards	(2, –1)	$x = 2$	1, 3	3
$y = -x^2 - 2x + 3$	**–1**	**Opens downwards**	**(–1, 4)**	$\boldsymbol{x = -1}$	**–3, 1**	**3**
$y = x^2 - 4x + 4$	**1**	**Opens upwards**	**(2, 0)**	$\boldsymbol{x = 2}$	**2**	**4**
$y = -4x^2 + 12x - 9$	**–4**	**Opens downwards**	**(1.5, 0)**	$\boldsymbol{x = 1.5}$	**1.5**	**–9**
$y = 2x^2 + 2x + 1$	**2**	**Opens upwards**	**(–0.5, 0.5)**	$\boldsymbol{x = -0.5}$	**None**	**1**
$y = -3x^2 + x - 4$	**–3**	**Opens downwards**	$\boldsymbol{\left(\frac{1}{6}, -3\frac{11}{12}\right)}$	$\boldsymbol{x = \frac{1}{6}}$	**None**	**–4**

Practise Now 7

When $y = 0$, $-2x^2 + x + 6 = 0$

$$-(2x^2 - x - 6) = 0$$
$$(2x^2 - x - 6) = 0$$
$$(2x + 3)(x - 2) = 0$$
$$2x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -1\frac{1}{2} \quad \text{or} \quad x = 2$$

$\therefore$ the coordinates of A and B are $\boldsymbol{(-1\frac{1}{2}, 0)}$ and **(2, 0)** respectively.

When $x = 0$, $y = -2(0)^2 + 0 + 6$
$= 6$

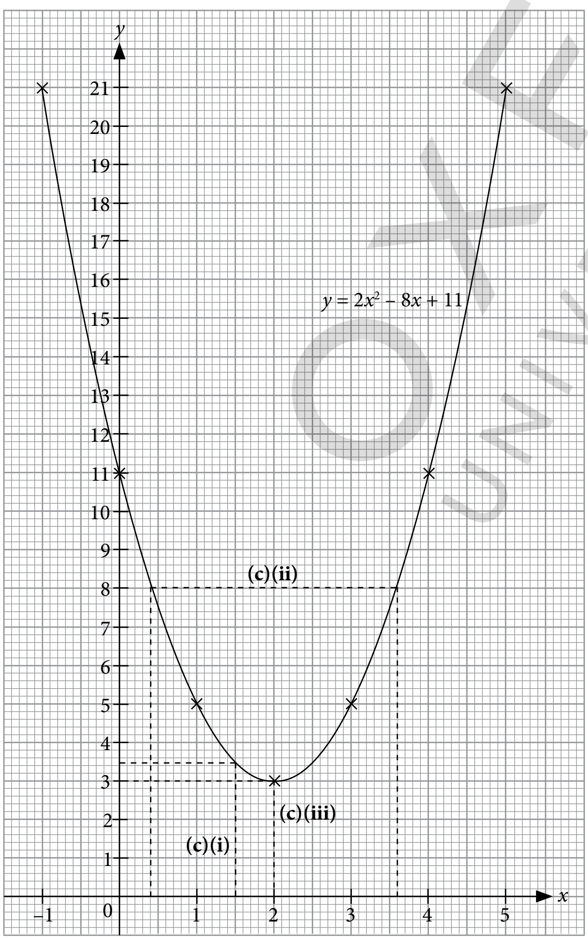
$\therefore$ the coordinates of C are **(0, 6)**.

Practise Now 8

(a) When $x = 3$, $y = 2(3)^2 - 8(3) + 11$
$= 5$

$\therefore$ $q = \mathbf{5}$

(b)



(c) **(i)** When $x = 1.5$, $y = \mathbf{3.5}$

(ii) When $y = 8$, $x = \mathbf{0.4}$ or $\mathbf{3.6}$

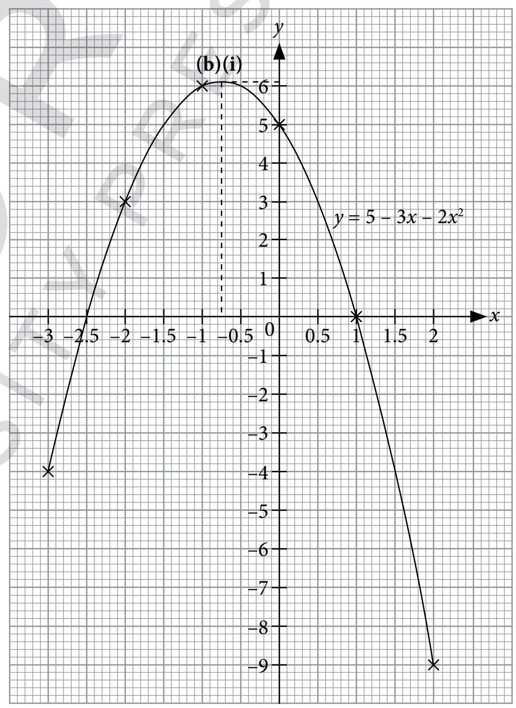
(iii) Minimum value of $\boldsymbol{y = 3}$

Minimum value of y occurs when $\boldsymbol{x = 2}$

(d) The equation of the line of symmetry of the graph is $\boldsymbol{x = 2}$.

Practise Now 9

(a)



(b) **(i)** The coordinates of the turning point are **(–0.75, 6.1)**.

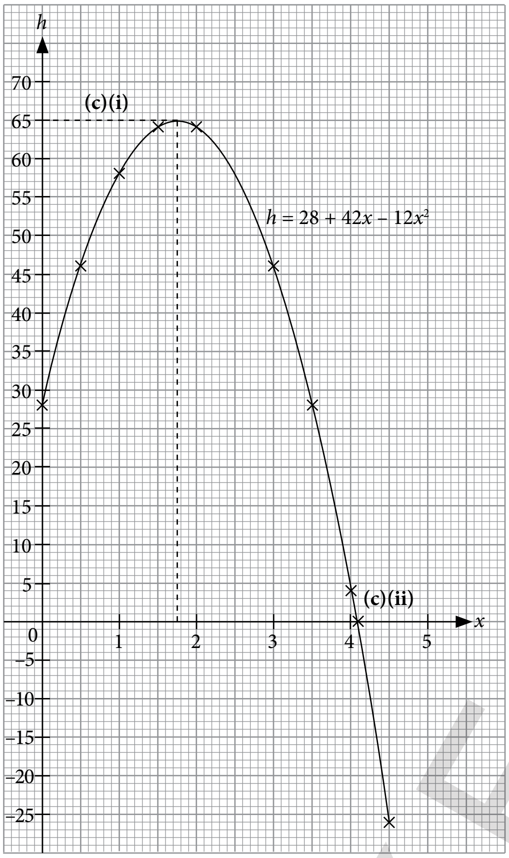
(ii) The equation of the line of symmetry of the graph is $\boldsymbol{x = -0.75}$.

Practise Now 10

1. **(a)** When $x = 0$, $h = 28 + 42(0) - 12(0)^2$
$= 28$
$\therefore$ the height of the cliff is **28 m**.

(b)

x	0	0.5	1	1.5	2	3	3.5	4	4.5
h	28	46	58	64	64	46	28	4	–26

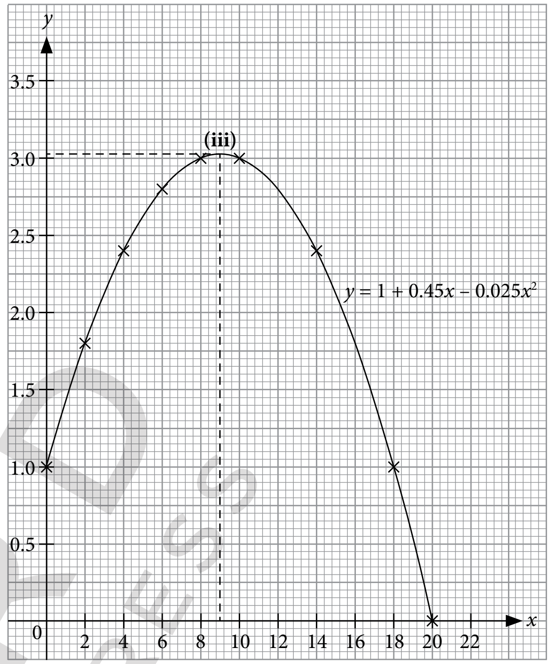


(c) **(i)** Maximum height of the stone = **65 m**
Horizontal distance travelled by the stone = **1.75 m**

(ii) Horizontal distance travelled before the stone hits the ground = **4.05 m**

2. **(i)**

x	0	2	4	6	8	10	14	18	20
y	1	1.8	2.4	2.8	3	3	2.4	1	0



(ii) It means that the farmhouse is 1 m above sea level.

(iii) Maximum height of the farmland = **3.025 m**
Horizontal distance from the farmhouse = **9 m**

Practise Now 11

1. **(i)** When $t = 12$, $y = 96(12) - 4(12)^2$
$= 576$
$\therefore$ height of the rocket 12 seconds after it leaves the ground = **576 m**

(ii) When $y = 0$, $96t - 4t^2 = 0$
$4t(24 - t) = 0$
$4t = 0$ or $24 - t = 0$
$t = 0$ or $t = 24$
$\therefore$ the rocket will strike the ground again after **24 seconds**.

2. **(i)** The value of h at $x = 0$ represents the height of the javelin when it was launched.

(ii) When $h = 0$,
$-\frac{1}{85}\left(x^2 - 65x - 204\right) = 0$
$x^2 - 65x - 204 = 0$
$(x - 68)(x + 3) = 0$
$x - 68 = 0$ or $x + 3 = 0$
$x = 68$ or $x = -3$ (rejected since $x > 0$)
Since the horizontal distance travelled by the javelin before it touches the ground is 68 m, the athlete did not achieve a new personal best.

(iii) The x-coordinate of the turning point $= \frac{68-3}{2}$
$= 32.5$
The y-coordinate of the turning point
$= -\frac{1}{85}\left[32.5^2 - 65(32.5) - 204\right]$
$=$ **14.8 m** (to 3 s.f.)

Exercise 2B

1. When $y = 0$, $x^2 + 7x + 6 = 0$

$$(x + 6)(x + 1) = 0$$

$$x + 6 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = -6 \quad \text{or} \quad x = -1$$

$\therefore$ the coordinates of A and B are **(–6, 0)** and **(–1, 0)** respectively.

When $x = 0$, $y = 0^2 + 7(0) + 6$
$= 6$

$\therefore$ the coordinates of C are **(0, 6)**.

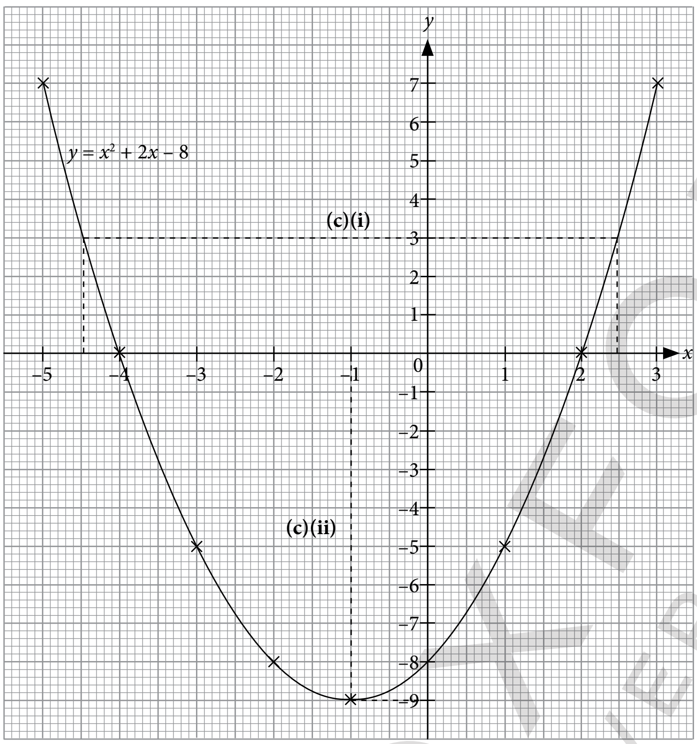
2. **(a)** When $x = -4$, $y = (-4)^2 + 2(-4) - 8$
$= 0$

$\therefore$ $\boldsymbol{a = 0}$

When $x = 1$, $y = 1^2 + 2(1) - 8$
$= -5$

$\therefore$ $\boldsymbol{b = -5}$

(b)



(c) **(i)** From **(b)**, when $y = 3$, $x \approx$ **2.45** or **–4.45**.

(ii) Minimum value of y = **–9**

(d) The equation of the line of symmetry of the graph is $\boldsymbol{x = -1}$.

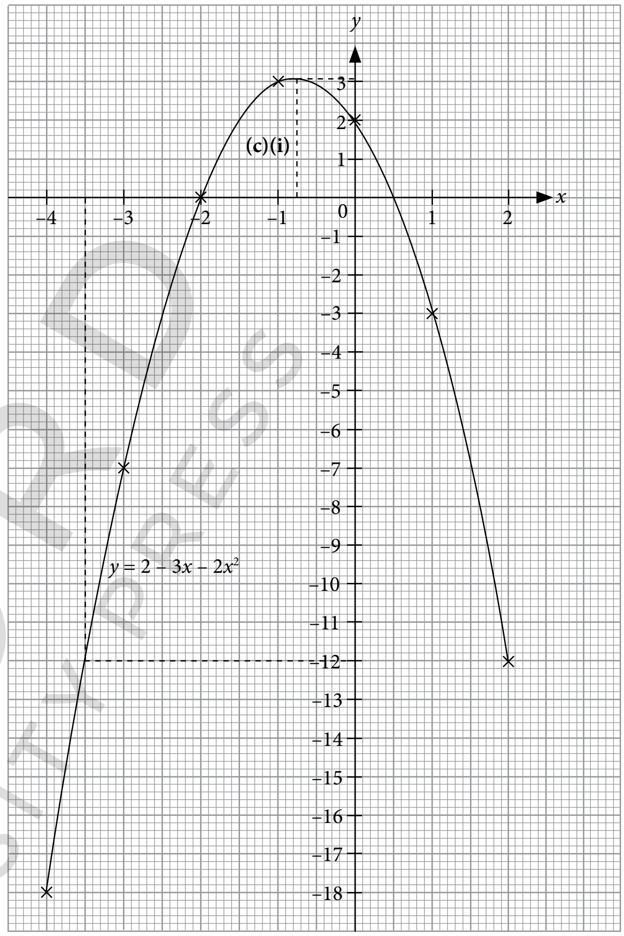
3. **(a)** When $x = -3$, $y = 2 - 3(-3) - 2(-3)^2$
$= -7$

$\therefore$ $\boldsymbol{p = -7}$

When $x = 1$, $y = 2 - 3(1) - 2(1)^2$
$= -3$

$\therefore$ $\boldsymbol{q = -3}$

(b)



(c) **(i)** From **(b)**, the coordinates of the turning point are **(–0.75, 3.1)**.

(ii) The equation of the line of symmetry of the graph is $\boldsymbol{x = -0.75}$.

4. **(i)** When $y = 0$, $-x^2 - x + 20 = 0$

$$-(x^2 + x - 20) = 0$$

$$x^2 + x - 20 = 0$$

$$(x - 4)(x + 5) = 0$$

$$x - 4 = 0 \quad \text{or} \quad x + 5 = 0$$

$$x = 4 \quad \text{or} \quad x = -5$$

$\therefore$ the coordinates of A and B are **(–5, 0)** and **(4, 0)** respectively.

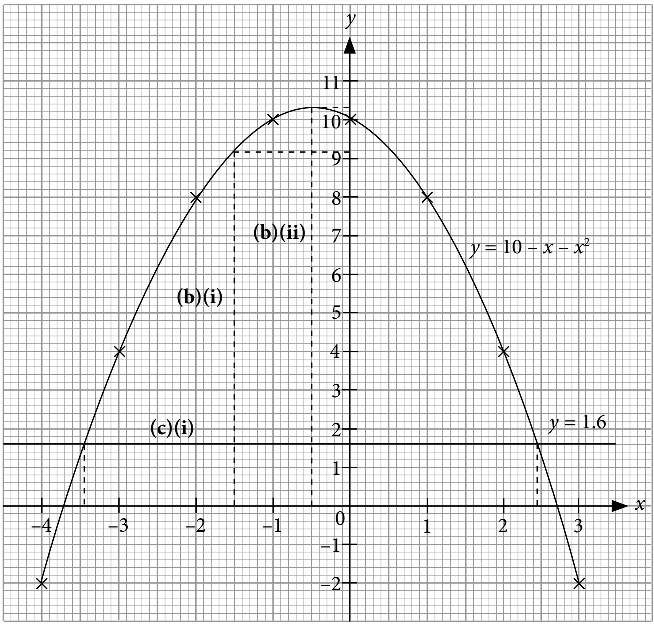
When $x = 0$, $y = -0^2 - 0 + 20$
$= 20$

$\therefore$ the coordinates of C are **(0, 20)**.

(ii) When $x = 3$, $y = -3^2 - 3 + 20$
$= 8$

$\therefore$ h = **8**

5. **(a)**



(b) **(i)** From **(a)**, the coordinates of the turning point are **(–0.5, 10.3)**.

(ii) The equation of the line of symmetry of the graph is $\boldsymbol{x = -0.5}$.

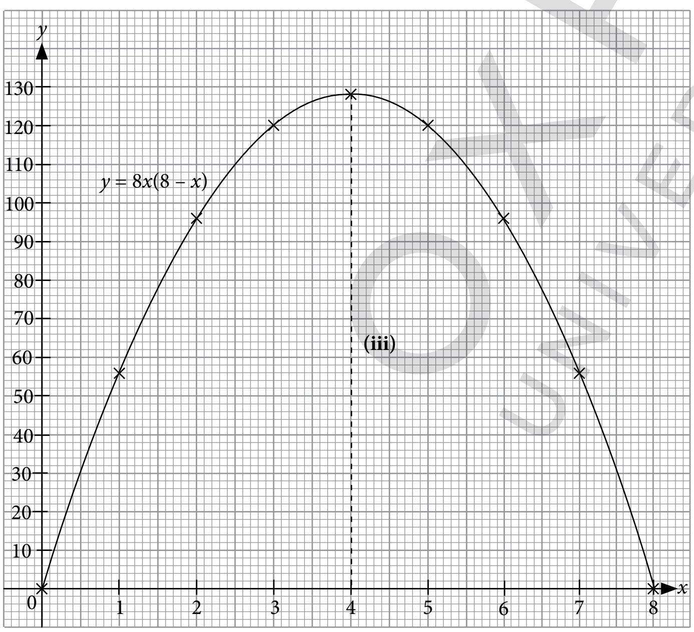
(c) **(ii)** From the graph, the solutions to the equation $10 - x - x^2 = 1.6$ are $x =$ **–3.45** or **2.45**.

6. **(i)** $y = x(64 - 8x)$

$\boldsymbol{y = 8x(8 - x)}$

(ii)

x	0	1	2	3	4	5	6	7	8
y	0	56	96	120	128	120	96	56	0



(iii) From **(ii)**, x-coordinate of the turning point = 4

$\therefore$ the store should price each book at **$4**.

7. **(i)** Area, A, of the triangle $= \frac{1}{2}(x+2)(7-x)$

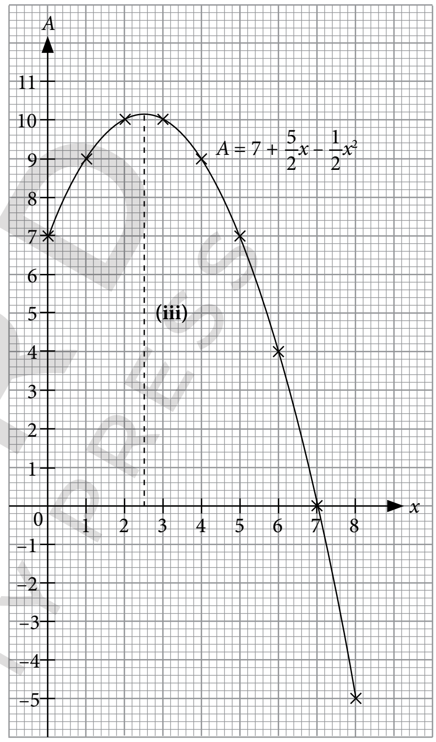
$= \frac{1}{2}(7x - x^2 + 14 - 2x)$

$= \frac{1}{2}(-x^2 + 5x + 14)$

$= 7 + \frac{5}{2}x - \frac{1}{2}x^2$ (shown)

(ii)

x	0	1	2	3	4	5	6	7	8
A	7	9	10	10	9	7	4	0	–5



(iii) From part **(ii)**, x-coordinate of the turning point = 2.5

Length of the side = 2.5 + 2

= **4.5 cm**

Length of the corresponding height = 7 – 2.5

= **4.5 cm**

8. **(i)** When $t = 0$, $h = 20 - 4(0) - 3(0)^2$

$= 20$

$\therefore$ the hawk dropped its prey **20 m** above the ground.

(ii) When $h = 0$, $20 - 4t - 3t^2 = 0$

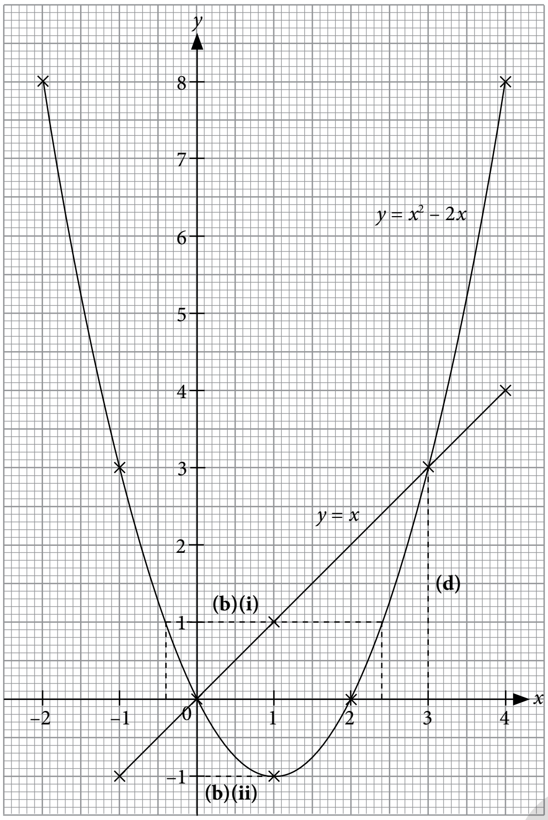
$(2 - t)(10 + 3t) = 0$

$2 - t = 0$ or $10 + 3t = 0$

$t = 2$ or $t = -\frac{10}{3}$ (rejected since $t > 0$)

$\therefore$ the prey will fall onto the ground after **2 seconds**.

9. **(a)**



(b) **(i)** From **(a)**, when $y = 1$, $x \approx$ **−0.4** or **2.4**

(ii) From **(a)**, the minimum value of $y =$ **−1**

(c) The equation of the line of symmetry of the graph is $\boldsymbol{x = 1}$.

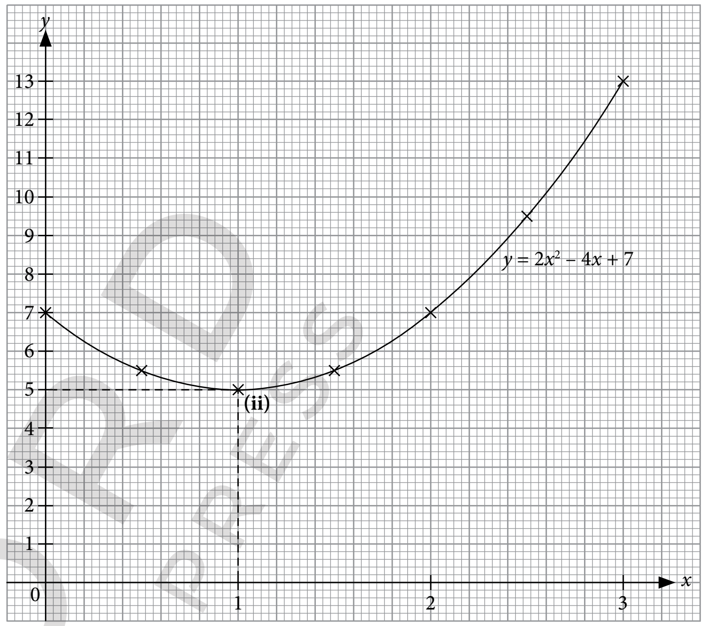
(d) By drawing $y = x$,

x	−1	1	4
y	−1	1	4

Solution to $x^2 - 2x = x$: $x =$ **0** or **3**

10. **(i)**

x	0	0.5	1	1.5	2	2.5	3
y	7	5.5	5	5.5	7	9.5	13

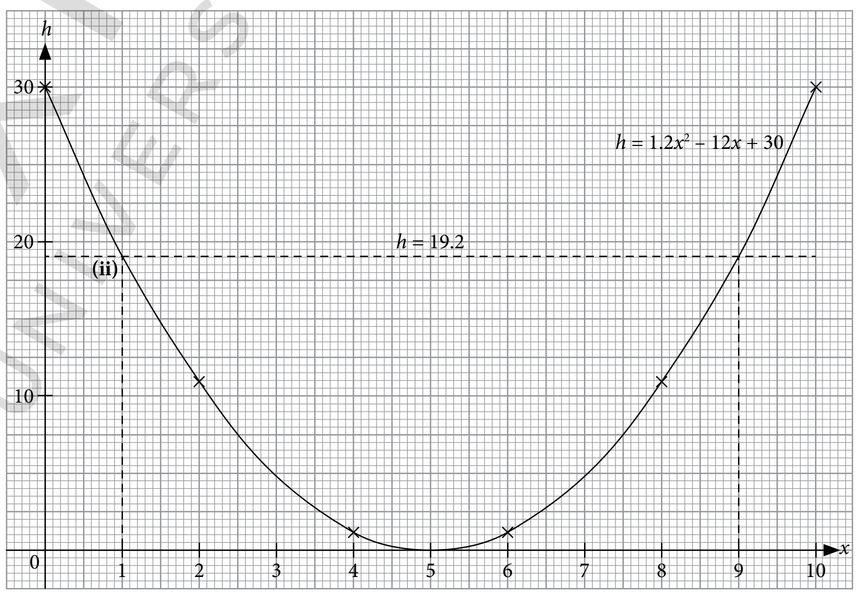


(ii) Minimum value of the asset = **$5000**

Time at which minimum value of asset occurs = **1 year**

11. **(i)**

x	0	2	4	6	8	10
h	30	10.8	1.2	1.2	10.8	30



(ii) It means that the ride begins at a height of 30 m above ground level.

(iii) From part **(i)**, when $h = 19.2$, $x = 1$ or $x = 9$

Horizontal distance a rider travels between the two splashes

$= 9 - 1$

= 8 m

12. **(i)** When $t = 2.5$, $y = 17(2.5) - 5(2.5)^2$

$= 11.25$

$\therefore$ height of the object 2.5 seconds after it leaves the ground $=$ **11.25 m**

(ii) When $y = 0$, $17t - 5t^2 = 0$

$$t(17 - 5t) = 0$$

$$t = 0 \quad \text{or} \quad 17 - 5t = 0$$

$$t = 3.4$$

$\therefore$ the object will strike the ground again after **3.4 seconds**.

(iii) When $t = 1$, $y = 17(1) - 5(1)^2$

$= 12$

$\therefore$ the distance between the two objects one second after the second object has been projected

$= 12 - 11.25$

$=$ **0.75 m**

13. **(i)** When $t = 3.5$, $h = 56(3.5) - 7(3.5)^2$

$= 110.25$

$\therefore$ height of the ball 3.5 seconds after it leaves the ground $=$ **110.25 m**

(ii) When $h = 0$, $56t - 7t^2 = 0$

$$7t(8 - t) = 0$$

$$7t = 0 \quad \text{or} \quad 8 - t = 0$$

$$t = 0 \quad \text{or} \quad t = 8$$

$\therefore$ the ball will strike the ground again after **8 seconds**.

(iii) When $h = 49$, $56t - 7t^2 = 49$

$$7t^2 - 56t + 49 = 0$$

$$t^2 - 8t + 7 = 0$$

$$(t - 7)(t - 1) = 0$$

$$t - 7 = 0 \quad \text{or} \quad t - 1 = 0$$

$$t = 7 \quad \text{or} \quad t = 1$$

$\therefore$ the ball will be 49 m above the ground again **1 s** and **7 s** after launch.

The ball reaches the maximum height at $t = \frac{7+1}{2}$

$= 4$

The graph is symmetrical about the line $t = 4$, which is equidistant from $t = 1$ and $t = 7$. Hence, there are two possible answers.

Chapter 3 Quadratic and Fractional Equations

TEACHING NOTES

Suggested Approach

In Chapter 2, students have learnt how to solve quadratic equations by factorisation. Teachers may want to begin this chapter by doing a recap and building up on what students have learnt so far. Once students have learnt the methods of solving quadratic equations, teachers may get the students to do a reflection on the methods of solving quadratic equations (see Journal Writing on page 60 of the textbook). Teachers should give students the opportunity to use a graphing software to explore the characteristics of the graphs of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$ and $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$.

Section 3.1: Solving quadratic equations by completing the square

Teachers can first get students to discuss with one another on what they have learnt about quadratic equations in Chapter 2. Students should be reminded that the right hand side of the equation should be zero before they factorise the equation. Teachers can ask students to solve a quadratic equation that cannot be easily solved by factorisation and highlight to them that such an equation can be rearranged into the form $(x + a)^2 = b$, where a and b are constants. The equation can then be solved easily by taking the square roots on both sides of the equation to obtain the solution.

Since students are familiar with algebra discs which they have come across in Book 1, teachers can first teach students how to complete the square for a quadratic expression with the use of the discs. When the students have understood how to complete the square, teachers can then teach them how to use a multiplication frame instead (see Investigation: Completing the square for quadratic expressions of the form $x^2 + bx$).

Section 3.2: Solving quadratic equations using formula

To give students a better understanding of how the general formula is derived, teachers can get students to work in small groups and use the method of completing the square for the general form of the quadratic equation. Teachers may guide students along when they have difficulty deriving the formula.

Teachers can raise the question to the class on how they can tell if a quadratic equation has no real solutions, one real solution or two real solutions (see Class Discussion: Number of real solutions to quadratic equations).

Section 3.3: Solving fractional equations reducible to quadratic equations

In Chapter 1, students have learnt how to solve equations involving algebraic fractions using the method of changing the subject of a formula. In this section, they will learn to solve equations which have one or more algebraic fractions. In particular, they will learn to solve fractional equations which can be reduced to quadratic equations. Also, teachers should highlight the common mistakes that students tend to make when solving fractional equations, especially when students cancel the common factors out.

Section 3.4: Solving quadratic equations by graphical method

The graphical method is another method of solving quadratic equations in which the solutions of the quadratic equation $ax^2 + bx + c = 0$ are the x-coordinates of the points of intersection of the graph $y = ax^2 + bx + c$ with the x-axis. However, students need to note that the solutions obtained by the graphical method can only be an approximation. Teachers should highlight to students that the answers obtained by the graphical method can only be accurate up to half of a small square grid. Teachers should give students more examples of the different types of quadratic graphs cutting the x-axis so that the students can better understand and identify the number of points of intersection between the graph and the x-axis.

Section 3.5: Sketching graphs of quadratic functions

In Chapter 2, students have learnt the properties of a quadratic graph of the form $y = ax^2 + bx + c$. Teachers should revise with them on what they have learnt (see Thinking Time on page 69 of the textbook) after teaching them to sketch the graphs of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$ and $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$.

For graphs of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$, teachers should guide students along and help them to recognise that the graphs cut the x-axis at $(h, 0)$ and $(k, 0)$ and are symmetrical about the vertical line that passes through the minimum or maximum point (see Investigation: Characteristics of graphs of $y = \pm(x - h)(x - k)$). Teachers should also highlight to students that the line of symmetry is halfway between the x-intercepts and how this can be used to find the coordinates of the minimum/maximum point.

For graphs of the form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$, teachers should guide the students along and help them to recognise that the minimum or maximum point is (p, q) and the line of symmetry is $x = p$ (see Investigation: Characteristics of graphs of $y = \pm(x - p)^2 + q$ or $y = -(x - p)^2 + q$).

Teachers can check on students' level of understanding of the graphs of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$ and $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$ by getting them to match graphs with their respective functions and justify their answers (see Class Discussion: Matching quadratic graphs with the corresponding functions).

Section 3.6: Applications of quadratic equations and functions in real-world contexts

Teachers should go through the problems given in the textbook and take a look at mathematics and real-life problems that involve the graphs of quadratic equations and functions. Students should use Pólya's 4-step Problem Solving Model to help them understand the problem and form an equation to solve the problem (see pages 73 and 74 of the textbook).

Worked Solutions

Introductory Problem

(a)
$$\begin{aligned} x^2 + 4x + 3 &= 0 \\ (x + 3)(x + 1) &= 0 \\ x + 3 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = -3 \quad &\text{or} \quad x = -1 \end{aligned}$$
$\therefore$ $\mathbf{x = -3}$ or $\mathbf{x = -1}$

(b) $x^2 + 4x - 3$ cannot be solved using the factorisation method.

3.1 Solving quadratic equations by completing the square

Practise Now 1

(a)
$$\begin{aligned} x^2 - 7x - 8 &= 0 \\ (x - 8)(x + 1) &= 0 \\ x - 8 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = \mathbf{8} \quad &\text{or} \quad x = \mathbf{-1} \end{aligned}$$

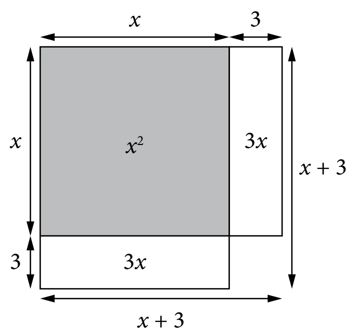
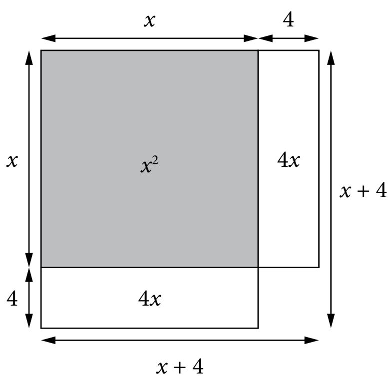
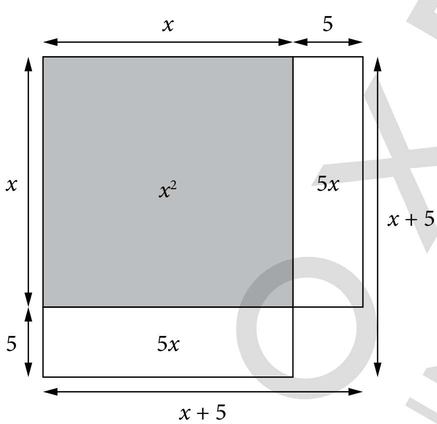
(b)
$$\begin{aligned} 2y^2 + 3y - 20 &= 0 \\ (2y - 5)(y + 4) &= 0 \\ 2y - 5 = 0 \quad &\text{or} \quad y + 4 = 0 \\ 2y = 5 \quad &\text{or} \quad \mathbf{y = -4} \\ y = \frac{5}{2} & \end{aligned}$$

Practise Now 2

(a)
$$\begin{aligned} (x + 7)^2 &= 100 \\ x + 7 &= \pm\sqrt{100} \\ &= \pm 10 \\ x &= -7 \pm 10 \\ &= \mathbf{3} \quad \text{or} \quad \mathbf{-17} \end{aligned}$$

(b)
$$\begin{aligned} (2y - 5)^2 &= 11 \\ 2y - 5 &= \pm\sqrt{11} \\ 2y &= 5 \pm \sqrt{11} \\ y &= \frac{5 \pm \sqrt{11}}{2} \\ &= \mathbf{4.16} \quad \text{or} \quad \mathbf{0.842} \text{ (to 3.s.f.)} \end{aligned}$$

Investigation (Completing the square for quadratic expressions of the form $x^2 + bx$)

<table>
<tr><th></th><th>Quadratic expression
$x^2 + bx$</th><th>Number, t, that must be added to complete the square</th><th>Half the coefficient of x, i.e. $\frac{b}{2}$</th><th>Completed square $(x + r)^2$</th><th>r</th></tr>
<tr><td>(a)</td><td>$x^2 + 6x$

</td><td>3^2</td><td>$\frac{6}{2} = 3$</td><td>$(x + 3)^2$</td><td>3</td></tr>
<tr><td>(b)</td><td>$x^2 + 8x$

</td><td>**4^2**</td><td>**$\frac{8}{2} = 4$**</td><td>**$(x + 4)^2$**</td><td>**4**</td></tr>
<tr><td>(c)</td><td>$x^2 + 10x$

</td><td>**5^2**</td><td>**$\frac{10}{2} = 5$**</td><td>**$(x + 5)^2$**</td><td>**5**</td></tr>
</table>

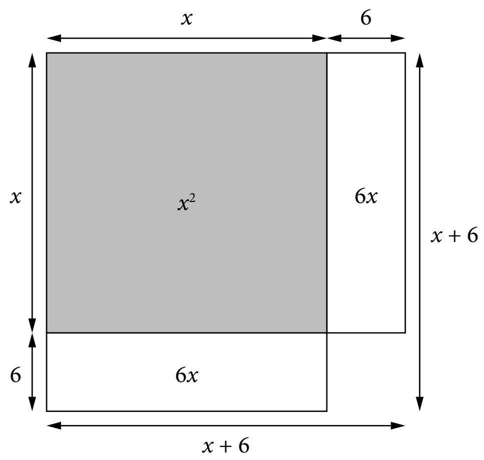
	Quadratic expression $x^2 + bx$	Number, t, that must be added to complete the square	Half the coefficient of x, i.e. $\frac{b}{2}$	Completed square $(x + r)^2$	r
(d)	$x^2 + 12x$ 	$\mathbf{6^2}$	$\mathbf{\frac{12}{2} = 6}$	$\boldsymbol{(x + 6)^2}$	**6**

Table 3.1

1. $\boldsymbol{t = \left(\frac{b}{2}\right)^2}$
2. $\boldsymbol{r = \frac{b}{2}}$
3. $t = \left(\frac{b}{2}\right)^2$
$= \left(\frac{18}{2}\right)^2$
$= 9^2$
$= 81$
$\therefore$ the number is **81**.
4. $r = \frac{b}{2}$
$= \frac{18}{2}$
$= 9$
$\therefore (x + r)^2 = \boldsymbol{(x + 9)^2}$

Practise Now 3

(a) $x^2 + 12x = \left[x^2 + 12x + \left(\frac{12}{2}\right)^2\right] - \left(\frac{12}{2}\right)^2$
$= (x^2 + 12x + 6^2) - 6^2$
$= \boldsymbol{(x + 6)^2 - 36}$

(b) $x^2 - 7x = \left[x^2 - 7x + \left(-\frac{7}{2}\right)^2\right] - \left(-\frac{7}{2}\right)^2$
$= \boldsymbol{\left(x - \frac{7}{2}\right)^2 - \frac{49}{4}}$

(c) $x^2 + 1.6x = \left[x^2 + 1.6x + \left(\frac{1.6}{2}\right)^2\right] - \left(\frac{1.6}{2}\right)^2$
$= (x^2 + 1.6x + 0.8^2) - 0.8^2$
$= \boldsymbol{(x + 0.8)^2 - 0.64}$

(d) $x^2 - \frac{3}{4}x = \left[x^2 - \frac{3}{4}x + \left(-\frac{3}{8}\right)^2\right] - \left(-\frac{3}{8}\right)^2$
$= \boldsymbol{\left(x - \frac{3}{8}\right)^2 - \frac{9}{64}}$

Practise Now 4

1. **(a)** $x^2 + 14x + 5 = (x^2 + 14x + 7^2) - 7^2 + 5$
$= \boldsymbol{(x + 7)^2 - 44}$
(b) $x^2 + 7x - 1.2 = (x^2 + 7x + 3.5^2) - 3.5^2 - 1.2$
$= \boldsymbol{(x + 3.5)^2 - 13.45}$
(c) $x^2 - 9x + 3 = \left[x^2 - 9x + \left(-\frac{9}{2}\right)^2\right] - \left(-\frac{9}{2}\right)^2 + 3$
$= \boldsymbol{\left(x - \frac{9}{2}\right)^2 - \frac{69}{4}}$
(d) $x^2 - \frac{6}{5}x - 4 = \left[x^2 - \frac{6}{5}x + \left(-\frac{3}{5}\right)^2\right] - \left(-\frac{3}{5}\right)^2 - 4$
$= \boldsymbol{\left(x - \frac{3}{5}\right)^2 - \frac{109}{25}}$

2. **(a)** $-x^2 + 6x - 2 = -(x^2 - 6x) - 2$
$= -[x^2 - 6x + (-3)^2 - (-3)^2] - 2$
$= -(x - 3)^2 + 9 - 2$
$= \boldsymbol{-(x - 3)^2 + 7}$
(b) $-x^2 + 9x - 3.5 = -(x^2 - 9x) - 3.5$
$= -[x^2 - 9x + (-4.5)^2 - (-4.5)^2] - 3.5$
$= -(x - 4.5)^2 + 20.25 - 3.5$
$= \boldsymbol{-(x - 4.5)^2 + 16.75}$
(c) $-x^2 - 7x + 5 = -(x^2 + 7x) + 5$
$= -\left[x^2 + 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2\right] + 5$
$= -\left(x + \frac{7}{2}\right)^2 + \frac{49}{4} + 5$
$= \boldsymbol{-\left(x + \frac{7}{2}\right)^2 + \frac{69}{4}}$
(d) $-x^2 - \frac{4}{9}x - 1 = -\left(x^2 + \frac{4}{9}x\right) - 1$
$= -\left[x^2 + \frac{4}{9}x + \left(\frac{2}{9}\right)^2 - \left(\frac{2}{9}\right)^2\right] - 1$
$= -\left(x + \frac{2}{9}\right)^2 + \frac{4}{81} - 1$
$= \boldsymbol{-\left(x + \frac{2}{9}\right)^2 - \frac{77}{81}}$

Practise Now 5

1. **(a)**
$$\begin{aligned} x^2 + 6x - 4 &= 0 \\ x^2 + 6x &= 4 \\ x^2 + 6x + 3^2 &= 4 + 3^2 \\ (x+3)^2 &= 13 \\ x + 3 &= \pm\sqrt{13} \\ x &= -3 \pm \sqrt{13} \\ &= \mathbf{0.61} \quad \text{or} \quad \mathbf{-6.61} \text{ (to 2 d.p.)} \end{aligned}$$

(b)
$$\begin{aligned} y^2 + 7y + 5 &= 0 \\ y^2 + 7y &= -5 \\ y^2 + 7y + \left(\frac{7}{2}\right)^2 &= -5 + \left(\frac{7}{2}\right)^2 \\ \left(y + \frac{7}{2}\right)^2 &= \frac{29}{4} \\ y + \frac{7}{2} &= \pm\sqrt{\frac{29}{4}} \\ y &= -\frac{7}{2} \pm \sqrt{\frac{29}{4}} \\ &= \mathbf{-0.81} \quad \text{or} \quad \mathbf{-6.19} \text{ (to 2 d.p.)} \end{aligned}$$

(c)
$$\begin{aligned} z^2 - z - 1 &= 0 \\ z^2 - z &= 1 \\ z^2 - z + \left(-\frac{1}{2}\right)^2 &= 1 + \left(-\frac{1}{2}\right)^2 \\ \left(z - \frac{1}{2}\right)^2 &= \frac{5}{4} \\ z - \frac{1}{2} &= \pm\sqrt{\frac{5}{4}} \\ z &= \frac{1}{2} \pm \sqrt{\frac{5}{4}} \\ &= \mathbf{1.62} \quad \text{or} \quad \mathbf{-0.62} \text{ (to 2 d.p.)} \end{aligned}$$

2.
$$\begin{aligned} (x+4)(x-3) &= 15 \\ x^2 + x - 12 &= 15 \\ x^2 + x &= 27 \\ x^2 + x + \left(\frac{1}{2}\right)^2 &= 27 + \left(\frac{1}{2}\right)^2 \\ \left(x + \frac{1}{2}\right)^2 &= \frac{109}{4} \\ x + \frac{1}{2} &= \pm\sqrt{\frac{109}{4}} \\ x &= -\frac{1}{2} \pm \sqrt{\frac{109}{4}} \\ &= \mathbf{4.72} \quad \text{or} \quad \mathbf{-5.72} \text{ (to 2 d.p.)} \end{aligned}$$

Exercise 3A

1. **(a)**
$$\begin{aligned} x^2 + 7x - 18 &= 0 \\ (x+9)(x-2) &= 0 \\ x + 9 = 0 \quad &\text{or} \quad x - 2 = 0 \\ x = \mathbf{-9} \quad &\text{or} \quad x = \mathbf{2} \end{aligned}$$

(b)
$$\begin{aligned} 2x^2 + 5x - 7 &= 0 \\ (2x+7)(x-1) &= 0 \\ 2x + 7 = 0 \quad &\text{or} \quad x - 1 = 0 \\ 2x = -7 \quad &\text{or} \quad x = \mathbf{1} \\ x = -\frac{\mathbf{7}}{\mathbf{2}} \end{aligned}$$

(c)
$$\begin{aligned} 5y^2 - 28y + 15 &= 0 \\ (5y-3)(y-5) &= 0 \\ 5y - 3 = 0 \quad &\text{or} \quad y - 5 = 0 \\ 5y = 3 \quad &\text{or} \quad y = \mathbf{5} \\ y = \frac{\mathbf{3}}{\mathbf{5}} \end{aligned}$$

(d)
$$\begin{aligned} 4z^2 - 49 &= 0 \\ (2z+7)(2z-7) &= 0 \\ 2z + 7 = 0 \quad &\text{or} \quad 2z - 7 = 0 \\ 2z = -7 \quad &\text{or} \quad 2z = 7 \\ z = -\frac{\mathbf{7}}{\mathbf{2}} \quad &\text{or} \quad z = \frac{\mathbf{7}}{\mathbf{2}} \end{aligned}$$

2. **(a)**
$$\begin{aligned} (x+1)^2 &= 9 \\ x + 1 &= \pm\sqrt{9} \\ &= \pm 3 \\ x + 1 = 3 \quad &\text{or} \quad x + 1 = -3 \\ x = \mathbf{2} \quad &\text{or} \quad x = \mathbf{-4} \end{aligned}$$

(b)
$$\begin{aligned} (2y+1)^2 &= 16 \\ 2y + 1 &= \pm\sqrt{16} \\ &= \pm 4 \\ 2y + 1 = 4 \quad &\text{or} \quad 2y + 1 = -4 \\ 2y = 3 \quad &\text{or} \quad 2y = -5 \\ y = \frac{\mathbf{3}}{\mathbf{2}} \quad &\text{or} \quad y = -\frac{\mathbf{5}}{\mathbf{2}} \end{aligned}$$

(c)
$$\begin{aligned} (5h-4)^2 &= 81 \\ 5h - 4 &= \pm\sqrt{81} \\ &= \pm 9 \\ 5h - 4 = 9 \quad &\text{or} \quad 5h - 4 = -9 \\ 5h = 13 \quad &\text{or} \quad 5h = -5 \\ h = \frac{\mathbf{13}}{\mathbf{5}} \quad &\text{or} \quad h = \mathbf{-1} \end{aligned}$$

(d)
$$\begin{aligned} (7-3k)^2 &= \frac{9}{16} \\ 7 - 3k &= \pm\sqrt{\frac{9}{16}} \\ &= \pm\frac{3}{4} \\ 7 - 3k = \frac{3}{4} \quad &\text{or} \quad 7 - 3k = -\frac{3}{4} \\ -3k = -\frac{25}{4} \quad &\text{or} \quad -3k = -\frac{31}{4} \\ k = \frac{\mathbf{25}}{\mathbf{12}} \quad &\text{or} \quad k = \frac{\mathbf{31}}{\mathbf{12}} \end{aligned}$$

(e)
$$\begin{aligned} (m+3)^2 &= 11 \\ m + 3 &= \pm\sqrt{11} \\ m + 3 = \sqrt{11} \quad &\text{or} \quad m + 3 = -\sqrt{11} \\ m = \sqrt{11} - 3 \quad &\text{or} \quad m = -\sqrt{11} - 3 \\ m = \mathbf{0.32} \quad &\text{or} \quad m = \mathbf{-6.32} \text{ (to 2 d.p.)} \end{aligned}$$

(f)
$$\begin{aligned} (2n-3)^2 &= 23 \\ 2n - 3 &= \pm\sqrt{23} \\ 2n - 3 = \sqrt{23} \quad &\text{or} \quad 2n - 3 = -\sqrt{23} \\ 2n = \sqrt{23} + 3 \quad &\text{or} \quad 2n = -\sqrt{23} + 3 \\ n = \mathbf{3.90} \quad &\text{or} \quad n = \mathbf{-0.90} \text{ (to 2 d.p.)} \end{aligned}$$

(g)
$$\begin{aligned} (5-w)^2 &= 7 \\ 5 - w &= \pm\sqrt{7} \\ 5 - w = \sqrt{7} \quad &\text{or} \quad 5 - w = -\sqrt{7} \\ -w = \sqrt{7} - 5 \quad &\text{or} \quad -w = -\sqrt{7} - 5 \\ w = \mathbf{2.35} \quad &\text{or} \quad w = \mathbf{7.65} \text{ (to 2 d.p.)} \end{aligned}$$

(h)
$$\begin{aligned} \left(\frac{1}{2} - t\right)^2 &= 10 \\ \frac{1}{2} - t &= \pm\sqrt{10} \\ \frac{1}{2} - t = \sqrt{10} \quad &\text{or} \quad \frac{1}{2} - t = -\sqrt{10} \\ -t = \sqrt{10} - \frac{1}{2} \quad &\text{or} \quad -t = -\sqrt{10} - \frac{1}{2} \\ t = \mathbf{-2.66} \quad &\text{or} \quad t = \mathbf{3.66} \text{ (to 2 d.p.)} \end{aligned}$$

3. (a) $x^2 + 20x = (x^2 + 20x + 10^2) - 10^2$

$= \mathbf{(x + 10)^2 - 100}$

(b) $x^2 - 15x = \left[x^2 - 15x + \left(-\frac{15}{2}\right)^2\right] - \left(-\frac{15}{2}\right)^2$

$= \boldsymbol{\left(x - \frac{15}{2}\right)^2 - \frac{225}{4}}$

(c) $x^2 + \frac{1}{2}x = \left[x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2\right] - \left(\frac{1}{4}\right)^2$

$= \boldsymbol{\left(x + \frac{1}{4}\right)^2 - \frac{1}{16}}$

(d) $x^2 - \frac{2}{9}x = \left[x^2 - \frac{2}{9}x + \left(-\frac{1}{9}\right)^2\right] - \left(-\frac{1}{9}\right)^2$

$= \boldsymbol{\left(x - \frac{1}{9}\right)^2 - \frac{1}{81}}$

(e) $x^2 + 0.2x = (x^2 + 0.2x + 0.1^2) - 0.1^2$

$= \mathbf{(x + 0.1)^2 - 0.01}$

(f) $x^2 - 1.4x = [x^2 - 1.4x + (-0.7)^2] - (-0.7)^2$

$= \mathbf{(x - 0.7)^2 - 0.49}$

(g) $-x^2 - 10x = -(x^2 + 10x)$

$= -[x^2 + 10x + 5^2 - 5^2]$

$= \mathbf{-(x + 5)^2 + 25}$

(h) $-x^2 + 11x = -(x^2 - 11x)$

$= -\left[x^2 - 11x + \left(-\frac{11}{2}\right)^2 - \left(-\frac{11}{2}\right)^2\right]$

$= \boldsymbol{-\left(x - \frac{11}{2}\right)^2 + \frac{121}{4}}$

4. (a) $x^2 - 6x + 1 = [x^2 - 6x + (-3)^2] - (-3)^2 + 1$

$= \mathbf{(x - 3)^2 - 8}$

(b) $x^2 + 3x - 2 = \left[x^2 + 3x + \left(\frac{3}{2}\right)^2\right] - \left(\frac{3}{2}\right)^2 - 2$

$= \boldsymbol{\left(x + \frac{3}{2}\right)^2 - \frac{17}{4}}$

(c) $x^2 + 9x - 1.8 = (x^2 + 9x + 4.5^2) - 4.5^2 - 1.8$

$= \mathbf{(x + 4.5)^2 - 22.05}$

(d) $x^2 - \frac{2}{7}x + 7 = \left[x^2 - \frac{2}{7}x + \left(-\frac{1}{7}\right)^2\right] - \left(-\frac{1}{7}\right)^2 + 7$

$= \boldsymbol{\left(x - \frac{1}{7}\right)^2 + \frac{342}{49}}$

(e) $-x^2 + 10x - 2 = -(x^2 - 10x) - 2$

$= -[x^2 - 10x + (-5)^2 - (-5)^2] - 2$

$= -(x - 5)^2 + 25 - 2$

$= \mathbf{-(x - 5)^2 + 23}$

(f) $-x^2 + 13x - \frac{13}{2} = -(x^2 - 13x) - \frac{13}{2}$

$= -\left[x^2 - 13x + \left(-\frac{13}{2}\right)^2 - \left(-\frac{13}{2}\right)^2\right] - \frac{13}{2}$

$= -\left(x - \frac{13}{2}\right)^2 + \frac{169}{4} - \frac{13}{2}$

$= \boldsymbol{-\left(x - \frac{13}{2}\right)^2 + \frac{143}{4}}$

(g) $-x^2 - 9x - 20.25 = -(x^2 + 9x) - 20.25$

$= -[x^2 + 9x + (4.5)^2 - (4.5)^2] - 20.25$

$= -(x + 4.5)^2 + 20.25 - 20.25$

$= \mathbf{-(x + 4.5)^2}$

(h) $-x^2 - \frac{3}{4}x + 3 = -\left(x^2 + \frac{3}{4}x\right) + 3$

$= -\left[x^2 + \frac{3}{4}x + \left(\frac{3}{8}\right)^2 - \left(\frac{3}{8}\right)^2\right] + 3$

$= -\left(x + \frac{3}{8}\right)^2 + \frac{9}{64} + 3$

$= \boldsymbol{-\left(x + \frac{3}{8}\right)^2 + \frac{201}{64}}$

5. (a) $x^2 + 2x - 5 = 0$

$x^2 + 2x = 5$

$x^2 + 2x + 1^2 = 5 + 1^2$

$(x + 1)^2 = 6$

$x + 1 = \pm\sqrt{6}$

$x = -1 \pm \sqrt{6}$

$= \mathbf{1.45}$ or $\mathbf{-3.45}$ (to 2 d.p.)

(b) $y^2 - 12y + 9 = 0$

$y^2 - 12y = -9$

$y^2 - 12y + (-6)^2 = -9 + (-6)^2$

$(y - 6)^2 = 27$

$y - 6 = \pm\sqrt{27}$

$y = 6 \pm \sqrt{27}$

$= \mathbf{11.20}$ or $\mathbf{0.80}$ (to 2 d.p.)

(c) $z^2 - 5z - 5 = 0$

$z^2 - 5z = 5$

$z^2 - 5z + \left(-\frac{5}{2}\right)^2 = 5 + \left(-\frac{5}{2}\right)^2$

$\left(z - \frac{5}{2}\right)^2 = \frac{45}{4}$

$z - \frac{5}{2} = \pm\sqrt{\frac{45}{4}}$

$z = \frac{5}{2} \pm \sqrt{\frac{45}{4}}$

$= \mathbf{5.85}$ or $\mathbf{-0.85}$ (to 2 d.p.)

(d) $p^2 + \frac{1}{4}p - 3 = 0$

$p^2 + \frac{1}{4}p = 3$

$p^2 + \frac{1}{4}p + \left(\frac{1}{8}\right)^2 = 3 + \left(\frac{1}{8}\right)^2$

$\left(p + \frac{1}{8}\right)^2 = \frac{193}{64}$

$p + \frac{1}{8} = \pm\sqrt{\frac{193}{64}}$

$p = -\frac{1}{8} \pm \sqrt{\frac{193}{64}}$

$= \mathbf{1.61}$ or $\mathbf{-1.86}$ (to 2 d.p.)

(e) $q^2 - \frac{6}{7}q + \frac{2}{49} = 0$

$q^2 - \frac{6}{7}q = -\frac{2}{49}$

$q^2 - \frac{6}{7}q + \left(-\frac{3}{7}\right)^2 = -\frac{2}{49} + \left(-\frac{3}{7}\right)^2$

$\left(q - \frac{3}{7}\right)^2 = \frac{1}{7}$

$q - \frac{3}{7} = \pm\sqrt{\frac{1}{7}}$

$q = \frac{3}{7} \pm \sqrt{\frac{1}{7}}$

$= \mathbf{0.81}$ or $\mathbf{0.05}$ (to 2 d.p.)

(f) $r^2 + 0.6r - 1 = 0$

$r^2 + 0.6r = 1$

$r^2 + 0.6r + 0.3^2 = 1 + 0.3^2$

$(r + 0.3)^2 = 1.09$

$r + 0.3 = \pm\sqrt{1.09}$

$r = -0.3 \pm \sqrt{1.09}$

$= \mathbf{0.74}$ or $\mathbf{-1.34}$ (to 2 d.p.)

6. **(i)** $x^2 + 17x - 30 = \left[x^2 + 17x + \left(\frac{17}{2}\right)^2\right] - \left(\frac{17}{2}\right)^2 - 30$

$= \left(\boldsymbol{x} + \mathbf{\frac{17}{2}}\right)^2 - \mathbf{\frac{409}{4}}$

(ii) $x^2 + 17x - 30 = 0$

$\left(x + \frac{17}{2}\right)^2 - \frac{409}{4} = 0$

$\left(x + \frac{17}{2}\right)^2 = \frac{409}{4}$

$x + \frac{17}{2} = \pm\sqrt{\frac{409}{4}}$

$x = -\frac{17}{2} \pm \sqrt{\frac{409}{4}}$

$= \mathbf{1.6}$ or $\mathbf{-18.6}$ (to 1 d.p.)

7. **(a)** $a(a + 4) = 3a + 1$

$a^2 + 4a = 3a + 1$

$a^2 + a = 1$

$a^2 + a + \left(\frac{1}{2}\right)^2 = 1 + \left(\frac{1}{2}\right)^2$

$\left(a + \frac{1}{2}\right)^2 = \frac{5}{4}$

$a + \frac{1}{2} = \pm\sqrt{\frac{5}{4}}$

$a = -\frac{1}{2} \pm \sqrt{\frac{5}{4}}$

$= \mathbf{0.618}$ or $\mathbf{-1.62}$ (to 3 s.f.)

(b) $(b + 1)^2 = 7b$

$b^2 + 2b + 1 = 7b$

$b^2 - 5b + 1 = 0$

$b^2 - 5b = -1$

$b^2 - 5b + \left(-\frac{5}{2}\right)^2 = -1 + \left(-\frac{5}{2}\right)^2$

$\left(b - \frac{5}{2}\right)^2 = \frac{21}{4}$

$b - \frac{5}{2} = \pm\sqrt{\frac{21}{4}}$

$b = \frac{5}{2} \pm \sqrt{\frac{21}{4}}$

$= \mathbf{4.79}$ or $\mathbf{0.209}$ (to 3 s.f.)

(c) $(c - 2)(c + 5) = c$

$c^2 + 3c - 10 = c$

$c^2 + 2c - 10 = 0$

$c^2 + 2c = 10$

$c^2 + 2c + 1^2 = 10 + 1^2$

$(c + 1)^2 = 11$

$c + 1 = \pm\sqrt{11}$

$c = -1 \pm \sqrt{11}$

$= \mathbf{2.32}$ or $\mathbf{-4.32}$ (to 3.s.f.)

(d) $d(d - 4) = 2(d + 7)$

$d^2 - 4d = 2d + 14$

$d^2 - 6d = 14$

$d^2 - 6d + (-3)^2 = 14 + (-3)^2$

$(d - 3)^2 = 23$

$d - 3 = \pm\sqrt{23}$

$d = 3 \pm \sqrt{23}$

$= \mathbf{7.80}$ or $\mathbf{-1.80}$ (to 3 s.f.)

8. $y^2 - ay - 6 = 0$

$y^2 - ay = 6$

$y^2 - ay + \left(-\frac{a}{2}\right)^2 = 6 + \left(-\frac{a}{2}\right)^2$

$\left(y - \frac{a}{2}\right)^2 = \frac{a^2 + 24}{4}$

$y - \frac{a}{2} = \pm\frac{\sqrt{a^2 + 24}}{2}$

$y = \frac{a}{2} \pm \frac{\sqrt{a^2 + 24}}{2}$

$= \boldsymbol{\frac{a \pm \sqrt{a^2 + 24}}{2}}$

3.2 Solving quadratic equations using formula

Practise Now 6

(a) $2x^2 + 3x - 7 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-3 \pm \sqrt{3^2 - 4(2)(-7)}}{2(2)}$

$= \frac{-3 \pm \sqrt{65}}{4}$

$= \mathbf{1.27}$ or $\mathbf{-2.77}$ (to 3 s.f.)

(b) $-5x^2 + 8x + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-8 \pm \sqrt{8^2 - 4(-5)(1)}}{2(-5)}$

$= \frac{-8 \pm \sqrt{84}}{-10}$

$= \mathbf{1.72}$ or $\mathbf{-0.117}$ (to 3 s.f.)

(c) $3x^2 - 5 - x = 0$

$3x^2 - x - 5 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)}$

$= \frac{1 \pm \sqrt{61}}{6}$

$= \mathbf{1.47}$ or $\mathbf{-1.14}$ (to 3 s.f.)

(d) $1 - x^2 - 7x = 0$

$-x^2 - 7x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-1)(1)}}{2(-1)}$$

$$= \frac{7 \pm \sqrt{53}}{-2}$$

$= \mathbf{0.140}$ or $\mathbf{-7.14}$ (to 3 s.f.)

(e) $(x - 1)^2 = 4x - 5$

$x^2 - 2x + 1 = 4x - 5$

$x^2 - 6x + 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{12}}{2}$$

$= \mathbf{4.73}$ or $\mathbf{1.27}$ (to 3 s.f.)

(f) $(x + 3)(x - 1) = 8x - 7$

$x^2 + 2x - 3 = 8x - 7$

$x^2 - 6x + 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{20}}{2}$$

$= \mathbf{5.24}$ or $\mathbf{0.764}$ (to 3 s.f.)

Class Discussion (Number of real solutions to quadratic equations)

(a) $3x^2 + 5x - 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4(3)(-4)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{73}}{6}$$

$= \mathbf{0.591}$ or $\mathbf{-2.26}$ (to 3 s.f.)

(b) $4x^2 - 12x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$

$$= \frac{12 \pm \sqrt{0}}{8}$$

$= \mathbf{1.5}$

(c) $2x^2 + 5x + 8 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4(2)(8)}}{2(2)}$$

$$= \frac{-5 \pm \sqrt{-39}}{4}$$

1. For $2x^2 + 5x + 8$, $b^2 - 4ac = -39$ is negative. Since we cannot take the square root of a negative number, the equation does not have any real solution.

2. The numbers of real solutions for **(a)**, **(b)** and **(c)** are **two**, **one** and **zero** respectively.

3. $4x^2 - 12x + 9 = 0$

$(2x - 3)^2 = 0$

$2x - 3 = 0$

$2x = 3$

$x = 1.5$

Yes, the equation can be solved by factorisation.

3.3 Solving fractional equations reducible to quadratic equations

Practise Now 7

(a) $\frac{4}{p} = 2p - 3$

$\frac{4}{p} \times p = (2p - 3) \times p$

$4 = p(2p - 3)$

$= 2p^2 - 3p$

$0 = 2p^2 - 3p - 4$

$2p^2 - 3p - 4 = 0$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{41}}{4}$$

$= \mathbf{2.35}$ or $\mathbf{-0.851}$ (to 3 s.f.)

(b) $\frac{2q}{10 - 3q} = 10 - 3q$

$2q = (10 - 3q)^2$

$= 9q^2 - 60q + 100$

$0 = 9q^2 - 62q + 100$

$9q^2 - 62q + 100 = 0$

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-62) \pm \sqrt{(-62)^2 - 4(9)(100)}}{2(9)}$$

$$= \frac{62 \pm \sqrt{244}}{18}$$

$= \mathbf{4.31}$ or $\mathbf{2.58}$ (to 3 s.f.)

(c)
$$\frac{6}{x+4} = x+3$$
$$\frac{6}{x+4} \times (x+4) = (x+3) \times (x+4)$$
$$6 = (x+3)(x+4)$$
$$= x^2 + 7x + 12$$
$$0 = x^2 + 7x + 6$$
$$x^2 + 7x + 6 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-7 \pm \sqrt{7^2 - 4(1)(6)}}{2(1)}$$
$$= \frac{-7 \pm \sqrt{25}}{2}$$
$= \mathbf{-1}$ or $\mathbf{-6}$

(d)
$$\frac{3}{12-y} = 3y - 1$$
$$\frac{3}{12-y} \times (12-y) = (3y-1) \times (12-y)$$
$$3 = (3y-1)(12-y)$$
$$= -3y^2 + 37y - 12$$
$$3y^2 - 37y + 15 = 0$$
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-37) \pm \sqrt{(-37)^2 - 4(3)(15)}}{2(3)}$$
$$= \frac{37 \pm \sqrt{1189}}{6}$$
$= \mathbf{11.9}$ or $\mathbf{0.420}$ (to 3 s.f.)

Practise Now 8

(a)
$$\frac{1}{x+6} + \frac{2}{3-x} = 5$$
$$\left(\frac{1}{x+6} + \frac{2}{3-x}\right) \times (x+6)(3-x) = 5 \times (x+6)(3-x)$$
$$3 - x + 2(x+6) = 5(-x^2 - 3x + 18)$$
$$3 - x + 2x + 12 = -5x^2 - 15x + 90$$
$$x + 15 = -5x^2 - 15x + 90$$
$$5x^2 + 16x - 75 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-16 \pm \sqrt{16^2 - 4(5)(-75)}}{2(5)}$$
$$= \frac{-16 \pm \sqrt{1756}}{10}$$
$= \mathbf{2.59}$ or $\mathbf{-5.79}$ (to 3 s.f.)

(b)
$$\frac{5}{y-3} + \frac{y-1}{y-2} = 7$$
$$\left(\frac{5}{y-3} + \frac{y-1}{y-2}\right) \times (y-3)(y-2) = 7 \times (y-3)(y-2)$$
$$5(y-2) + (y-1)(y-3) = 7(y^2 - 5y + 6)$$
$$5y - 10 + y^2 - 4y + 3 = 7y^2 - 35y + 42$$
$$y^2 + y - 7 = 7y^2 - 35y + 42$$
$$0 = 6y^2 - 36y + 49$$
$$6y^2 - 36y + 49 = 0$$
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-36) \pm \sqrt{(-36)^2 - 4(6)(49)}}{2(6)}$$
$$= \frac{36 \pm \sqrt{120}}{12}$$
$= \mathbf{3.91}$ or $\mathbf{2.09}$ (to 3 s.f.)

(c)
$$\frac{3}{n-2} - \frac{1}{(n-2)^2} = -2$$
$$\left[\frac{3}{n-2} - \frac{1}{(n-2)^2}\right] \times (n-2)^2 = -2 \times (n-2)^2$$
$$3(n-2) - 1 = -2(n^2 - 4n + 4)$$
$$3n - 6 - 1 = -2n^2 + 8n - 8$$
$$3n - 7 = -2n^2 + 8n - 8$$
$$2n^2 - 5n + 1 = 0$$
$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{5 \pm \sqrt{17}}{4}$$
$= \mathbf{2.28}$ or $\mathbf{0.219}$ (to 3 s.f.)

(d)
$$\frac{6}{1-2t} + \frac{3t}{1-4t^2} = 1$$
$$\frac{6}{1-2t} + \frac{3t}{(1+2t)(1-2t)} = 1$$
$$\left[\frac{6}{1-2t} + \frac{3t}{(1+2t)(1-2t)}\right] \times (1+2t)(1-2t) = 1 \times (1+2t)(1-2t)$$
$$6(1+2t) + 3t = 1 - 4t^2$$
$$6 + 12t + 3t = 1 - 4t^2$$
$$15t + 6 = 1 - 4t^2$$
$$4t^2 + 15t + 5 = 0$$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-15 \pm \sqrt{15^2 - 4(4)(5)}}{2(4)}$$
$$= \frac{-15 \pm \sqrt{145}}{8}$$
$= \mathbf{-0.370}$ or $\mathbf{-3.38}$ (to 3 s.f.)

Exercise 3B

1. **(a)** $x^2 + 4x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{12}}{2}$$

$= \mathbf{-0.268}$ or $\mathbf{-3.73}$ (to 3 s.f.)

(b) $3x^2 + 6x - 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{6^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{-6 \pm \sqrt{48}}{6}$$

$= \mathbf{0.155}$ or $\mathbf{-2.15}$ (to 3 s.f.)

(c) $3x^2 - 5x - 17 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-17)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{229}}{6}$$

$= \mathbf{3.36}$ or $\mathbf{-1.69}$ (to 3 s.f.)

(d) $-3x^2 - 7x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-3)(9)}}{2(-3)}$$

$$= \frac{7 \pm \sqrt{157}}{-6}$$

$= \mathbf{0.922}$ or $\mathbf{-3.25}$ (to 3 s.f.)

(e) $2 + 2x^2 - 7x = 0$

$2x^2 - 7x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(2)}}{2(2)}$$

$$= \frac{7 \pm \sqrt{33}}{4}$$

$= \mathbf{3.19}$ or $\mathbf{0.314}$ (to 3 s.f.)

(f) $10x - 5x^2 - 2 = 0$

$-5x^2 + 10x - 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-10 \pm \sqrt{10^2 - 4(-5)(-2)}}{2(-5)}$$

$$= \frac{-10 \pm \sqrt{60}}{-10}$$

$= \mathbf{1.77}$ or $\mathbf{0.225}$ (to 3 s.f.)

2. **(a)** $x^2 + 5x = 21$

$x^2 + 5x - 21 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4(1)(-21)}}{2(1)}$$

$$= \frac{-5 \pm \sqrt{109}}{2}$$

$= \mathbf{2.72}$ or $\mathbf{-7.72}$ (to 3 s.f.)

(b) $10x^2 - 12x = 15$

$10x^2 - 12x - 15 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(10)(-15)}}{2(10)}$$

$$= \frac{12 \pm \sqrt{744}}{20}$$

$= \mathbf{1.96}$ or $\mathbf{-0.764}$ (to 3 s.f.)

(c) $8x^2 = 3x + 6$

$8x^2 - 3x - 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(8)(-6)}}{2(8)}$$

$$= \frac{3 \pm \sqrt{201}}{16}$$

$= \mathbf{1.07}$ or $\mathbf{-0.699}$ (to 3 s.f.)

(d) $4x^2 + 1 = -4x$

$4x^2 + 4x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(4)(1)}}{2(4)}$$

$$= \frac{-4 \pm \sqrt{0}}{8}$$

$= \mathbf{-0.5}$

(e) $9 - 5x^2 = -3x$

$-5x^2 + 3x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{3^2 - 4(-5)(9)}}{2(-5)}$$

$$= \frac{-3 \pm \sqrt{189}}{-10}$$

$= \mathbf{1.67}$ or $\mathbf{-1.07}$ (to 3 s.f.)

(f) $16x - 61 = x^2$

$-x^2 + 16x - 61 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-16 \pm \sqrt{16^2 - 4(-1)(-61)}}{2(-1)}$$

$$= \frac{-16 \pm \sqrt{12}}{-2}$$

$= \mathbf{6.27}$ or $\mathbf{9.73}$ (to 3 s.f.)

3. **(a)**
$$\frac{8}{x} = 2x + 1$$
$$\frac{8}{x} \times x = (2x+1) \times x$$
$$8 = x(2x+1)$$
$$= 2x^2 + x$$
$$0 = 2x^2 + x - 8$$
$$2x^2 + x - 8 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1^2 - 4(2)(-8)}}{2(2)}$$
$$= \frac{-1 \pm \sqrt{65}}{4}$$
$= \mathbf{1.77}$ or $\mathbf{-2.27}$ (to 3 s.f.)

(b)
$$x + \frac{7}{x} = 9$$
$$\left(x + \frac{7}{x}\right) \times x = 9 \times x$$
$$x^2 + 7 = 9x$$
$$x^2 - 9x + 7 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(7)}}{2(1)}$$
$$= \frac{9 \pm \sqrt{53}}{2}$$
$= \mathbf{8.14}$ or $\mathbf{0.860}$ (to 3 s.f.)

(c)
$$\frac{x+1}{5-x} = x$$
$$\frac{x+1}{5-x} \times (5-x) = x \times (5-x)$$
$$x + 1 = x(5 - x)$$
$$= 5x - x^2$$
$$x^2 - 4x + 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{12}}{2}$$
$= \mathbf{0.268}$ or $\mathbf{3.73}$ (to 3 s.f.)

(d)
$$3x - 1 = -\frac{9}{3x+5}$$
$$(3x-1) \times (3x+5) = -\frac{9}{3x+5} \times (3x+5)$$
$$(3x-1)(3x+5) = -9$$
$$9x^2 + 12x - 5 = -9$$
$$9x^2 + 12x + 4 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-12 \pm \sqrt{12^2 - 4(9)(4)}}{2(9)}$$
$$= \frac{-12 \pm \sqrt{0}}{18}$$
$$= -\mathbf{\frac{2}{3}}$$

(e)
$$2x + 1 = \frac{x+1}{x-5}$$
$$(2x+1) \times (x-5) = \frac{x+1}{x-5} \times (x-5)$$
$$(2x+1)(x-5) = x + 1$$
$$2x^2 - 9x - 5 = x + 1$$
$$2x^2 - 10x - 6 = 0$$
$$x^2 - 5x - 3 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)}$$
$$= \frac{5 \pm \sqrt{37}}{2}$$
$= \mathbf{5.54}$ or $\mathbf{-0.541}$ (to 3 s.f.)

(f)
$$\frac{5x}{x+4} = 4x + 1$$
$$\frac{5x}{x+4} \times (x+4) = (4x+1) \times (x+4)$$
$$5x = (4x+1)(x+4)$$
$$= 4x^2 + 17x + 4$$
$$0 = 4x^2 + 12x + 4$$
$$4x^2 + 12x + 4 = 0$$
$$x^2 + 3x + 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{5}}{2}$$
$= \mathbf{-0.382}$ or $\mathbf{-2.62}$ (to 3 s.f.)

4. **(a)**
$$x(x+1) = 1$$
$$x^2 + x = 1$$
$$x^2 + x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$
$= \mathbf{0.618}$ or $\mathbf{-1.62}$ (to 3 s.f.)

(b)
$$3(x+1)(x-1) = 7x$$
$$3(x^2 - 1) = 7x$$
$$3x^2 - 3 = 7x$$
$$3x^2 - 7x - 3 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-3)}}{2(3)}$$
$$= \frac{7 \pm \sqrt{85}}{6}$$
$= \mathbf{2.70}$ or $\mathbf{-0.370}$ (to 3 s.f.)

(c) $(x-1)^2 - 2x = 0$

$x^2 - 2x + 1 - 2x = 0$

$x^2 - 4x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$= \mathbf{3.73}$ or $\mathbf{0.268}$ (to 3 s.f.)

(d) $x(x-5) = 7 - 2x$

$x^2 - 5x = 7 - 2x$

$x^2 - 3x - 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-7)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{37}}{2}$$

$= \mathbf{4.54}$ or $\mathbf{-1.54}$ (to 3 s.f.)

(e) $(5x-9)(x-1) - x(x-2) = 0$

$5x^2 - 14x + 9 - (x^2 - 2x) = 0$

$5x^2 - 14x + 9 - x^2 + 2x = 0$

$4x^2 - 12x + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$

$$= \frac{12 \pm \sqrt{0}}{8}$$

$= \mathbf{1.5}$

(f) $(4x-3)^2 + (4x+3)^2 = 25$

$16x^2 - 24x + 9 + 16x^2 + 24x + 9 = 25$

$32x^2 + 18 = 25$

$32x^2 = 7$

$$x^2 = \frac{7}{32}$$

$$x = \pm\sqrt{\frac{7}{32}}$$

$= \mathbf{0.468}$ or

$\mathbf{-0.468}$ (to 3 s.f.)

5. **(a)**
$$\frac{x-1}{x+1} = \frac{8x}{1-x}$$

$$\frac{x-1}{x+1} \times (x+1)(1-x) = \frac{8x}{1-x} \times (x+1)(1-x)$$

$(x-1)(1-x) = 8x(x+1)$

$x - x^2 - 1 + x = 8x^2 + 8x$

$0 = 9x^2 + 6x + 1$

$9x^2 + 6x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{6^2 - 4(9)(1)}}{2(9)}$$

$$= \frac{-6 \pm \sqrt{0}}{18}$$

$$= \mathbf{-\frac{1}{3}}$$

(b)
$$\frac{(x-2)(x-3)}{(x-1)(x+2)} = \frac{2}{3}$$

$$\frac{(x-2)(x-3)}{(x-1)(x+2)} \times 3(x-1)(x+2) = \frac{2}{3} \times 3(x-1)(x+2)$$

$3(x-2)(x-3) = 2(x-1)(x+2)$

$3(x^2 - 5x + 6) = 2(x^2 + x - 2)$

$3x^2 - 15x + 18 = 2x^2 + 2x - 4$

$x^2 - 17x + 22 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-17) \pm \sqrt{(-17)^2 - 4(1)(22)}}{2(1)}$$

$$= \frac{17 \pm \sqrt{201}}{2}$$

$= \mathbf{15.6}$ or $\mathbf{1.41}$ (to 3 s.f.)

6.
$$\frac{x(x-3)}{(x+1)^2} = \frac{3}{5}$$

$$\frac{x(x-3)}{(x+1)^2} \times 5(x+1)^2 = \frac{3}{5} \times 5(x+1)^2$$

$5x(x-3) = 3(x+1)^2$

$5x^2 - 15x = 3(x^2 + 2x + 1)$

$= 3x^2 + 6x + 3$

$2x^2 - 21x - 3 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-21) \pm \sqrt{(-21)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{21 \pm \sqrt{465}}{4}$$

$= \mathbf{10.6}$ or $\mathbf{-0.141}$ (to 3 s.f.)

7. **(a)**
$$\frac{x}{2} = \frac{4}{x} - 1$$

$$\frac{x}{2} \times 2x = \left(\frac{4}{x} - 1\right) \times 2x$$

$x^2 = 8 - 2x$

$x^2 + 2x - 8 = 0$

$(x-2)(x+4) = 0$

$x - 2 = 0$ or $x + 4 = 0$

$x = \mathbf{2}$ or $x = \mathbf{-4}$

(b)
$$\frac{2}{x+5} = 1 - \frac{x+1}{5}$$

$$\frac{2}{x+5} \times 5(x+5) = \left(1 - \frac{x+1}{5}\right) \times 5(x+5)$$

$10 = 5(x+5) - (x+1)(x+5)$

$= 5x + 25 - (x^2 + 6x + 5)$

$= 5x + 25 - x^2 - 6x - 5$

$= -x^2 - x + 20$

$x^2 + x - 10 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2 - 4(1)(-10)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{41}}{2}$$

$= \mathbf{2.70}$ or $\mathbf{-3.70}$ (to 3 s.f.)

(c)
$$\frac{x-2}{5}+\frac{1}{2x-3}=2$$
$$\left(\frac{x-2}{5}+\frac{1}{2x-3}\right)\times 5(2x-3)=2)5\times 2x-3)$$
$$(x-2)(2x-3)+5=10(2x-3)$$
$$2x^2-7x+6+5=20x-30$$
$$2x^2-7x+11=20x-30$$
$$2x^2-27x+41=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-27)\pm\sqrt{(-27)^2-4(2)(41)}}{2(2)}$$
$$=\frac{27\pm\sqrt{401}}{4}$$
$= \mathbf{11.8}$ or $\mathbf{1.74}$ (to 3 s.f.)

(d)
$$\frac{4}{x}-\frac{1}{x-1}=9$$
$$\left(\frac{4}{x}-\frac{1}{x-1}\right)\times x(x-1)=9\times x(x-1)$$
$$4(x-1)-x=9x(x-1)$$
$$4x-4-x=9x^2-9x$$
$$3x-4=9x^2-9x$$
$$0=9x^2-12x+4$$
$$9x^2-12x+4=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-12)\pm\sqrt{(-12)^2-4(9)(4)}}{2(9)}$$
$$=\frac{12\pm 0}{18}$$
$$=\mathbf{\frac{2}{3}}$$

(e)
$$\frac{1}{x+2}+\frac{1}{x-2}=\frac{3}{11}$$
$$\left(\frac{1}{x+2}+\frac{1}{x-2}\right)\times 11(x+2)(x-2)=\frac{3}{11}\times 11(x+2)(x-2)$$
$$11(x-2)+11(x+2)=3(x+2)(x-2)$$
$$11x-22+11x+22=3(x^2-4)$$
$$22x=3x^2-12$$
$$0=3x^2-22x-12$$
$$3x^2-22x-12=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-22)\pm\sqrt{(-22)^2-4(3)(-12)}}{2(3)}$$
$$=\frac{22\pm\sqrt{628}}{6}$$
$= \mathbf{7.84}$ or $\mathbf{-0.510}$ (to 3 s.f.)

(f)
$$\frac{7}{x-1}-\frac{x+1}{x+3}=\frac{1}{2}$$
$$\left(\frac{7}{x-1}-\frac{x+1}{x+3}\right)\times 2(x-1)(x+3)=\frac{1}{2}\times 2(x-1)(x+3)$$
$$14(x+3)-2(x+1)(x-1)=(x-1)(x+3)$$
$$14x+42-2(x^2-1)=x^2+2x-3$$
$$14x+42-2x^2+2=x^2+2x-3$$
$$-2x^2+14x+44=x^2+2x-3$$
$$0=3x^2-12x-47$$
$$3x^2-12x-47=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-12)\pm\sqrt{(-12)^2-4(3)(-47)}}{2(3)}$$
$$=\frac{12\pm\sqrt{708}}{6}$$
$= \mathbf{6.43}$ or $\mathbf{-2.43}$ (to 3 s.f.)

(g)
$$\frac{5}{x-2}=2-\frac{4}{(x-2)^2}$$
$$\left(\frac{5}{x-2}\right)\times(x-2)^2=\left[2-\frac{4}{(x-2)^2}\right]\times(x-2)^2$$
$$5(x-2)=2(x-2)^2-4$$
$$5x-10=2(x^2-4x+4)-4$$
$$=2x^2-8x+8-4$$
$$=2x^2-8x+4$$
$$0=2x^2-13x+14$$
$$2x^2-13x+14=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-13)\pm\sqrt{(-13)^2-4(2)(14)}}{2(2)}$$
$$=\frac{13\pm\sqrt{57}}{4}$$
$= \mathbf{5.14}$ or $\mathbf{1.36}$ (to 3 s.f.)

(h)
$$\frac{5}{x-1}+\frac{x}{(x-1)^2}=1$$
$$\left[\frac{5}{x-1}+\frac{x}{(x-1)^2}\right]\times(x-1)^2=1\times(x-1)^2$$
$$5(x-1)+x=(x-1)^2$$
$$5x-5+x=x^2-2x+1$$
$$6x-5=x^2-2x+1$$
$$0=x^2-8x+6$$
$$x^2-8x+6=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(6)}}{2(1)}$$
$$=\frac{8\pm\sqrt{40}}{2}$$
$= \mathbf{7.16}$ or $\mathbf{0.838}$ (to 3 s.f.)

8. **(a)**
$$\frac{1}{x}+\frac{2}{x-1}+\frac{3}{x+1}=0$$
$$\left(\frac{1}{x}+\frac{2}{x-1}+\frac{3}{x+1}\right)\times x(x-1)(x+1)=0\times x(x-1)(x+1)$$
$$(x-1)(x+1)+2x(x+1)+3x(x-1)=0$$
$$x^2-1+2x^2+2x+3x^2-3x=0$$
$$6x^2-x-1=0$$
$$(2x-1)(3x+1)=0$$
$$2x-1=0 \quad\text{or}\quad 3x+1=0$$
$$2x=1 \quad\text{or}\quad 3x=-1$$
$$x=\mathbf{\frac{1}{2}} \quad\text{or}\quad x=\mathbf{-\frac{1}{2}}$$

(b)
$$\frac{1}{x^2-9}-\frac{2}{3-x}=1$$
$$\frac{1}{(x+3)(x-3)}+\frac{2}{x-3}=1$$
$$\left[\frac{1}{(x+3)(x-3)}+\frac{2}{x-3}\right]\times(x+3)(x-3)=1\times(x+3)(x-3)$$
$$1+2(x+3)=(x+3)(x-3)$$
$$1+2x+6=x^2-9$$
$$2x+7=x^2-9$$
$$0=x^2-2x-16$$
$$x^2-2x-16=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-16)}}{2(1)}$$
$$=\frac{2\pm\sqrt{68}}{2}$$
$= \mathbf{5.12}$ or $\mathbf{-3.12}$ (to 3 s.f.)

(c)
$$\frac{3}{x-3}+\frac{x+1}{x^2-5x+6}=1$$
$$\frac{3}{x-3}+\frac{x+1}{(x-3)(x-2)}=1$$
$$\left[\frac{3}{x-3}+\frac{x+1}{(x-3)(x-2)}\right]\times(x-3)(x-2)=1\times(x-3)(x-2)$$
$$3(x-2)+x+1=(x-3)(x-2)$$
$$3x-6+x+1=x^2-5x+6$$
$$4x-5=x^2-5x+6$$
$$0=x^2-9x+11$$
$$x^2-9x+11=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-9)\pm\sqrt{(-9)^2-4(1)(11)}}{2(1)}$$
$$=\frac{9\pm\sqrt{37}}{2}$$
$= \mathbf{7.54}$ or $\mathbf{1.46}$ (to 3 s.f.)

(d)
$$\frac{4}{x-1}+2=\frac{x+2}{2x^2+3x-5}$$
$$=\frac{x+2}{(2x+5)(x-1)}$$
$$\left(\frac{4}{x-1}+2\right)\times(2x+5)(x-1)=\frac{x+2}{(2x+5)(x-1)}\times(2x+5)(x-1)$$
$$4(2x+5)+2(2x+5)(x-1)=x+2$$
$$8x+20+2(2x^2+3x-5)=x+2$$
$$8x+20+4x^2+6x-10=x+2$$
$$4x^2+14x+10=x+2$$
$$4x^2+13x+8=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-13\pm\sqrt{13^2-4(4)(8)}}{2(4)}$$
$$=\frac{-13\pm\sqrt{41}}{8}$$
$= \mathbf{-0.825}$ or $\mathbf{-2.43}$ (to 3 s.f.)

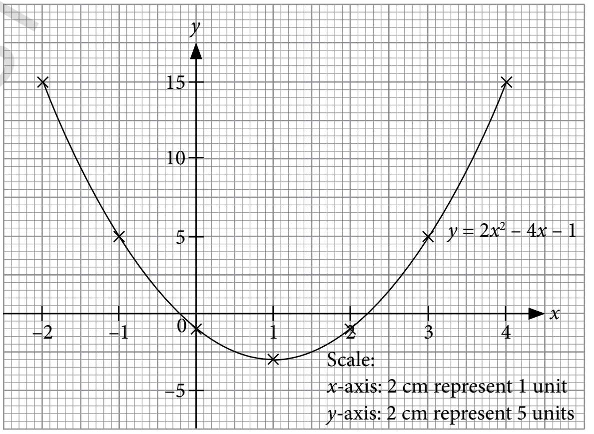
3.4 Solving quadratic equations by graphical method

Practise Now 9

1. **(i)** $y = 2x^2 - 4x - 1$

x	−2	−1	0	1	2	3	4
y	**15**	**5**	**−1**	**−3**	**−1**	**5**	**15**

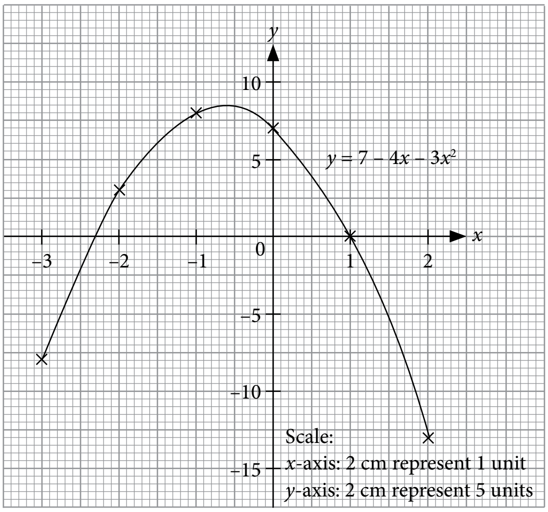
(ii)



(iii) From the graph, the x-coordinates of the points of intersection of $y = 2x^2 - 4x - 1$ and the x-axis are $x = -0.2$ and $x = 2.2$.
$\therefore$ the solutions of the equation $2x^2 - 4x - 1 = 0$ are $x = \mathbf{-0.2}$ and $x = \mathbf{2.2}$.

2. $y = 7 - 4x - 3x^2$

x	−3	−2	−1	0	1	2
y	−8	3	8	7	0	−13



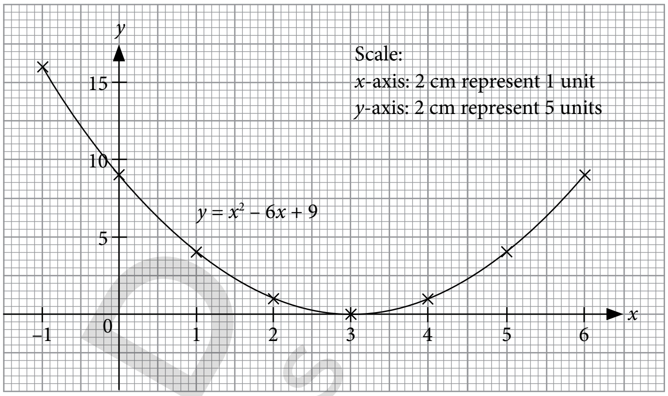
From the graph, the x-coordinates of the points of intersection of $y = 7 - 4x - 3x^2$ and the x-axis are $x = -2.3$ and $x = 1$.
$\therefore$ the solutions of the equation $7 - 4x - 3x^2 = 0$ are $x = \mathbf{-2.3}$ and $x = \mathbf{1}$.

Practise Now 10

1. **(i)** $y = x^2 - 6x + 9$

x	−1	0	1	2	3	4	5	6
y	**16**	**9**	**4**	**1**	**0**	**1**	**4**	**9**

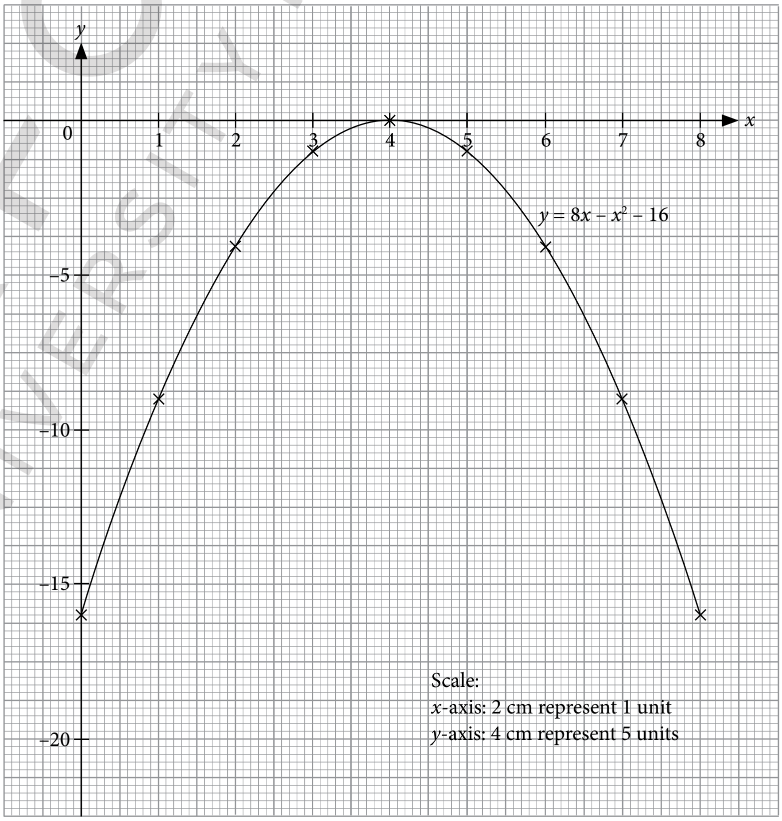
(ii)



(iii) From the graph, the x-coordinate of the point of intersection of $y = x^2 - 6x + 9$ and the x-axis is $x = 3$.
$\therefore$ the solution of the equation $x^2 - 6x + 9 = 0$ is $x = \mathbf{3}$.

2. $y = 8x - x^2 - 16$

x	0	1	2	3	4	5	6	7	8
y	−16	−9	−4	−1	0	−1	−4	−9	−16

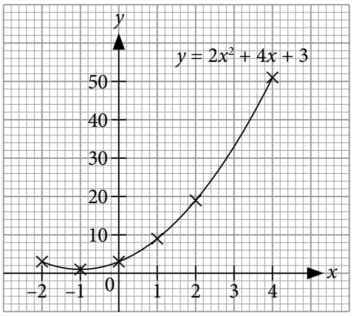


From the graph, the x-coordinate of the point of intersection of $y = 8x - x^2 - 16$ and the x-axis is $x = 4$.
$\therefore$ the solution of the equation $8x - x^2 - 16 = 0$ is $x = \mathbf{4}$.

Thinking Time (Page 60)

$y = 2x^2 + 4x + 3$

x	−2	−1	0	1	2	4
y	3	1	3	9	19	51



1. **(a)** From the graph, there are **no points of intersection** between the graph and the x-axis.
   (b) There are no real solutions to the equation $2x^2 + 4x + 3 = 0$. There are no points of intersection between the graph and the x-axis.
2. The number of real solutions of a quadratic equation is equal to the number of points of intersection of the graph of the corresponding quadratic function and the x-axis.

Journal Writing (Page 60)

(a) Factorisation
Advantage: Can be solved easily by factorising the quadratic expression.
Disadvantage: Can only be used when the quadratic expression can be factorised.

(b) Completing the square
Advantage: Methodical process that can solve all quadratic equations with real root(s).
Disadvantage: May involve computation of fractions or decimals, increasing the chances of making computational errors.

(c) Quadratic formula
Advantage: Methodical process that can solve all quadratic equations with real root(s).
Disadvantage: May involve computation of fractions or decimals, increasing the chances of making computational errors.

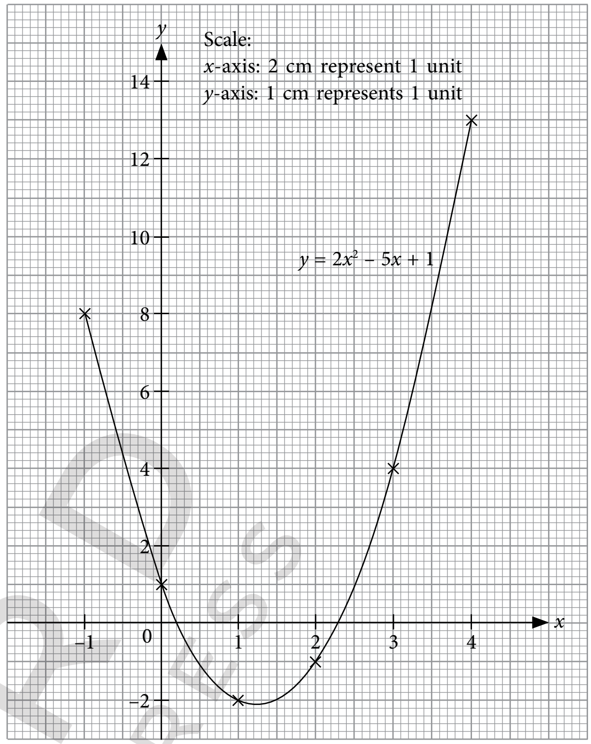
(d) Graphical method
Advantage: The answer can be observed from the graph, given that the graph has been drawn accurately.
Disadvantage: The exact value cannot be obtained and it is a more tedious method.

Exercise 3C

1. (i) $y = 2x^2 - 5x + 1$

x	−1	0	1	2	3	4
y	**8**	**1**	**−2**	**−1**	**4**	**13**

(ii)

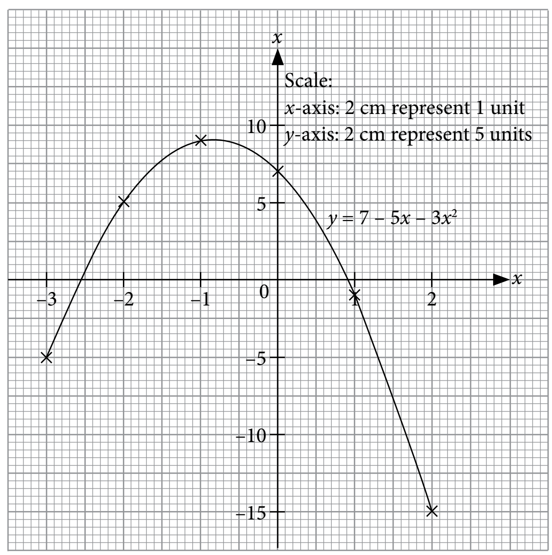


(iii) From the graph, the x-coordinates of the points of intersection of $y = 2x^2 - 5x + 1$ and the x-axis are $x = 2.3$ and $x = 0.2$.
$\therefore$ the solutions of the equation $2x^2 - 5x + 1 = 0$ are $x = \mathbf{2.3}$ and $x = \mathbf{0.2}$.

2. (i) $y = 7 - 5x - 3x^2$

x	−3	−2	−1	0	1	2
y	**−5**	**5**	**9**	**7**	**−1**	**−15**

(ii)

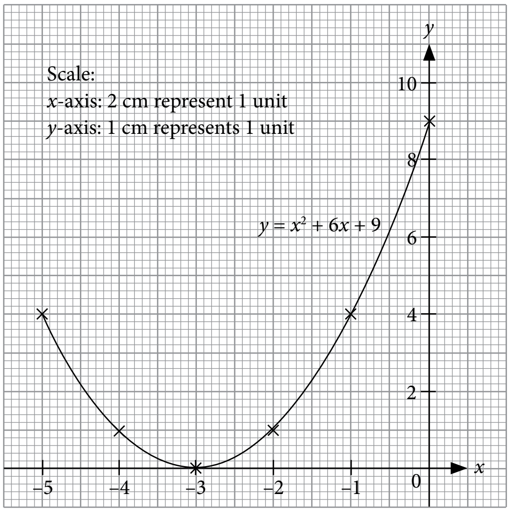


(iii) From the graph, the x-coordinates of the points of intersection of $y = 7 - 5x - 3x^2$ and the x-axis are $x = -2.55$ and $x = 0.9$.
$\therefore$ the solutions of the equation $7 - 5x - 3x^2 = 0$ are $x = \mathbf{-2.55}$ and $x = \mathbf{0.9}$.

3. **(i)** $y = x^2 + 6x + 9$

x	−5	−4	−3	−2	−1	0
y	**4**	**1**	**0**	**1**	**4**	**9**

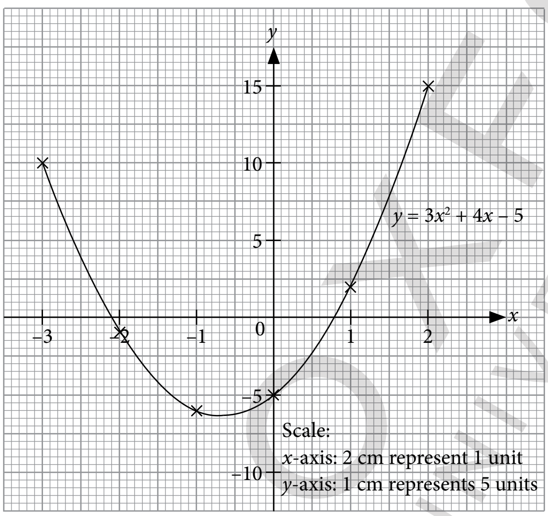
(ii)



(iii) From the graph, the x-coordinate of the point of intersection of $y = x^2 + 6x + 9$ and the x-axis is $x = -3$.
$\therefore$ the solution of the equation $x^2 + 6x + 9 = 0$ is $x = \mathbf{-3}$.

4. **(i)** $y = 3x^2 + 4x - 5$

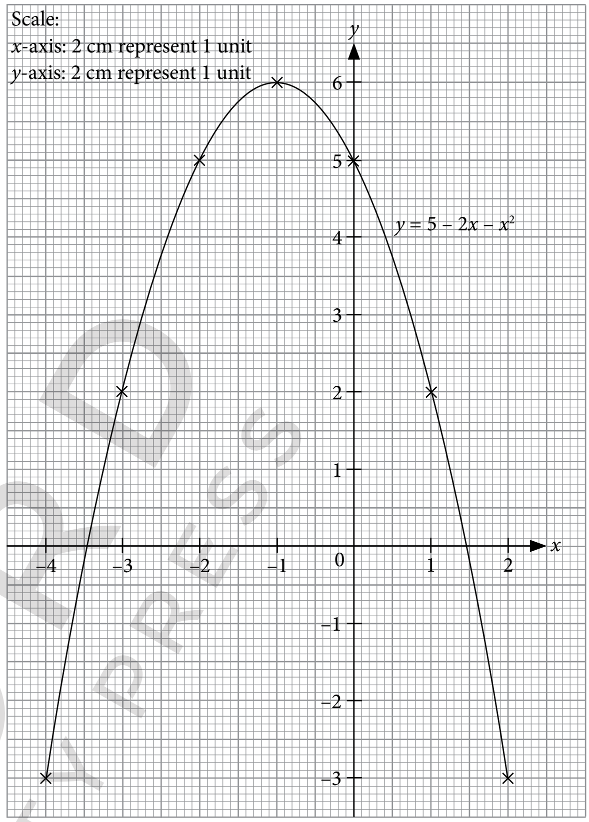
x	−3	−2	−1	0	1	2
y	10	−1	−6	−5	2	15



(ii) From the graph, the x-coordinates of the points of intersection of $y = 3x^2 + 4x - 5$ and the x-axis are $x = -2.1$ and $x = 0.8$.
$\therefore$ the solutions of the equation $3x^2 + 4x - 5 = 0$ are $x = \mathbf{-2.1}$ and $x = \mathbf{0.8}$.

5. $y = 5 - 2x - x^2$

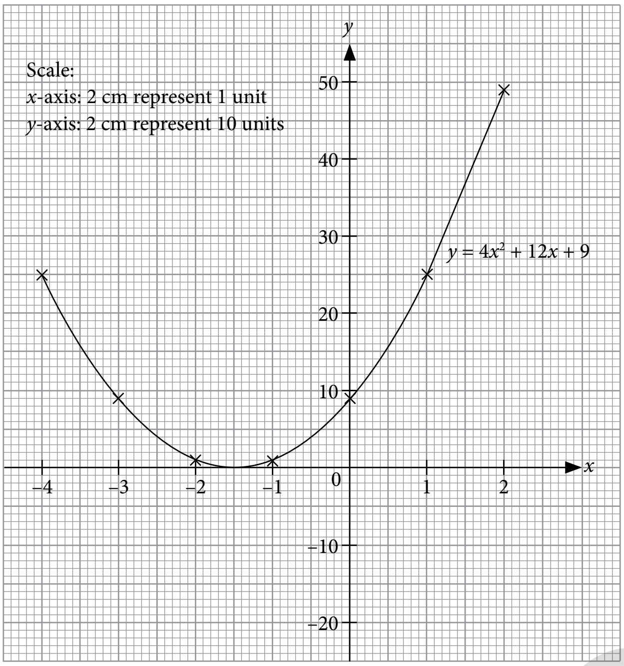
x	−4	−3	−2	−1	0	1	2
y	−3	2	5	6	5	2	−3



From the graph, the x-coordinates of the points of intersection of $y = 5 - 2x - x^2$ and the x-axis are $x = -3.45$ and $x = 1.45$.
$\therefore$ the solutions of the equation $5 - 2x - x^2$ are $x = \mathbf{-3.45}$ and $x = \mathbf{1.45}$.

6. **(i)** $y = 4x^2 + 12x + 9$

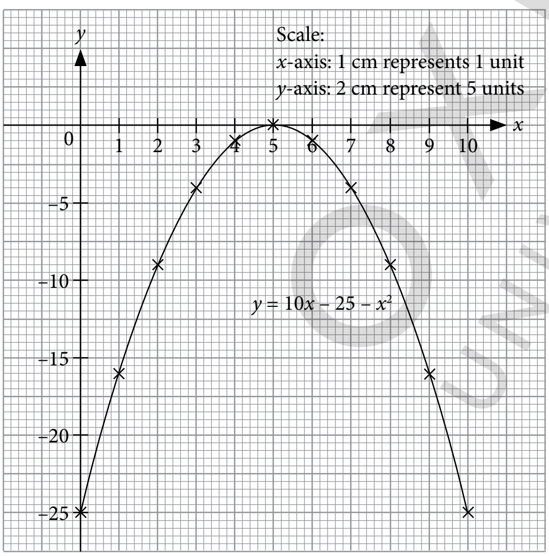
x	−4	−3	−2	−1	0	1	2
y	25	9	1	1	9	25	49



(ii) From the graph, the x-coordinate of the point of intersection of $y = 4x^2 + 12x + 9$ and the x-axis is $x = -1.5$.
$\therefore$ the solution of the equation $4x^2 + 12x + 9 = 0$ is $x = \mathbf{-1.5}$.

7. $y = 10x - 25 - x^2$

x	0	1	2	3	4	5	6	7	8	9	10
y	−25	−16	−9	−4	−1	0	−1	−4	−9	−16	−25

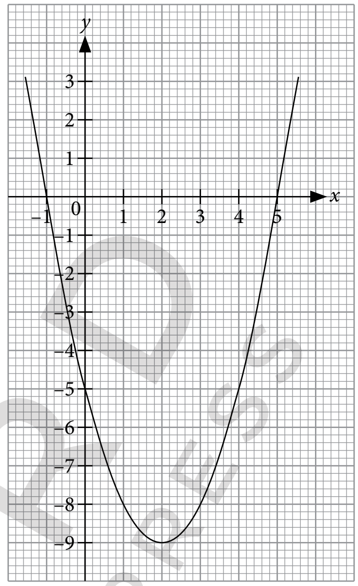


From the graph, the x-coordinate of the point of intersection of $y = 10x - 25 - x^2$ and the x-axis is $x = 5$.
$\therefore$ the solution of the equation $10x - 25 - x^2 = 0$ is $x = \mathbf{5}$.

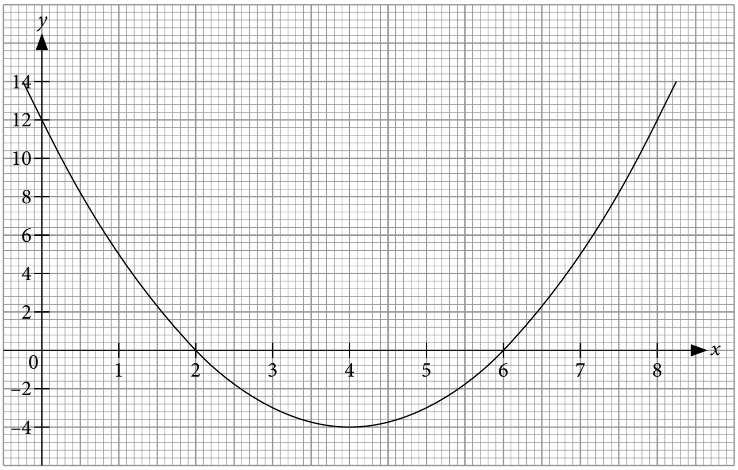
3.5 Sketching graphs of quadratic functions

Investigation (Characteristics of graphs of $y = \pm(x - h)(x - k)$)

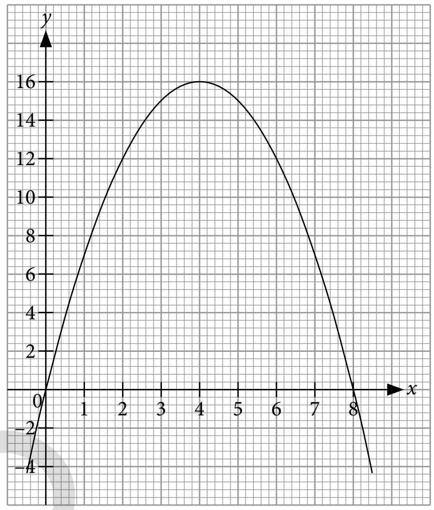
1. $y = (x + 1)(x - 5)$



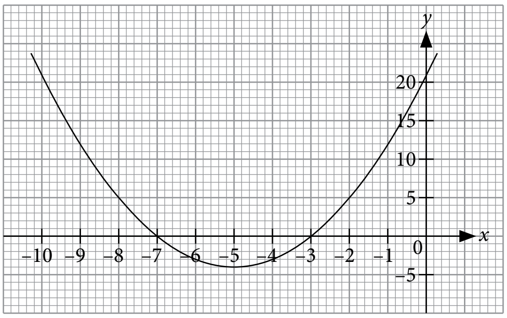
$y = (x - 2)(x - 6)$



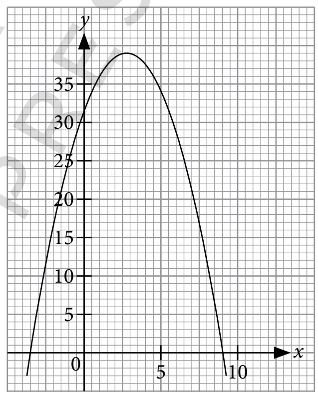
$y = x(8 - x)$



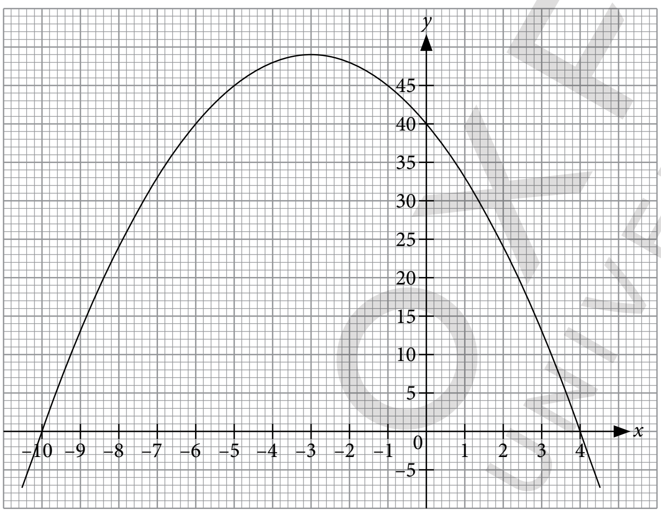
$y = (x + 7)(x + 3)$



$y = -\left(x + \frac{7}{2}\right)(x - 9)$



$y = -(x - 4)(x + 10)$



Function	h	k	x-intercepts	Average of x-intercepts	Equation of line of symmetry	Turning point (state maximum or minimum)
$y = (x + 1)(x - 5)$	-1	5	$-1, 5$	$\frac{-1+5}{2} = 2$	$x = 2$	Minimum, $(2, -9)$
$y = (x - 2)(x - 6)$	**2**	**6**	**2, 6**	$\frac{2+6}{2} = 4$	$x = 4$	**Minimum, (4, –4)**
$y = (x + 7)(x + 3)$	**–7**	**–3**	**–7, –3**	$\frac{-7+(-3)}{2} = -5$	$x = -5$	**Minimum, (–5, –4)**
$y = -(x - 4)(x + 10)$	**4**	**–10**	**4, –10**	$\frac{4+(-10)}{2} = -3$	$x = -3$	**Maximum, (–3, 49)**
$y = x(8 - x)$	**0**	**8**	**0, 8**	$\frac{0+8}{2} = 4$	$x = 4$	**Maximum, (4, 16)**
$y = -\left(x + \frac{7}{2}\right)(x - 9)$	**–3.5**	**9**	**–3.5, 9**	$\frac{-3.5+9}{2} = 2.75$	$x = 2.75$	**Maximum, (2.75, 39.0625)**

Table 3.3

2. **(a)** h and k are the x-intercepts.
(b) The equation of the line of symmetry, $x = a$, is such that a is the midpoint of the two x-intercepts.
(c) The equation of the line of symmetry, $x = a$, is such that a is the x-coordinate of the turning point.
(d) Substitute the x-coordinate into the equation of the function to find y.

3.

Function	h	k	x-intercepts	Equation of line of symmetry	Turning point (state maximum or minimum)
$y = (x - 1)(x - 9)$	**1**	**9**	**1, 9**	$x = 5$	**Minimum, (5, –16)**
$y = (x + 2)(x - 7)$	**–2**	**7**	**–2, 7**	$x = 2\frac{1}{2}$	**Minimum,** $\left(2\frac{1}{2}, -20\frac{1}{4}\right)$
$y = (3 - x)(x + 4)$	**3**	**–4**	**3, –4**	$x = -\frac{1}{2}$	**Maximum,** $\left(-\frac{1}{2}, 12\frac{1}{4}\right)$
$y = -x\left(x + \frac{5}{3}\right)$	**0**	$-1\frac{2}{3}$	$0, -1\frac{2}{3}$	$x = -\frac{5}{6}$	**Maximum,** $\left(-\frac{5}{6}, \frac{25}{36}\right)$

Table 3.4

Practise Now 11

(a) $y = (x - 2)(x - 6)$

Find y-intercept:

When $x = 0$, $y = (x - 2)(x - 6)$
$= (0 - 2)(0 - 6)$
$= 12$

Find x-intercept(s):

When $y = 0$,

$(x - 2)(x - 6) = 0$

$x - 2 = 0$ or $x - 6 = 0$

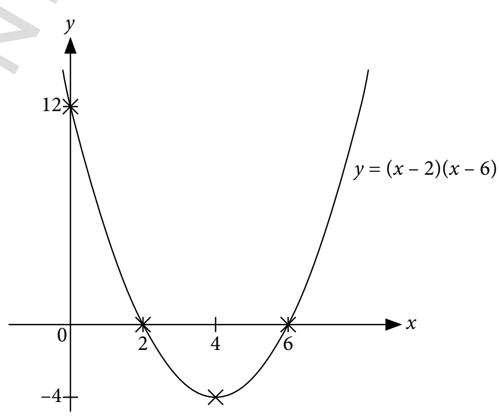
$x = 2$ or $x = 6$

Find turning point:

x-coordinate of turning point $= \frac{2+6}{2}$
$= 4$

y-coordinate of turning point $= (4 - 2)(4 - 6)$
$= -4$

Sketch:



(b) $y = (x + 4)(x + 7)$

Find y-intercept:

When $x = 0$, $y = (x + 4)(x + 7)$

$= (0 + 4)(0 + 7)$

$= 28$

Find x-intercept(s):

When $y = 0$,

$(x + 4)(x + 7) = 0$

$x + 4 = 0$ or $x + 7 = 0$

$x = -4$ or $x = -7$

Find turning point:

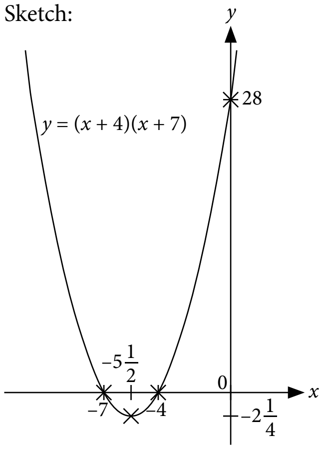
x-coordinate of turning point $= \dfrac{-4 + (-7)}{2}$

$= -5\frac{1}{2}$

y-coordinate of turning point $= \left(-5\frac{1}{2} + 4\right)\left(-5\frac{1}{2} + 7\right)$

$= -2\frac{1}{4}$

Sketch:



(c) $y = -x(x - 5)$

Find y-intercept:

When $x = 0$, $y = -x(x - 5)$

$= -0(0 - 5)$

$= 0$

Find x-intercept(s):

When $y = 0$, $-x(x - 5) = 0$

$x = 0$ or $x - 5 = 0$

$x = 5$

Find turning point:

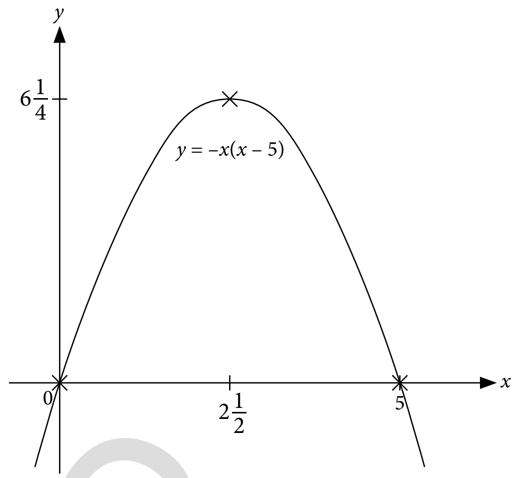
x-coordinate of turning point $= \dfrac{0 + 5}{2}$

$= 2\frac{1}{2}$

y-coordinate of turning point $= -2\frac{1}{2}\left(2\frac{1}{2} - 5\right)$

$= 6\frac{1}{4}$

Sketch:



(d) $y = (3 - x)(x + 8)$

Find y-intercept:

When $x = 0$, $y = (3 - x)(x + 8)$

$= (3 - 0)(0 + 8)$

$= 24$

Find x-intercept(s):

When $y = 0$,

$(3 - x)(x + 8) = 0$

$3 - x = 0$ or $x + 8 = 0$

$x = 3$ or $x = -8$

Find turning point:

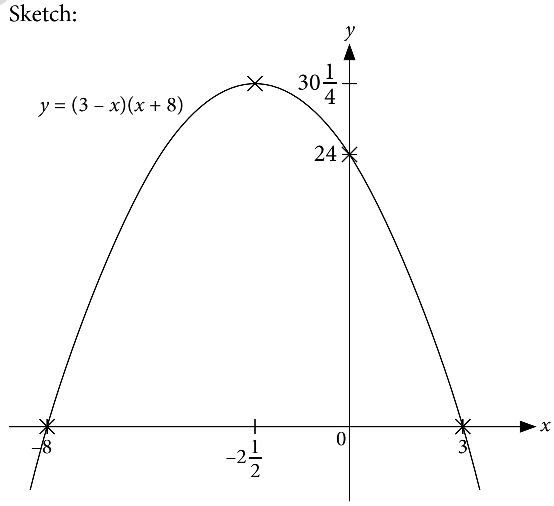
x-coordinate of turning point $= \dfrac{3 + (-8)}{2}$

$= -2\frac{1}{2}$

y-coordinate of turning point $= \left[3 - \left(-2\frac{1}{2}\right)\right]\left(-2\frac{1}{2} + 8\right)$

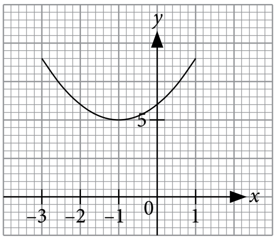
$= 30\frac{1}{4}$

Sketch:

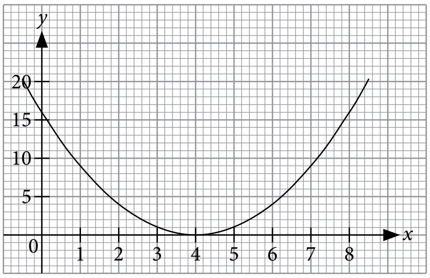


Investigation (Characteristics of graphs of $y = \pm(x - p)^2 + q$)

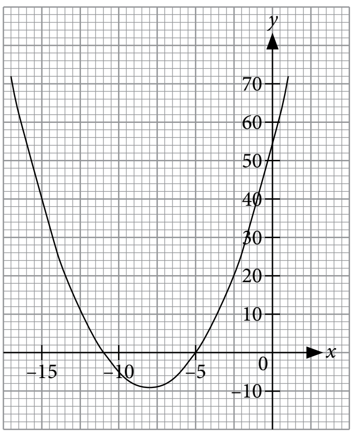
1. $y = (x + 1)^2 + 5$



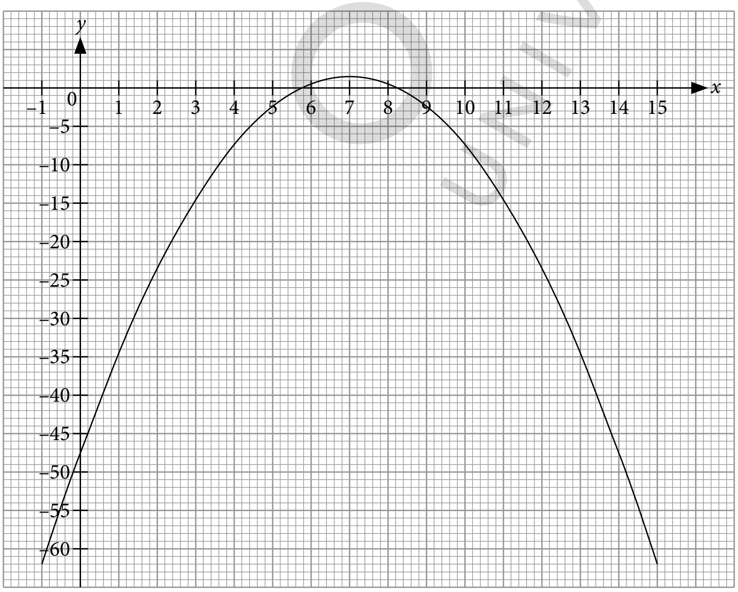
$y = (x - 4)^2$



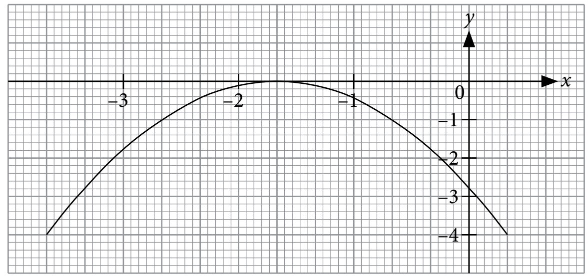
$y = (x + 8)^2 - 9$



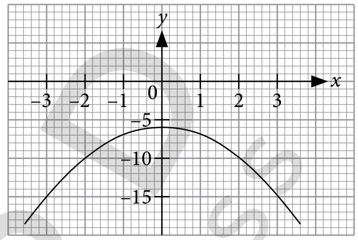
$y = -(x - 7)^2 + \frac{3}{2}$



$y = -\left(x + \frac{5}{3}\right)^2$



$y = -x^2 - 6$



Function	p	q	Turning point (state maximum or minimum)	Equation of line of symmetry	Number of x-intercepts
$y = (x + 1)^2 + 5$	–1	5	Minimum, (–1, 5)	$x = -1$	0
$y = (x - 4)^2$	**4**	**0**	**Minimum, (4, 0)**	$\boldsymbol{x = 4}$	**1**
$y = (x + 8)^2 - 9$	**–8**	**–9**	**Minimum, (–8, –9)**	$\boldsymbol{x = -8}$	**2**
$y = -(x - 7)^2 + \frac{3}{2}$	**7**	$\boldsymbol{1\frac{1}{2}}$	**Maximum,** $\boldsymbol{\left(7, 1\frac{1}{2}\right)}$	$\boldsymbol{x = 7}$	**2**
$y = -\left(x + \frac{5}{3}\right)^2$	$\boldsymbol{-1\frac{2}{3}}$	**0**	**Maximum,** $\boldsymbol{\left(-1\frac{2}{3}, 0\right)}$	$\boldsymbol{x = -1\frac{2}{3}}$	**1**
$y = -x^2 - 6$	**0**	**–6**	**Maximum, (0, –6)**	$\boldsymbol{x = 0}$	**0**

Table 3.5

2. **(a)** p and q are the respective x- and y-coordinate of the turning point.
(b) The equation of the line of symmetry, $x = a$, is such that a is the x-coordinate of the turning point.
(c) Since the curve has only one turning point, it will have only one x-intercept if the turning point lies on the x-axis, i.e. $q = 0$.
(d) If the coefficient of x^2 is positive, the curve will cut the x-axis at two points when q is less than 0, and will not cut the x-axis when q is greater than 0.
(e) If the coefficient of x^2 is negative, the curve will cut the x-axis at two points when q is greater than 0, and will not cut the x-axis when q is less than 0.
(f) The solution to the equation when $y = 0$ gives the x-intercept(s).
(g) Substitute the x-coordinate into the equation of the function to find y.

3.

Function	p	q	Turning point (state maximum or minimum)	Equation of line of symmetry	Number of x-intercepts
$y = (x-7)^2 + 2$	**7**	**2**	**Minimum, (7, 2)**	$\boldsymbol{x = 7}$	**0**
$y = x^2 - \frac{8}{3}$	**0**	$\boldsymbol{-2\frac{2}{3}}$	**Minimum,** $\boldsymbol{\left(0, -2\frac{2}{3}\right)}$	$\boldsymbol{x = 0}$	**2**
$y = -(x+14)^2 + 16$	**−14**	**16**	**Maximum, (−14, 16)**	$\boldsymbol{x = -14}$	**2**
$y = -\left(x - \frac{5}{2}\right)^2$	$\boldsymbol{2\frac{1}{2}}$	**0**	**Maximum,** $\boldsymbol{\left(2\frac{1}{2}, 0\right)}$	$\boldsymbol{x = 2\frac{1}{2}}$	**1**

Table 3.6

Practise Now 12

1. $y = (x-2)^2 - 9$

Find y-intercept:

When $x = 0$, $y = (x-2)^2 - 9$
$= (0-2)^2 - 9$
$= -5$

Find x-intercept(s):

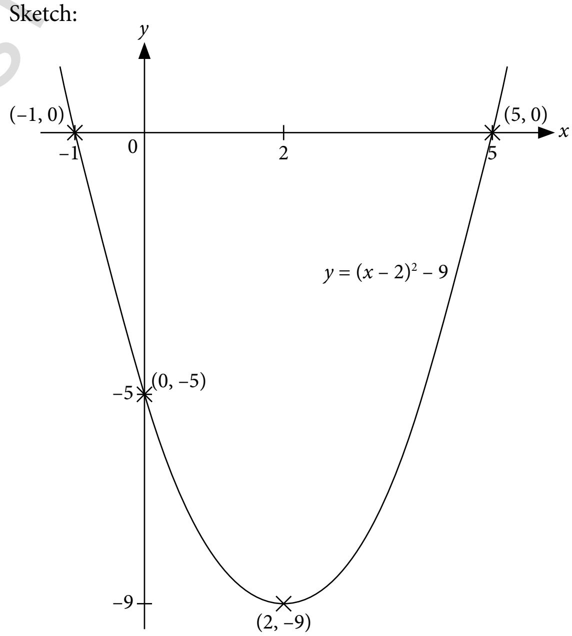
When $y = 0$, $(x-2)^2 - 9 = 0$
$(x-2)^2 = 9$
$x - 2 = \pm 3$
$x = 2 \pm 3$
$= -1$ or 5

Find turning point:

Comparing $y = (x-2)^2 - 9$ and $y = (x-p)^2 + q$, $p = 2$ and $q = -9$.

$\therefore$ coordinates of minimum point $= (2, -9)$

Sketch:



2. $y = -(x + 5)^2 + 1$

Find y-intercept:

When $x = 0$, $y = -(x + 5)^2 + 1$
$= -(0 + 5)^2 + 1$
$= -24$

Find x-intercept(s):

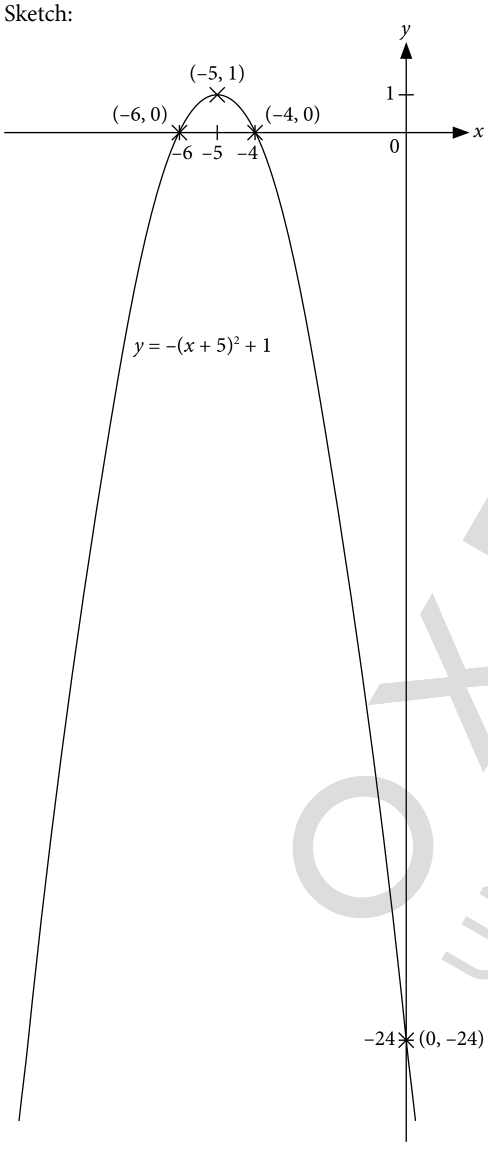
When $y = 0$, $-(x + 5)^2 + 1 = 0$
$(x + 5)^2 = 1$
$x + 5 = \pm 1$
$x = -5 \pm 1$
$= -6$ or -4

Find turning point:

Comparing $y = -(x + 5)^2 + 1$ and $y = -(x - p)^2 + q$, $p = -5$ and $q = 1$.

$\therefore$ coordinates of maximum point = $(-5, 1)$

Sketch:



Practise Now 13

1. **(i)** $y = -x^2 + 4x - 6$
$= -(x^2 - 4x) - 6$
$= -[x^2 - 4x + (-2)^2 - (-2)^2] - 6$
$= -(x - 2)^2 + 4 - 6$
$= \mathbf{-(x - 2)^2 - 2}$

(ii) Find y-intercept:

When $x = 0$, $y = -(x - 2)^2 - 2$
$= -(0 - 2)^2 - 2$
$= -6$

Find x-intercept(s):

When $y = 0$, $-(x - 2)^2 - 2 = 0$
$(x - 2)^2 = -2$

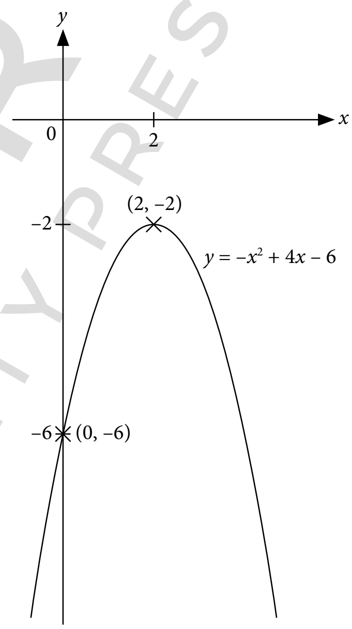
Since $(x - 2)^2 \geqslant 0$, there are no real solutions.

Find turning point:

Comparing $y = -(x - 2)^2 - 2$ and $y = -(x - p)^2 + q$, $p = 2$ and $q = -2$.

$\therefore$ coordinates of maximum point = $(2, -2)$

Sketch:



2. **(i)** $y = x^2 + x + 1$
$= x^2 + x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$
$= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$
$= \mathbf{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$

(ii) Find y-intercept:

When $x = 0$, $y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

$= \left(0 + \frac{1}{2}\right)^2 + \frac{3}{4}$

$= 1$

Find x-intercept(s):

When $y = 0$, $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$

$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}$

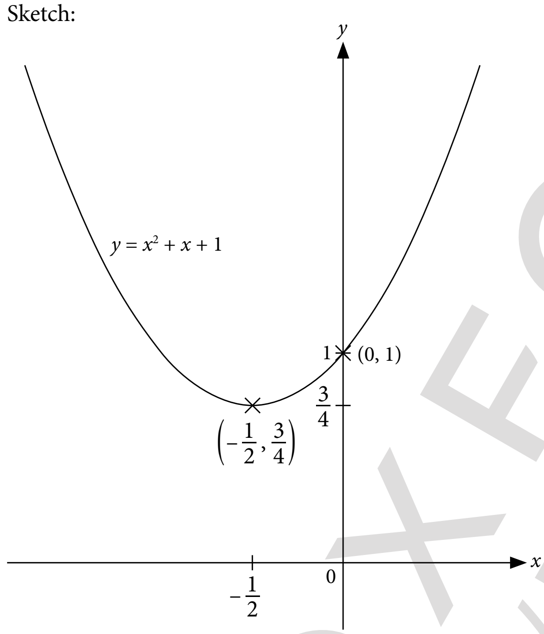
Since $\left(x + \frac{1}{2}\right)^2 \geqslant 0$, there are no real solutions.

Find turning point:

Comparing $y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ and $y = (x - p)^2 + q$, $p = -\frac{1}{2}$ and $q = \frac{3}{4}$.

$\therefore$ coordinates of minimum point $= \left(-\frac{1}{2}, \frac{3}{4}\right)$

Sketch:



3. **(i)** $y = -x^2 - 6x - 9$

$= -(x^2 + 6x) - 9$

$= -[x^2 + 6x + 3^2 - 3^2] - 9$

$= -(x + 3)^2 + 9 - 9$

$= \mathbf{-(x + 3)^2}$

(ii) Find y-intercept:

When $x = 0$, $y = -(x + 3)^2$

$= -(0 + 3)^2$

$= -9$

Find x-intercept(s):

When $y = 0$, $-(x + 3)^2 = 0$

$x + 3 = 0$

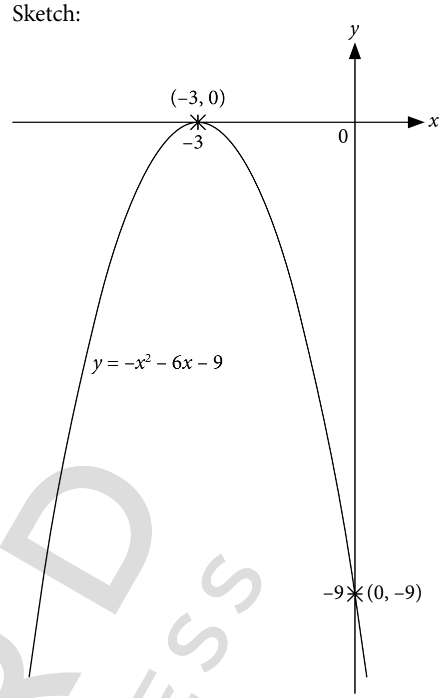
$x = -3$

Find turning point:

Comparing $y = -(x + 3)^2$ and $y = -(x - p)^2 + q$, $p = -3$ and $q = 0$.

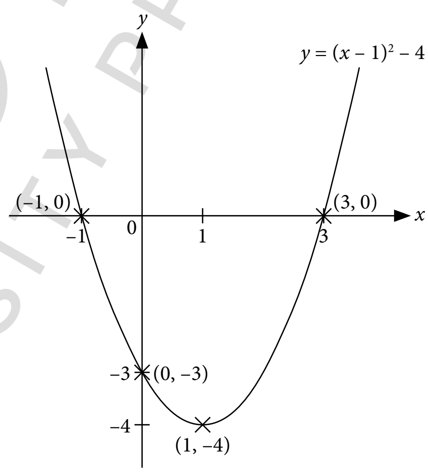
$\therefore$ coordinates of maximum point $= (-3, 0)$

Sketch:



Thinking Time (Page 69)

1.



(i) The graph cuts the x-axis **2** times.

(ii) **Yes**, $y = (x - 1)^2 - 4$ can be expressed in the form of $y = \pm(x - h)(x - k)$ since the graph cuts the x-axis.

There are two ways to do this. If the graph is not drawn, we can manipulate the expression algebraically by expanding $(x - 1)^2$ and then factorising the RHS.

$y = (x - 1)^2 - 4$

$= x^2 - 2x + 1 - 4$

$= x^2 - 2x - 3$

$= (x - 3)(x + 1)$

If the graph is already drawn as in Worked Example 12 and that the coefficient of x^2 is ±1, we can read off the x-intercepts to determine the values of h and k. Since the coefficient of x^2 is $+1$ in $y = (x - 1)^2 - 4$, then the coefficient of x^2 for $y = \pm(x - h)(x - k)$ must also be $+1$.

From the graph, the x-intercepts are -1 and 3, therefore $y = (x - 3)(x + 1)$.

2. **(i)** $y = (x - 3)(x - 3)$ resembles the form $y = \pm(x - h)(x - k)$, where h and k have the same value of 3. Since the two factors of the quadratic function are the same (the quadratic function cuts the x-axis once), the quadratic function can also be seen as belonging to the form of $y = \pm(x - p)^2 + q$, where $p = h = k = 3$, $q = 0$.

$\therefore$ $y = (x - 3)(x - 3)$ is in **both forms**.

(ii) When $y = 0$, $(x - 3)(x - 3) = 0$

$$(x - 3)^2 = 0$$
$$x - 3 = 0$$
$$x = 3$$

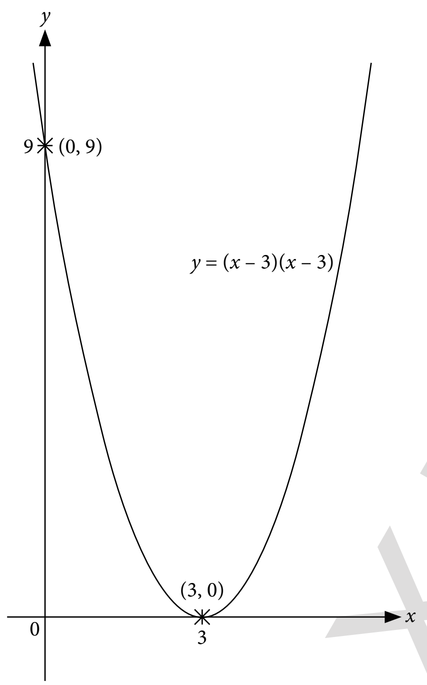
$\therefore$ the graph cuts the x-axis **1** time.

(iii) When $x = 0$, $y = (x - 3)^2$

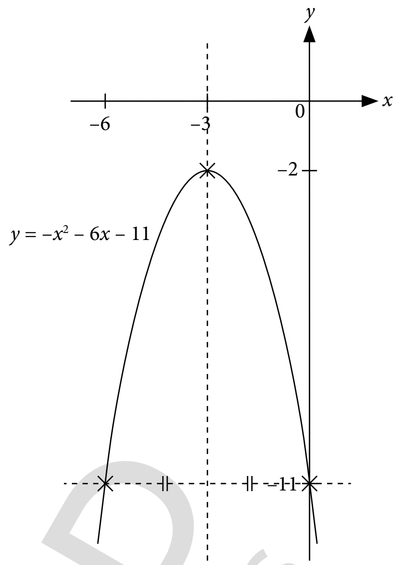
$$= (0 - 3)^2$$
$$= 9$$

Comparing $y = (x - 3)^2$ and $y = (x - p)^2 + q$, $p = 3$ and $q = 0$.

$\therefore$ coordinates of minimum point = (3, 0)



3.



(i) The graph **does not** cut the x-axis.

(ii) Recall that in the graph of $y = \pm(x - h)(x - k)$, h and k are the x-intercepts. Since the graph of $y = -x^2 - 6x - 11$ does not cut the x-axis, it **cannot** be expressed in the form of $\pm(x - h)(x - k)$.

4. For a quadratic function to be expressed in the factorised form, the function must cut the x-axis at least once and the x-intercept(s) of the function must be a rational number.

This is not the case for the completed-square form, since all quadratic functions can be expressed that way.

Hence, unless it is obvious that the quadratic expression can be factorised, it may be more straightforward to express the quadratic function in the completed-square form.

Class Discussion (Matching quadratic graphs with the corresponding functions)

For Graphs 1, 2, 4, 8 and 11, the graphs have a minimum point as well as x-intercept(s) and so, the function is of the form $y = (x - h)(x - k)$, where h and k are the x-intercepts.

For Graph 12, the graph has a minimum point with no x-intercept and so, the function is of the form $y = x^2 + bx + c$, where c is the y-intercept.

For Graphs 3, 5, 6, 7 and 9, the graphs have a maximum point as well as x-intercept(s) and so, the function is of the form $y = -(x - h)(x - k)$, where h and k are the x-intercepts.

For Graph 10, the graph has a maximum point with no x-intercept and so, the function is of the form $y = -x^2 + bx + c$, where c is the y-intercept.

Graph 1: $y = (x - 1)(x + 6) \rightarrow$ **C**

Graph 2: $y = (x - 1)(x - 6) = x^2 - 7x + 6 \rightarrow$ **E**

Graph 3: $y = -(x - 1)(x - 6) \rightarrow$ **D**

Graph 4: $y = (x + 1)(x - 6) = x^2 - 5x - 6 \rightarrow$ **G**

Graph 5: $y = -(x - 1)(x + 6) = -x^2 - 5x + 6 \rightarrow$ **F**

Graph 6: $y = -(x + 1)(x + 6) \rightarrow$ **A**

Graph 7: $y = -(x + 1)(x - 6) = -x^2 + 5x + 6 \rightarrow$ **H**

Graph 8: $y = (x + 1)(x + 6) \rightarrow$ **B**

Graph 9: $y = -(x - 1)^2 \rightarrow$ **L**

Graph 10: $y = -x^2 + 4x - 6 \rightarrow$ **J**

Graph 11: $y = (x + 1)^2 = x^2 + 2x + 1 \rightarrow$ **K**

Graph 12: $y = x^2 + 4x + 6 \rightarrow$ **I**

Exercise 3D

1. (a) $y = (x + 1)(x + 3)$

Find y-intercept:

When $x = 0$, $y = (x + 1)(x + 3)$
$= (0 + 1)(0 + 3)$
$= 3$

Find x-intercept(s):

When $y = 0$,

$(x + 1)(x + 3) = 0$

$x + 1 = 0$ or $x + 3 = 0$

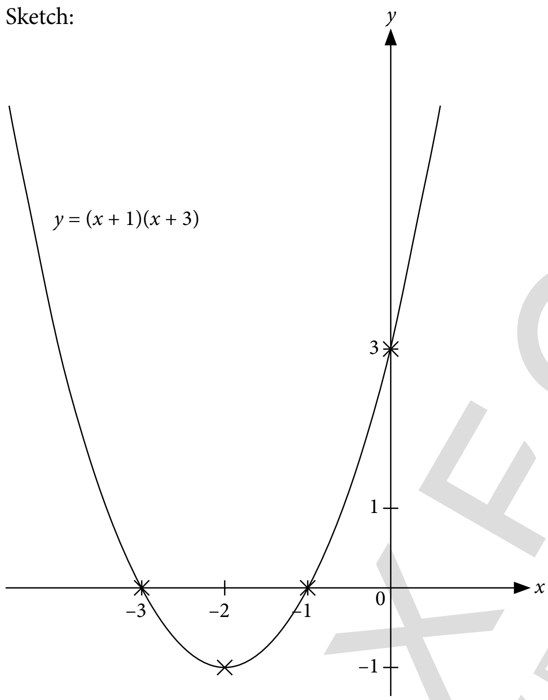
$x = -1$ or $x = -3$

Find turning point:

x-coordinate of turning point $= \dfrac{-1 + (-3)}{2}$
$= -2$

y-coordinate of turning point $= (-2 + 1)(-2 + 3)$
$= -1$

Sketch:



(b) $y = (x - 2)(x + 4)$

Find y-intercept:

When $x = 0$, $y = (x - 2)(x + 4)$
$= (0 - 2)(0 + 4)$
$= -8$

Find x-intercept(s):

When $y = 0$,

$(x - 2)(x + 4) = 0$

$x - 2 = 0$ or $x + 4 = 0$

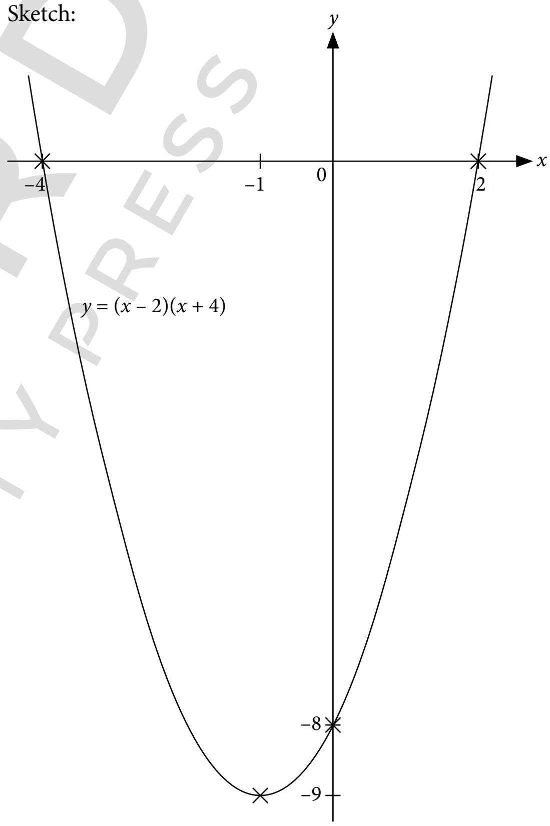
$x = 2$ or $x = -4$

Find turning point:

x-coordinate of turning point $= \dfrac{2 + (-4)}{2}$
$= -1$

y-coordinate of turning point $= (-1 - 2)(-1 + 4)$
$= -9$

Sketch:



(c) $y = -(x + 1)(x - 5)$

Find y-intercept:

When $x = 0$, $y = -(x + 1)(x - 5)$

$= -(0 + 1)(0 - 5)$

$= 5$

Find x-intercept(s):

When $y = 0$,

$-(x + 1)(x - 5) = 0$

$x + 1 = 0$ or $x - 5 = 0$

$x = -1$ or $x = 5$

Turning point:

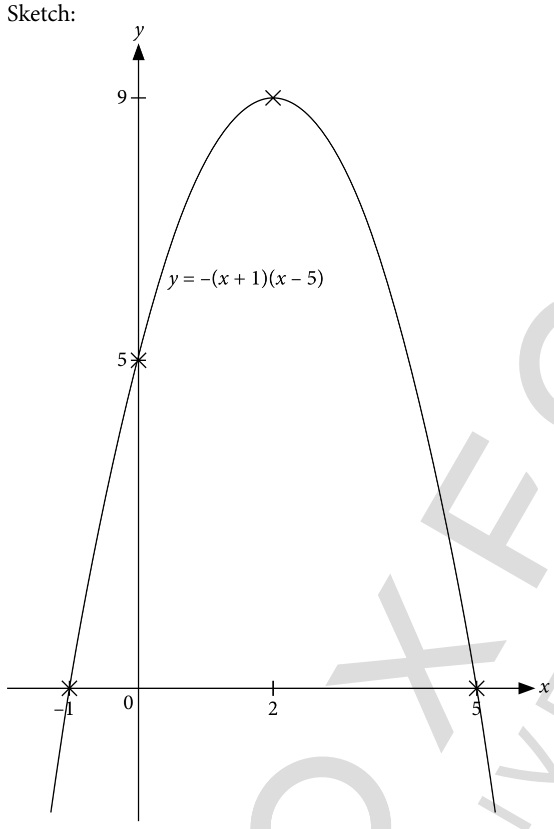
x-coordinate of turning point $= \dfrac{-1 + 5}{2}$

$= 2$

y-coordinate of turning point $= -(2 + 1)(2 - 5)$

$= 9$

Sketch:



(d) $y = -(x - 1)(x + 6)$

Find y-intercept:

When $x = 0$, $y = -(x - 1)(x + 6)$

$= -(0 - 1)(0 + 6)$

$= 6$

Find x-intercept(s):

When $y = 0$,

$-(x - 1)(x + 6) = 0$

$x - 1 = 0$ or $x + 6 = 0$

$x = 1$ or $x = -6$

Find turning point:

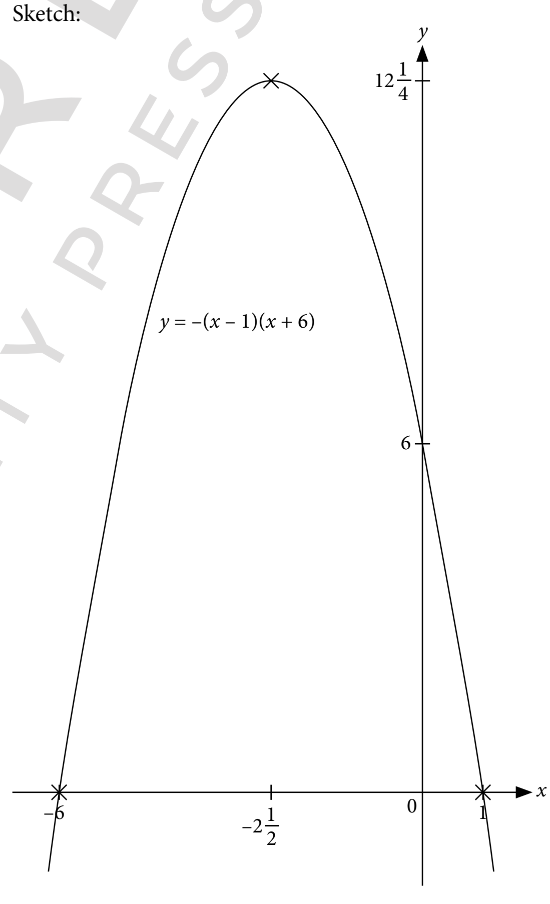
x-coordinate of turning point $= \dfrac{1 + (-6)}{2}$

$= -2\frac{1}{2}$

y-coordinate of turning point $= -\left(-2\frac{1}{2} - 1\right)\left(-2\frac{1}{2} + 6\right)$

$= 12\frac{1}{4}$

Sketch:



(e) $y = (3 - x)(x + 2)$

Find y-intercept:

When $x = 0$, $y = (3 - x)(x + 2)$

$= (3 - 0)(0 + 2)$

$= 6$

Find x-intercept(s):

When $y = 0$,

$(3 - x)(x + 2) = 0$

$3 - x = 0$ or $x + 2 = 0$

$x = 3$ or $x = -2$

Find turning point:

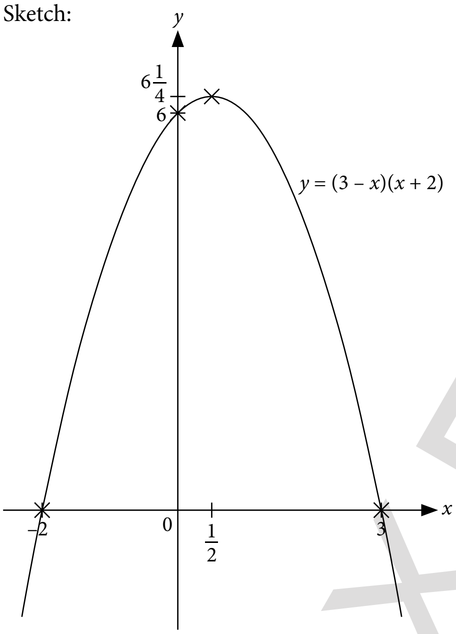
x-coordinate of turning point $= \dfrac{3 + (-2)}{2}$

$= \dfrac{1}{2}$

y-coordinate of turning point $= \left(3 - \dfrac{1}{2}\right)\left(\dfrac{1}{2} + 2\right)$

$= 6\dfrac{1}{4}$

Sketch:



(f) $y = (2 - x)(4 - x)$

Find y-intercept:

When $x = 0$, $y = (2 - x)(4 - x)$

$= (2 - 0)(4 - 0)$

$= 8$

Find x-intercept(s):

When $y = 0$,

$(2 - x)(4 - x) = 0$

$2 - x = 0$ or $4 - x = 0$

$x = 2$ or $x = 4$

Find turning point:

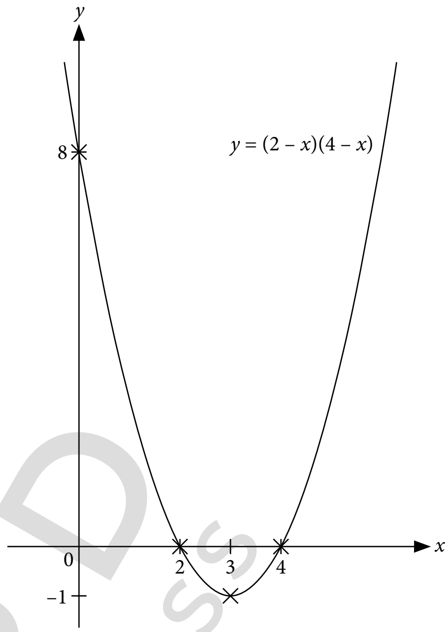
x-coordinate of turning point $= \dfrac{2 + 4}{2}$

$= 3$

y-coordinate of turning point $= (2 - 3)(4 - 3)$

$= -1$

Sketch:



2. **(a)** $y = x^2 - 4$

Find y-intercept:

When $x = 0$, $y = x^2 - 4$

$= 0^2 - 4$

$= -4$

Find x-intercept(s):

When $y = 0$, $x^2 - 4 = 0$

$x^2 = 4$

$x = -2$ or 2

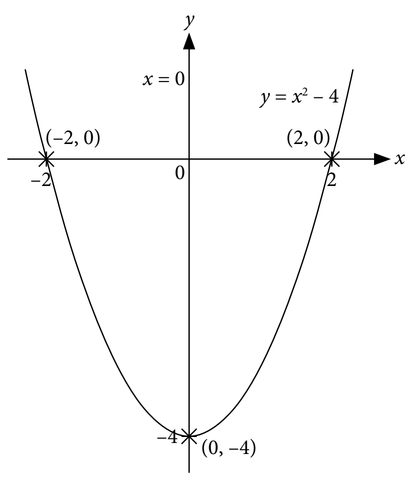
Find turning point:

Comparing $y = x^2 - 4$ and $y = (x - p)^2 + q$, $p = 0$ and $q = -4$.

$\therefore$ coordinates of minimum point $= (0, -4)$

The equation of the line of symmetry is $x = 0$.

Sketch:



(b) $y = -x^2 + 6$

Find y-intercept:

When $x = 0$, $y = -x^2 + 6$
$= -0^2 + 6$
$= 6$

Find x-intercept(s):

When $y = 0$, $-x^2 + 6 = 0$
$x^2 = 6$
$x = \pm\sqrt{6}$
$x = -2.45$ or 2.45 (to 3 s.f.)

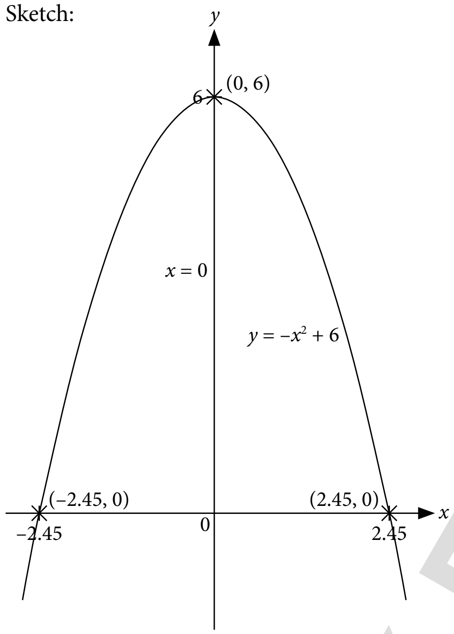
Find turning point:

Comparing $y = -x^2 + 6$ and $y = -(x - p)^2 + q$, $p = 0$ and $q = 6$.

$\therefore$ coordinates of maximum point = (0, 6)

The equation of the line of symmetry is $x = 0$.

Sketch:



(c) $y = (x - 3)^2 - 2$

Find y-intercept:

When $x = 0$, $y = (x - 3)^2 - 2$
$= (0 - 3)^2 - 2$
$= 7$

Find x-intercept(s):

When $y = 0$, $(x - 3)^2 - 2 = 0$
$(x - 3)^2 = 2$
$x - 3 = \pm\sqrt{2}$
$x = 3 \pm \sqrt{2}$
$x = 1.59$ or 4.41 (to 3 s.f.)

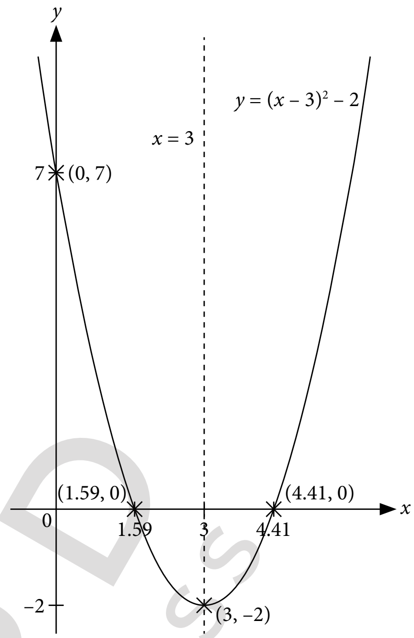
Find turning point:

Comparing $y = (x - 3)^2 - 2$ and $y = (x - p)^2 + q$, $p = 3$ and $q = -2$.

$\therefore$ coordinates of minimum point = (3, –2)

The equation of the line of symmetry is $x = 3$.

Sketch:



(d) $y = (x + 1)^2 - 3$

Find y-intercept:

When $x = 0$, $y = (x + 1)^2 - 3$
$= (0 + 1)^2 - 3$
$= -2$

Find x-intercept(s):

When $y = 0$, $(x + 1)^2 - 3 = 0$
$(x + 1)^2 = 3$
$x + 1 = \pm\sqrt{3}$
$x = -1 \pm \sqrt{3}$
$x = -2.73$ or 0.732 (to 3 s.f.)

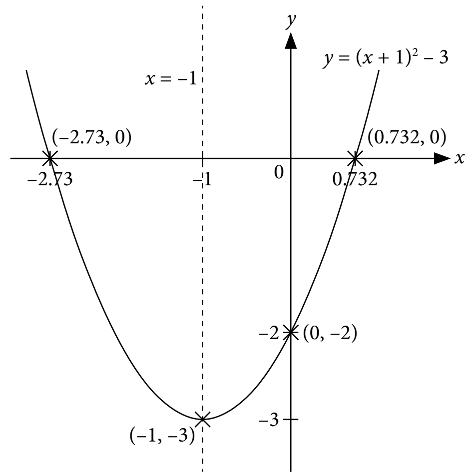
Find turning point:

Comparing $y = (x + 1)^2 - 3$ and $y = (x - p)^2 + q$, $p = -1$ and $q = -3$.

$\therefore$ coordinates of minimum point = (–1, –3)

The equation of the line of symmetry is $x = -1$.

Sketch:



(e) $y = -(x + 2)^2 + 3$

Find y-intercept:

When $x = 0$, $y = -(x + 2)^2 + 3$

$= -(0 + 2)^2 + 3$

$= -1$

Find x-intercept(s):

When $y = 0$, $-(x + 2)^2 + 3 = 0$

$(x + 2)^2 = 3$

$x + 2 = \pm\sqrt{3}$

$x = -2 \pm \sqrt{3}$

$x = -3.73$ or -0.268 (to 3 s.f.)

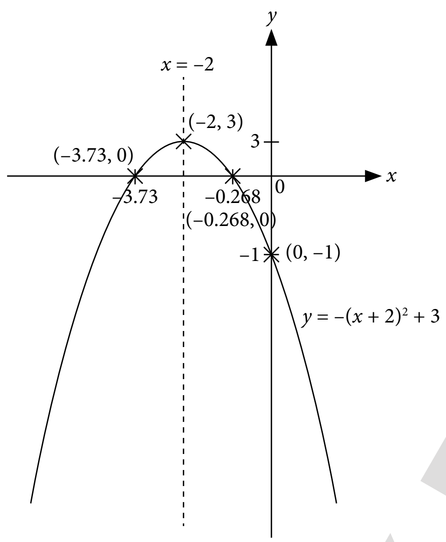
Find turning point:

Comparing $y = -(x + 2)^2 + 3$ and $y = -(x - p)^2 + q$, $p = -2$ and $q = 3$.

$\therefore$ coordinates of maximum point $= (-2, 3)$

The equation of the line of symmetry is $x = -2$.

Sketch:



(f) $y = -(x - 4)^2 + 5$

Find y-intercept:

When $x = 0$, $y = -(x - 4)^2 + 5$

$= -(0 - 4)^2 + 5$

$= -11$

Find x-intercept(s):

When $y = 0$, $-(x - 4)^2 + 5 = 0$

$(x - 4)^2 = 5$

$x - 4 = \pm\sqrt{5}$

$x = 4 \pm \sqrt{5}$

$x = 1.76$ or 6.24 (to 3 s.f.)

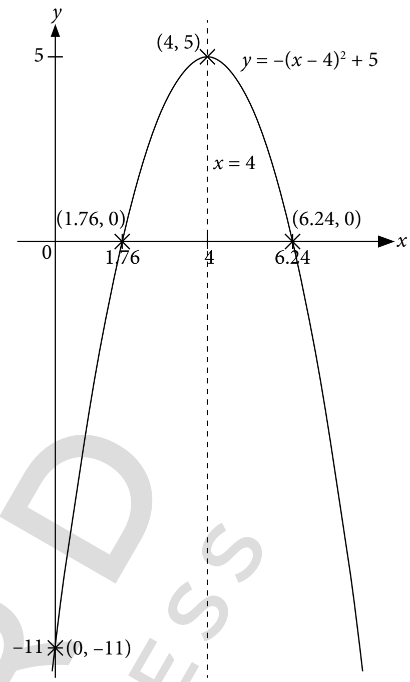
Find turning point:

Comparing $y = -(x - 4)^2 + 5$ and $y = -(x - p)^2 + q$, $p = 4$ and $q = 5$.

$\therefore$ coordinates of maximum point $= (4, 5)$

The equation of the line of symmetry is $x = 4$.

Sketch:



3. **(i)** $x^2 + \frac{3}{4}x = \boldsymbol{x\left(x + \frac{3}{4}\right)}$

(ii) Find y-intercept:

When $x = 0$, $y = x\left(x + \frac{3}{4}\right)$

$= 0\left(0 + \frac{3}{4}\right)$

$= 0$

Find x-intercept(s):

When $y = 0$,

$x\left(x + \frac{3}{4}\right) = 0$

$x = 0$ or $x + \frac{3}{4} = 0$

$x = -\frac{3}{4}$

Find turning point:

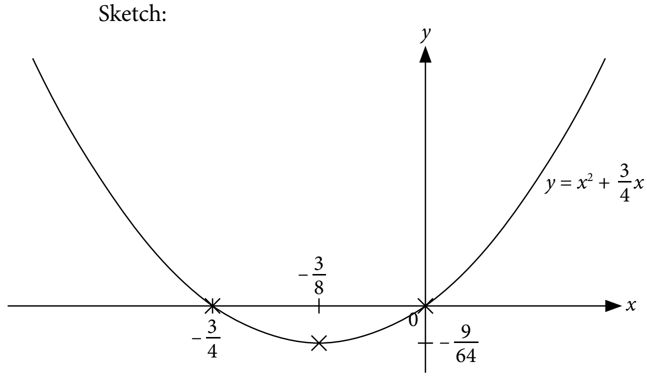
x-coordinate of turning point $= \dfrac{0 + \left(-\frac{3}{4}\right)}{2}$

$= -\frac{3}{8}$

y-coordinate of turning point $= -\frac{3}{8}\left(-\frac{3}{8} + \frac{3}{4}\right)$

$= -\frac{9}{64}$

Sketch:



4. $y = -(x^2 - x)$

$= -x(x - 1)$

Find y-intercept:

When $x = 0$, $y = -x(x - 1)$

$= -0(0 - 1)$

$= 0$

Find x-intercept(s):

When $y = 0$,

$-x(x - 1) = 0$

$x = 0$ or $x - 1 = 0$

$x = 1$

Find turning point:

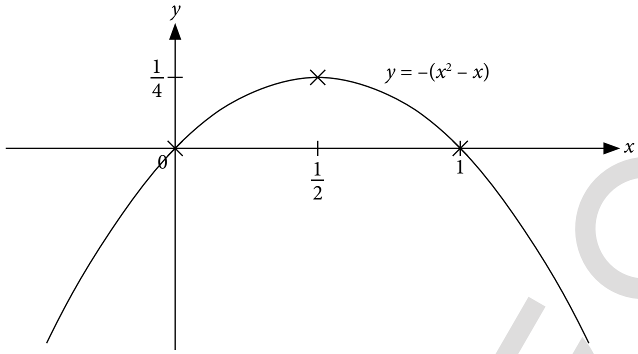
x-coordinate of turning point $= \frac{0 + 1}{2}$

$= \frac{1}{2}$

y-coordinate of turning point $= -\frac{1}{2}\left(\frac{1}{2} - 1\right)$

$= \frac{1}{4}$

Sketch:



5. **(i)** $x^2 + x - 6 = \mathbf{(x + 3)(x - 2)}$

(ii) Find y-intercept:

When $x = 0$, $y = (x + 3)(x - 2)$

$= (0 + 3)(0 - 2)$

$= -6$

Find x-intercept(s):

When $y = 0$,

$(x + 3)(x - 2) = 0$

$x + 3 = 0$ or $x - 2 = 0$

$x = -3$ or $x = 2$

Find turning point:

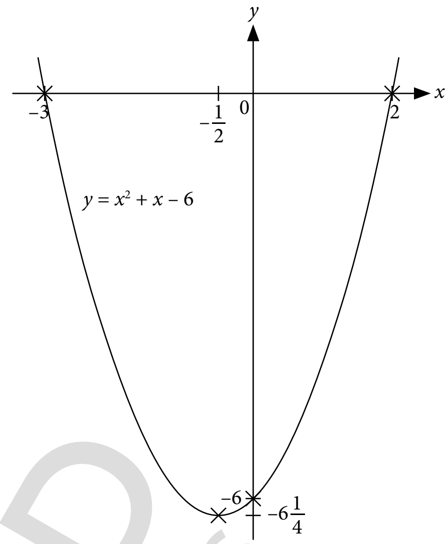
x-coordinate of turning point $= \frac{-3 + 2}{2}$

$= -\frac{1}{2}$

y-coordinate of turning point $= \left(-\frac{1}{2} + 3\right)\left(-\frac{1}{2} - 2\right)$

$= -6\frac{1}{4}$

Sketch:



6. $y = x^2 - 4x + 3$

$= (x - 3)(x - 1)$

Find y-intercept:

When $x = 0$, $y = (x - 3)(x - 1)$

$= (0 - 3)(0 - 1)$

$= 3$

Find x-intercept(s):

When $y = 0$,

$(x - 3)(x - 1) = 0$

$x - 3 = 0$ or $x - 1 = 0$

$x = 3$ or $x = 1$

Find turning point:

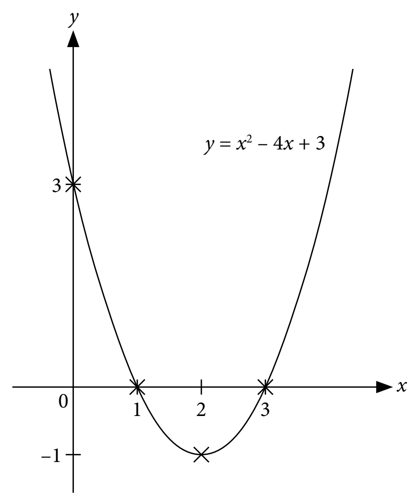
x-coordinate of turning point $= \frac{3 + 1}{2}$

$= 2$

y-coordinate of turning point $= (2 - 3)(2 - 1)$

$= -1$

Sketch:



7. **(a)** **(i)** $y = -x^2 - 7x - 15$

$= -(x^2 + 7x) - 15$

$= -\left[x^2 + 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2\right] - 15$

$= -\left(x + \frac{7}{2}\right)^2 + \frac{49}{4} - 15$

$= -\left(\boldsymbol{x} + \mathbf{\frac{7}{2}}\right)^2 - \mathbf{\frac{11}{4}}$

(ii) Find y-intercept:

When $x = 0$, $y = -\left(x + \frac{7}{2}\right)^2 - \frac{11}{4}$

$= -\left(0 + \frac{7}{2}\right)^2 - \frac{11}{4}$

$= -15$

Find x-intercept(s):

When $y = 0$, $-\left(x + \frac{7}{2}\right)^2 - \frac{11}{4} = 0$

$\left(x + \frac{7}{2}\right)^2 = -\frac{11}{4}$

Since $\left(x + \frac{7}{2}\right)^2 \geqslant 0$, there are no real solutions.

Find turning point:

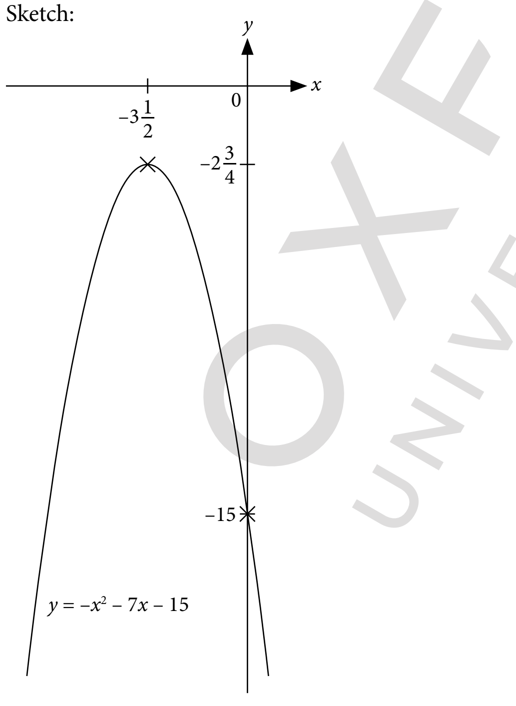
Comparing $y = -\left(x + \frac{7}{2}\right)^2 - \frac{11}{4}$ and $y = -(x - p)^2 + q$,

$p = -\frac{7}{2} = -3\frac{1}{2}$ and

$q = -\frac{11}{4} = -2\frac{3}{4}$.

$\therefore$ coordinates of maximum point $= \left(-3\frac{1}{2}, -2\frac{3}{4}\right)$

Sketch:



(b) **(i)** $y = x^2 - 3x + 4$

$= x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 + 4$

$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 4$

$= \left(\boldsymbol{x} - \mathbf{\frac{3}{2}}\right)^2 + \mathbf{\frac{7}{4}}$

(ii) Find y-intercept:

When $x = 0$, $y = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$

$= \left(0 - \frac{3}{2}\right)^2 + \frac{7}{4}$

$= 4$

Find x-intercept(s):

When $y = 0$, $\left(x - \frac{3}{2}\right)^2 + \frac{7}{4} = 0$

$\left(x - \frac{3}{2}\right)^2 = -\frac{7}{4}$

Since $\left(x - \frac{3}{2}\right)^2 \geqslant 0$, there are no real solutions.

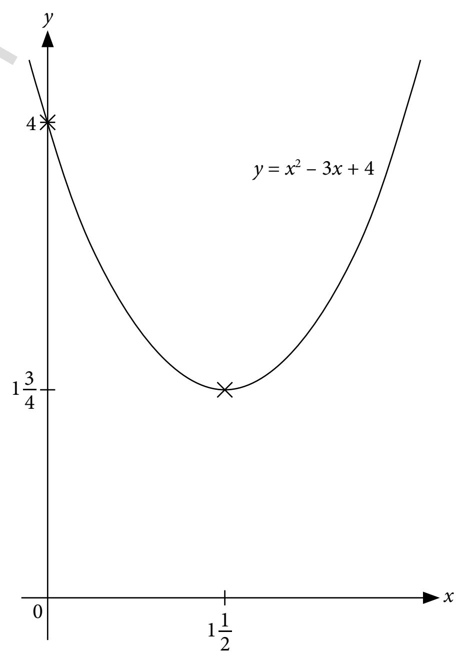
Find turning point:

Comparing $y = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$ and $y = (x - p)^2 + q$,

$p = \frac{3}{2} = 1\frac{1}{2}$ and $q = \frac{7}{4} = 1\frac{3}{4}$.

$\therefore$ coordinates of minimum point $= \left(1\frac{1}{2}, 1\frac{3}{4}\right)$

Sketch:



8. **(a)** **(i)** $y = -x^2 - 10x - 25$

$= -(x^2 + 10x) - 25$

$= -(x^2 + 10x + 5^2 - 5^2) - 25$

$= -(x + 5)^2 + 25 - 25$

$= \mathbf{-(x + 5)^2}$

(ii) Find y-intercept:

When $x = 0$, $y = -(x + 5)^2$

$= -(0 + 5)^2$

$= -25$

Find x-intercept(s):

When $y = 0$, $-(x + 5)^2 = 0$

$x + 5 = 0$

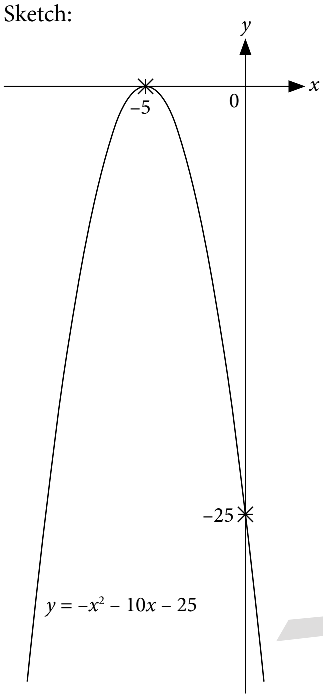
$x = -5$

Find turning point:

Comparing $y = -(x + 5)^2$ and $y = -(x - p)^2 + q$, $p = -5$ and $q = 0$.

$\therefore$ coordinates of maximum point = $(-5, 0)$

Sketch:



(b) **(i)** $y = x^2 - 8x + 16$

$= x^2 - 8x + (-4)^2 - (-4)^2 + 16$

$= (x - 4)^2 - 16 + 16$

$= \mathbf{(x - 4)^2}$

(ii) Find y-intercept:

When $x = 0$, $y = (x - 4)^2$

$= (0 - 4)^2$

$= 16$

Find x-intercept(s):

When $y = 0$, $(x - 4)^2 = 0$

$x - 4 = 0$

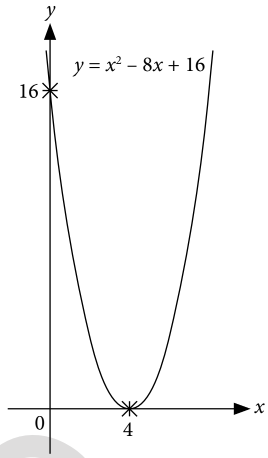
$x = 4$

Find turning point:

Comparing $y = (x - 4)^2$ and $y = (x - p)^2 + q$, $p = 4$ and $q = 0$.

$\therefore$ coordinates of minimum point = $(4, 0)$

Sketch:



9. **(a)** **(i)** $y = -x^2 + 6x - 6$

$= -(x^2 - 6x) - 6$

$= -[x^2 - 6x + (-3)^2 - (-3)^2] - 6$

$= -(x - 3)^2 + 9 - 6$

$= \mathbf{-(x - 3)^2 + 3}$

(ii) Find y-intercept:

When $x = 0$, $y = -(x - 3)^2 + 3$

$= -(0 - 3)^2 + 3$

$= -6$

Find x-intercept(s):

When $y = 0$, $-(x - 3)^2 + 3 = 0$

$(x - 3)^2 = 3$

$x - 3 = \pm\sqrt{3}$

$x = 3 \pm \sqrt{3}$

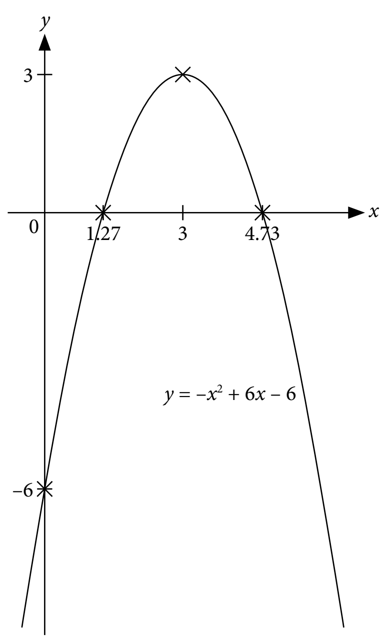
$x = 1.27$ or 4.73 (to 3 s.f.)

Find turning point:

Comparing $y = -(x - 3)^2 + 3$ and $y = -(x - p)^2 + q$, $p = 3$ and $q = 3$.

$\therefore$ coordinates of maximum point = $(3, 3)$

Sketch:



(b) (i) $y = x^2 - 8x + 5$

$= x^2 - 8x + (-4)^2 - (-4)^2 + 5$

$= (x - 4)^2 - 16 + 5$

$= \mathbf{(x - 4)^2 - 11}$

(ii) Find y-intercept:

When $x = 0$, $y = (x - 4)^2 - 11$

$= (0 - 4)^2 - 11$

$= 5$

Find x-intercept(s):

When $y = 0$, $(x - 4)^2 - 11 = 0$

$(x - 4)^2 = 11$

$x - 4 = \pm\sqrt{11}$

$x = 4 \pm \sqrt{11}$

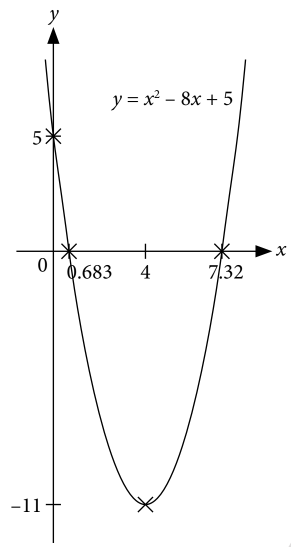
$x = 0.683$ or 7.32 (to 3 s.f.)

Find turning point:

Comparing $y = (x - 4)^2 - 11$ and $y = (x - p)^2 + q$, $p = 4$ and $q = -11$.

$\therefore$ coordinates of minimum point = $(4, -11)$

Sketch:



10. (i) Since the minimum point is $\left(-\frac{1}{2}, \frac{3}{4}\right)$,

$y = \left[x - \left(-\frac{1}{2}\right)\right]^2 + \frac{3}{4}$

$= (x - h)^2 + k$

$\therefore \boldsymbol{h = -\frac{1}{2}, k = \frac{3}{4}}$

(ii) Find y-intercept:

When $x = 0$, $y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

$= \left(0 + \frac{1}{2}\right)^2 + \frac{3}{4}$

$= 1$

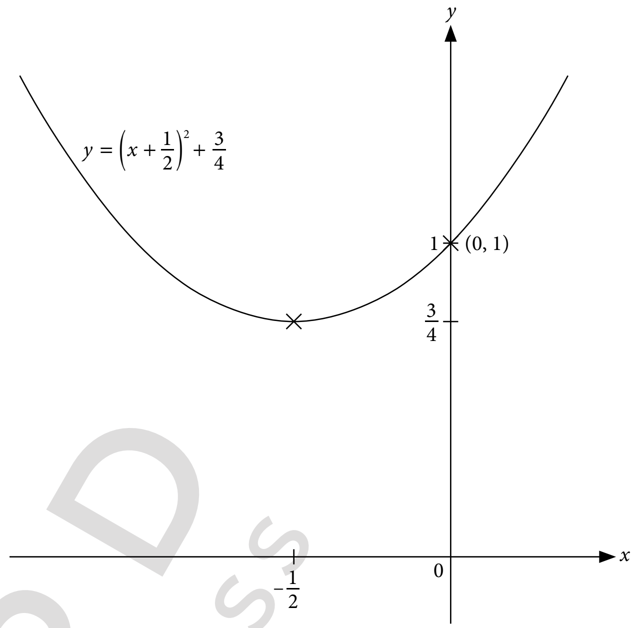
Find x-intercept(s):

When $y = 0$, $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$

$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}$

Since $\left(x + \frac{1}{2}\right)^2 \geqslant 0$, there are no real solutions.

Sketch:



3.6 Applications of quadratic equations and functions in real-world contexts

Practise Now 14

(i) Perimeter of triangle = 17 m

$AB + 8 + x = 17$

$AB = \mathbf{(9 - x)\ m}$

(ii) $8^2 = (9 - x)^2 + x^2$

$64 = 81 - 18x + x^2 + x^2$

$= 2x^2 - 18x + 81$

$2x^2 - 18x + 17 = 0$ (shown)

(iii) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-(-18) \pm \sqrt{(-18)^2 - 4(2)(17)}}{2(2)}$

$= \dfrac{18 \pm \sqrt{188}}{4}$

$= \mathbf{7.928}$ or $\mathbf{1.072}$ (to 3 d.p.)

(iv) Since AB is longer than BC, 7.928 must be rejected.

(v) $AB = 9 - x$

$= 9 - 1.072$

$= 7.928$ m

Area of $\triangle ABC = \frac{1}{2} \times 7.928 \times 1.072$

$= 4.249\ 408\ \text{m}^2$

Volume of structure = $4.249\ 408 \times 7$

$= \mathbf{29.7\ m^3}$ (to 3 s.f.)

Practise Now 15

(i) Average speed of return journey = $(x + 7)$ km/h

Time taken for journey from City P to City Q $= \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{600}{x}$ h

Time taken for return journey $= \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{600}{x+7}$ h

15 min $= \dfrac{15}{60}$ h $= \dfrac{1}{4}$ h

$$\therefore \frac{600}{x} - \frac{600}{x+7} = \frac{1}{4}$$

$$\left(\frac{600}{x} - \frac{600}{x+7}\right) \times 4x(x+7) = \frac{1}{4} \times 4x(x+7)$$

$$2400(x + 7) - 2400x = x(x + 7)$$

$$16\ 800 = x^2 + 7x$$

$$x^2 + 7x - 16\ 800 = 0 \text{ (shown)}$$

(ii) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-7 \pm \sqrt{7^2 - 4(1)(-16\ 800)}}{2(1)}$

$= \dfrac{-7 \pm \sqrt{67\ 249}}{2}$

$=$ **126.16** or **−133.16** (to 2 d.p.)

(iii) Time taken for return journey $= \dfrac{600}{x+7}$

$= \dfrac{600}{126.16 + 7}$ (since speed $x > 0$)

$= 4.5059$ h (to 5 s.f.)

$= 4$ h $+ 0.5059 \times 60$ min

$=$ **4 h 30 min** (to nearest minute)

Practise Now 16

1. When $x = 1$, $y = -x^2 + 2x + 0.5$

$= -(1)^2 + 2(1) + 0.5$

$= 1.5$

Since Vasi's height is 1.6 m, distance from the top of Vasi's head to the ground when $x = 1$

$= 1.5 + 1.6$

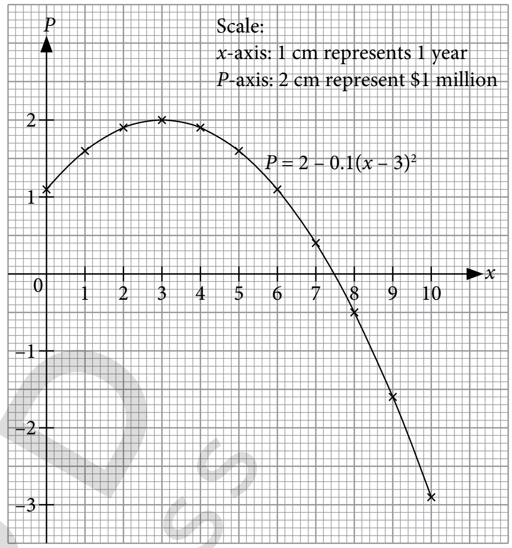
$= 3.1$ m

Since 3.1 m $>$ 3 m,

$\therefore$ **yes**, Vasi will be able to reach the bell.

2. (i)

x	0	1	2	3	4	5	6	7	8	9	10
P	1.1	1.6	1.9	2	1.9	1.6	1.1	0.4	−0.5	−1.6	−2.9



(ii) From the graph, when $P = 0$, $x =$ **7.5**.

Exercise 3E

1. (i) Perimeter of rectangular poster = 112 cm

$2 \times$ (Length of rectangular poster $+ x$) $= 112$

Length of rectangular poster $+ x = 56$

Length of rectangular poster $=$ **$(56 - x)$ cm**

(ii) Area of rectangular poster = 597 cm^2

$(56 - x) \times x = 597$

$56x - x^2 = 597$

$x^2 - 56x + 597 = 0$ (shown)

(iii) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-(-56) \pm \sqrt{(-56)^2 - 4(1)(597)}}{2(1)}$

$= \dfrac{56 \pm \sqrt{748}}{2}$

$=$ **41.67** or **14.33** (to 2 d.p.)

(iv) Since the length of the poster must be longer than the breadth of the poster, $x = 14.33$.

4 m = 400 cm

We can get the maximum number of posters by placing the shorter side (i.e. the side measuring 14.33 cm) along the length of the wall.

Maximum number of posters that can be placed in one row

$= \dfrac{400}{14.33}$

$=$ **27** (round down to the nearest integer)

2. (i) Number of pages printed by Printer A in 1 minute = $\mathbf{\frac{60}{x}}$

(ii) Time taken for Printer B to print 60 pages = $(x + 2)$ min

Number of pages printed by Printer B in 1 minute = $\mathbf{\frac{60}{x+2}}$

(iii)
$$\frac{60}{x} + \frac{60}{x+2} = 144$$
$$\left(\frac{60}{x} + \frac{60}{x+2}\right) \times x(x+2) = 144 \times x(x+2)$$
$$60(x+2) + 60x = 144x(x+2)$$
$$60x + 120 + 60x = 144x^2 + 288x$$
$$120x + 120 = 144x^2 + 288x$$
$$144x^2 + 168x - 120 = 0$$
$$6x^2 + 7x - 5 = 0 \text{ (shown)}$$

(iv)
$$6x^2 + 7x - 5 = 0$$
$$(3x+5)(2x-1) = 0$$
$$3x + 5 = 0 \quad \text{or} \quad 2x - 1 = 0$$
$$3x = -5 \quad \text{or} \quad 2x = 1$$
$$x = -\mathbf{\frac{5}{3}} \quad \text{or} \quad x = \mathbf{\frac{1}{2}}$$

(v) Number of pages printed by Printer B in 1 minute
$$= \frac{60}{x+2}$$
$$= \frac{60}{\frac{1}{2}+2} \text{ (since time } x > 0)$$
$$= 24$$

Time taken by Printer B to print 144 pages = $\frac{144}{24}$

= **6 min**

3. (i) Amount of rice ordered in January = $\mathbf{\frac{600}{x}}$ **kg**

(ii) Price of each kg of rice from Brand B = \$$(x - 0.4)$

Amount of rice ordered in February = $\mathbf{\frac{600}{x-0.4}}$ **kg**

(iii)
$$\frac{600}{x-0.4} - \frac{600}{x} = 45$$
$$\left(\frac{600}{x-0.4} - \frac{600}{x}\right) \times x(x-0.4) = 45 \times x(x-0.4)$$
$$600x - 600(x-0.4) = 45x(x-0.4)$$
$$600x - 600x + 240 = 45x^2 - 18x$$
$$240 = 45x^2 - 18x$$
$$45x^2 - 18x - 240 = 0$$
$$15x^2 - 6x - 80 = 0 \text{ (shown)}$$

(iv)
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(15)(-80)}}{2(15)}$$
$$= \frac{6 \pm \sqrt{4836}}{30}$$
$$= 2.52 \quad \text{or} \quad -2.12 \text{ (to 2 d.p.)}$$

Price of each kg of rice from Brand B = $x - 0.4$

= $2.52 - 0.4$ (since price $x > 0$)

= **\$2.12**

4. When $t = 0$, $h = -t^2 + 5t + 10$
$$= 10$$
$$h = -t^2 + 5t + 10$$
$$= -(t^2 - 5t) + 10$$
$$= -\left[t^2 - 5t + \left(-\frac{5}{2}\right)^2 - \left(-\frac{5}{2}\right)^2\right] + 10$$
$$= -\left(t - \frac{5}{2}\right)^2 + \frac{25}{4} + 10$$
$$= -\left(t - \frac{5}{2}\right)^2 + \frac{65}{4}$$

$\therefore$ h-coordinate of maximum point = $\frac{65}{4}$

$\therefore$ difference in height = $\frac{65}{4} - 10$

= **6.25 m**

5. (i) Area of $\triangle ABC = \frac{1}{2} \times 2x \times (x+5)$
$$= x(x+5)$$
$$= (x^2 + 5x) \text{ cm}^2$$

Area of square base = $(2x)^2$
$$= 4x^2 \text{ cm}^2$$

Total surface area of model = $4(x^2 + 5x) + 4x^2$
$$= 4x^2 + 20x + 4x^2$$
$$= \mathbf{(8x^2 + 20x) \text{ cm}^2}$$

(ii) Total surface area of model = 50 cm^2
$$8x^2 + 20x = 50$$
$$8x^2 + 20x - 50 = 0$$
$$4x^2 + 10x - 25 = 0 \text{ (shown)}$$

(iii)
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-10 \pm \sqrt{10^2 - 4(4)(-25)}}{2(4)}$$
$$= \frac{-10 \pm \sqrt{500}}{8}$$
$$= \mathbf{1.55} \quad \text{or} \quad \mathbf{-4.05} \text{ (to 3 s.f.)}$$

(iv) The sides of the triangle must be more than 0 cm, hence $2x > 0$ (i.e. $x > 0$) and -4.05 must be rejected.

(v) Floor area = area of square base
$$= 4x^2$$
$$= 4(1.55)^2$$
$$= 9.61 \text{ cm}^2$$

Actual floor area = $\left(\frac{100}{1}\right)^2 \times 9.61$ cm^2
$$= 96\,100 \text{ cm}^2$$
$$= \mathbf{9.61 \text{ m}^2}$$

6. (i) Time taken by Waseem to walk the first 2 km = $\frac{\text{distance}}{\text{speed}}$

$= \frac{2}{x}$ h

Time taken by Waseem to walk the remaining 8 km

$= \frac{\text{distance}}{\text{speed}}$

$= \frac{8}{x+1}$ h

$\therefore$ time taken by Waseem to complete the hike

$= \mathbf{\left(\frac{2}{x} + \frac{8}{x+1}\right) h}$

(ii) Time taken by Yasir to complete the hike = $\frac{10}{x}$ h

30 min = $\frac{30}{60}$ h = $\frac{1}{2}$ h

$$\therefore \frac{10}{x} - \left(\frac{2}{x} + \frac{8}{x+1}\right) = \frac{1}{2}$$
$$\frac{10}{x} - \frac{2}{x} - \frac{8}{x+1} = \frac{1}{2}$$
$$\frac{8}{x} - \frac{8}{x+1} = \frac{1}{2}$$
$$\left(\frac{8}{x} - \frac{8}{x+1}\right) \times 2x(x+1) = \frac{1}{2} \times 2x(x+1)$$
$$16(x+1) - 16x = x(x+1)$$
$$16x + 16 - 16x = x^2 + x$$
$$16 = x^2 + x$$
$$x^2 + x - 16 = 0 \text{ (shown)}$$

(iii) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1^2 - 4(1)(-16)}}{2(1)}$

$= \frac{-1 \pm \sqrt{65}}{2}$

= **3.53** or **–4.53** (to 3 s.f.)

Since speed must be more than 0 km/h, hence $x > 0$ and –4.53 must be rejected.

(iv) Time taken by Waseem to complete the hike

$= \frac{2}{x} + \frac{8}{x+1}$

$= \frac{2}{3.53} + \frac{8}{3.53+1}$

= 2.33 h (to 3 s.f.)

= 2 h 20 min (to nearest minute)

∴ time when Waseem finished the hike = **10.20 a.m.**

7. (i) Average speed of coach = $(x - 30)$ km/h

Time taken to travel by coach = $\frac{\text{distance}}{\text{speed}}$

$= \frac{700}{x-30}$ h

Time taken to travel by car = $\frac{\text{distance}}{\text{speed}}$

$= \frac{700}{x}$ h

$$\therefore \frac{700}{x-30} + \frac{700}{x} = 20$$
$$\left(\frac{700}{x-30} + \frac{700}{x}\right) \times x(x-30) = 20 \times x(x-30)$$
$$700x + 700(x-30) = 20x(x-30)$$
$$700x + 700x - 21\,000 = 20x^2 - 600x$$
$$1400x - 21\,000 = 20x^2 - 600x$$
$$20x^2 - 2000x + 21\,000 = 0$$
$$x^2 - 100x + 1050 = 0 \text{ (shown)}$$

(ii) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-100) \pm \sqrt{(-100)^2 - 4(1)(1050)}}{2(1)}$

$= \frac{100 \pm \sqrt{5800}}{2}$

= **88.08** or **11.92** (to 2 d.p.)

(iii) If x = 11.92, average speed of coach

$= x - 30$

= 11.92 – 30

= –18.08 km/h (rejected since speed is positive)

Time taken for return journey = $\frac{700}{x}$

$= \frac{700}{88.08}$

= **7.95 h** (to 3 s.f.)

8. (i) Time taken by Pump A = $\mathbf{\frac{1500}{x}}$ **min**

(ii) Time taken by Pump B = $\mathbf{\frac{1500}{x+50}}$ **min**

(iii) 30 s = $\frac{30}{60}$ min = $\frac{1}{2}$ min

$$\therefore \frac{1500}{x} - \frac{1500}{x+50} = \frac{1}{2}$$
$$\left(\frac{1500}{x} - \frac{1500}{x+50}\right) \times 2x(x+50) = \frac{1}{2} \times 2x(x+50)$$
$$3000(x+50) - 3000x = x(x+50)$$
$$3000x + 150\,000 - 3000x = x^2 + 50x$$
$$150\,000 = x^2 + 50x$$
$$x^2 + 50x - 150\,000 = 0 \text{ (shown)}$$

(iv) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-50 \pm \sqrt{50^2 - 4(1)(-150\,000)}}{2(1)}$

$= \frac{-50 \pm \sqrt{602\,500}}{2}$

= **363.10** or **–413.10** (to 2 d.p.)

(v) Time taken for Pump B to fill the tank completely

$= \frac{1500}{x+50}$

$= \frac{1500}{363.10+50}$ (since volume x is positive)

= 3.63 min (to 3 s.f.)

= **3 min 38 s** (to nearest second)

9. (i) At XYZ Money Exchange,

S\$$x$ = US\$1

S\$2000 = US\$$\left(\frac{1}{x} \times 2000\right)$

= **US\$**$\left(\mathbf{\frac{2000}{x}}\right)$

(ii) At ABC Money Exchange,

S\$$(x + 0.05)$ = US\$1

S\$1000 = US\$$\left(\frac{1}{x+0.05} \times 1000\right)$

= **US\$**$\left(\mathbf{\frac{1000}{x+0.05}}\right)$

(iii)
$$\frac{2000}{x} + \frac{1000}{x+0.05} = 2180$$
$$\left(\frac{2000}{x} + \frac{1000}{x+0.05}\right) \times x(x+0.05) = 2180 \times x(x+0.05)$$
$$2000(x+0.05) + 1000x = 2180x(x+0.05)$$
$$2000x + 100 + 1000x = 2180x^2 + 109x$$
$$3000x + 100 = 2180x^2 + 109x$$
$$2180x^2 - 2891x - 100 = 0$$
$$218x^2 - 289.1x - 10 = 0 \text{ (shown)}$$

(iv) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-(-289.1) \pm \sqrt{(-289.1)^2 - 4(218)(-18)}}{2(218)}$

$= \dfrac{289.1 \pm \sqrt{92\ 298.81}}{436}$

$=$ **1.3599** or **−0.0337** (to 4 d.p.)

(v) At ABC Money Exchange,
US\$1 = S\$(1.3599 + 0.05) (since amount of money x is positive)
US\$1 = S\$1.41 (to 2 d.p.)

10. Let the speed of the wind be x km/h.
Assumptions:

1. The aircraft is flying in the same direction as the wind from Sandy Land to White City such that the speed of the aircraft = $(165 + x)$ km/h.

2. The aircraft is flying against the wind from White City to Sandy Land such that the speed of the aircraft = $(165 - x)$ km/h.

5 h 30 min = 5.5 h

$$\frac{450}{165 + x} + \frac{450}{165 - x} = 5.5$$

$$\left(\frac{450}{165 + x} + \frac{450}{165 - x}\right) \times (165 + x)(165 - x) = 5.5 \times (165 + x)(165 - x)$$

$$450(165 - x) + 450(165 + x) = 5.5(165 + x)(165 - x)$$

$$74\ 250 - 450x + 74\ 250 + 450x = 5.5(27\ 225 - x^2)$$

$$148\ 500 = 149\ 737.5 - 5.5x^2$$

$$5.5x^2 = 1237.5$$

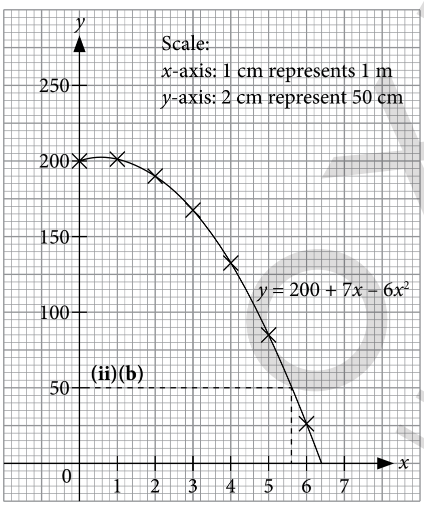
$$x^2 = 225$$

$$x = \sqrt{225} \text{ (since speed is positive)}$$

$$= 15$$

$\therefore$ speed of the wind = **15 km/h**

11. (i) $y = 200 + 7x - 6x^2$



(ii) (a) From the graph, the maximum height the balloon reaches is **202.5 cm**.

(b) From the graph, when the balloon is 50 cm above the ground, $y = 50$, $x = 5.6$.
$\therefore$ the horizontal distance from the foot of the platform when the balloon is 50 cm above the ground is **5.6 m**.

(iii) t = **6.4**. After $t = 6.4$, the flight of the ball will be below ground level, which is not valid in this case.

Chapter 4 Indices, Surds, Standard Form and Exponential Growth and Decay

TEACHING NOTES

Suggested Approach

In Book 1, students have been introduced to writing numbers in index notation. In this chapter, they will learn the laws of indices, zero, negative and rational indices.

Teachers should consider using the Investigation activities provided in the textbook to allow students to explore the laws of indices for numbers before moving on to variables. It is not advisable to state all the laws of indices to the students when teaching this chapter. After the students are familiar with laws of indices introduced in Section 4.2, where all the indices are positive integers, teachers can extend it to Section 4.3: Zero and negative indices, and Section 4.4: Rational indices.

Section 4.5: Surds is an extension of what students have learnt about laws of indices in Section 4.2 and rational indices in Section 4.4. Teachers may consider using examples in Section 4.4 to rewrite them as surds to show the similarities.

Teacher should also conduct more discussions on how exponential growth and decay (e.g. compound interest), and standard form are used or manifested in real life.

Section 4.1: Indices

This section gives students a better understanding on the meanings of the base and the index represented in an index notation. Teachers may start on this chapter by giving scenarios where indices are involved and ask students to represent their answers in index notation, like what they have learnt in Book 1. Teachers should guide students along as they learn how to describe and compare numbers written in index form.

Section 4.2: Laws of Indices

Teachers should provide simple numerical examples to illustrate each law of indices. Ample examples should be given to the students to master each law first before moving on to the next law (see Investigation: Discovering Law 1 of Indices, Investigation: Discovering Law 2 of Indices, Investigation: Discovering Law 3 of Indices, Investigation: Discovering Law 4 of Indices and Investigation: Discovering Law 5 of Indices).

Teachers should clarify any common misconceptions students may have or difficulties they may encounter when working on questions involving the use of a few laws of indices (see Worked Example 5 on page 89 of the textbook).

Section 4.3: Zero and Negative Indices

Teachers may ask students to explore the meaning of zero and negative indices through activities instead of only asking them to state the definition of such indices (see Investigation: Making sense of the zero index and see Investigation: Making sense of negative indices).

It is important to emphasise to the students the meaning of 'evaluate' and 'leaving your answer in positive index form. Teachers should also highlight the importance of recognising the conditions under which the five laws of indices apply (see Thinking Time on page 95 of the textbook).

Section 4.4: Rational Indices

In Book 1, students have learnt about the square root and cube root of a number. Teachers may wish to extend on this by introducing the meaning of the positive n^{th} root and a radical expression.

Teachers may ask students to explore the meaning of rational indices through examples (see Investigation: Making sense of rational indices). In addition, teachers should highlight to students the importance to consider the various cases and conditions in rational indices (see Thinking Time on page 97 and on page 99 of the textbook).

Section 4.5: Surds

Teachers can clarify any misconceptions students may have about the laws of indices before moving to this section. Students may also explore the similarities between the laws of surds and the laws of indices (see Investigation: Simplifying surds). It is also important for students to know the conditions under which the laws of surds apply (see Class Discussion: Mathematical fallacy: 1 = –1?)

Some students may find it difficult to perform operations on expressions involving surds. Teachers may advise them to substitute the surds with algebraic letters, perform the operations, and then reverse the substitution. Teachers should also remind students to rationalise the denominator of an expression involving surds after obtaining the answer.

Section 4.6: Exponential growth and decay

As an introduction, teachers may give examples of exponential growth and decay in real life, such as compound interest, bacterial growth and radioactive decay, before asking students to try writing a mathematical statement or formula which describes these examples.

While some students may be familiar with the mathematical concepts required to solve problems involving exponential growth and decay, they may find it difficult to apply the concepts. Teachers should give them time and guide them through some real-life examples to apply what they have learnt.

Section 4.7: Standard form

Teacher may begin this section by getting students to explore how standard forms are being expressed by giving them some examples of very large and small numbers for them to express these numbers in standard form (see Class Discussion: Exploring standard form). Teachers should highlight the difference between numbers expressed in standard form and numbers not expressed in standard form so that students have a clear idea of how to write a number in standard form.

As an introduction to common prefixes used in daily lives (see page 114 of the textbook), teachers may get students to give examples of prefixes that they encounter in their daily lives.

Some students may find it difficult to manipulate numbers in standard form using a calculator. Teachers should give them time and guide them through some examples on using the calculator to evaluate numbers represented in standard form.

Worked Solutions

Introductory Problem

Day of the Month	Amount donated by Joyce ($)	Amount donated by Bernard ($)
1st	10	0.01
2nd	20	0.02
3rd	30	0.04
4th	40	0.08
5th	50	0.16
6th	60	0.32
7th	70	0.64
8th	80	1.28
9th	90	2.56
10th	100	5.12
11th	110	10.24
12th	120	20.48
13th	130	40.96
14th	140	81.92
15th	150	163.84
16th	160	327.68
17th	170	655.36
18th	180	1310.72
19th	190	2621.44
20th	200	5242.88
21st	210	10 485.76
22nd	220	20 971.52
23rd	230	41 943.04
24th	240	83 886.08
25th	250	167 772.16
26th	260	335 544.32
27th	270	671 088.64
28th	280	1 342 177.28
29th	290	2 684 354.56
30th	300	5 368 709.12
31st	310	10 737 418.24
Total	**4960**	**21 474 836.47**

∴ **Bernard** will donate more money in total by the end of the month.

4.2 Laws of Indices

Investigation (Discovering Law 1 of Indices)

1. $7^2 \times 7^4 = \underbrace{(7 \times 7)}_{\text{2 factors}} \times \underbrace{(7 \times 7 \times 7 \times 7)}_{\textbf{4}\text{ factors}}$

$= \underbrace{7 \times 7 \times \ldots \times 7}_{\text{6 factors}}$

$= 7^{\mathbf{6}}$

$= 7^{2 + \mathbf{4}}$

2. $(-6)^5 \times (-6)^3$
$= \underbrace{[(\mathbf{-6}) \times (\mathbf{-6}) \times (\mathbf{-6}) \times (\mathbf{-6}) \times (\mathbf{-6})]}_{\mathbf{5}\text{ factors}} \times \underbrace{[(\mathbf{-6}) \times (\mathbf{-6}) \times (\mathbf{-6})]}_{\mathbf{3}\text{ factors}}$
$= \underbrace{(-6) \times (-6) \times \ldots \times (-6)}_{\mathbf{8}\text{ factors}}$
$= (-6)^{\mathbf{8}}$
$= (-6)^{\mathbf{5+3}}$

3. $a^3 \times a^4 = \underbrace{(a \times a \times a)}_{3\text{ factors}} \times \underbrace{(\boldsymbol{a} \times \boldsymbol{a} \times \boldsymbol{a} \times \boldsymbol{a})}_{\mathbf{4}\text{ factors}}$
$= \underbrace{a \times a \times \ldots \times a}_{\mathbf{7}\text{ factors}}$
$= a^{\mathbf{7}}$
$= a^{\mathbf{3+4}}$

4. $a^m \times a^n = \underbrace{(a \times a \times a \times \ldots \times a \times a)}_{m\text{ factors}} \times \underbrace{(a \times a \times \ldots \times a \times a)}_{\boldsymbol{n}\text{ factors}}$
$= \underbrace{a \times a \times \ldots \times a}_{(m+\boldsymbol{n})\text{ factors}}$
$= a^{m+\boldsymbol{n}}$

Practise Now 1

(a) $7^2 \times 7^5 = 7^{2+5}$
$= \mathbf{7^7}$

(b) $(-3)^5 \times (-3) = (-3)^{5+1}$
$= \mathbf{(-3)^6}$

(c) $a^{12} \times a^8 = a^{12+8}$
$= \boldsymbol{a^{20}}$

(d) $2xy^4 \times 3x^5y^3 = 6x^{1+5}y^{4+3}$
$= \boldsymbol{6x^6y^7}$

Investigation (Discovering Law 2 of Indices)

1. $5^7 \div 5^3 = \dfrac{\overbrace{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}^{7\text{ factors}}}{\underbrace{\mathbf{5 \times 5 \times 5}}_{\mathbf{3}\text{ factors}}}$
$= 5^{\mathbf{4}}$
$= 5^{\mathbf{7-3}}$

2. $\dfrac{(-10)^5}{(-10)^2} = \dfrac{\overbrace{(-10)\times(-10)\times(-10)\times(-10)\times(-10)}^{5\text{ factors}}}{\underbrace{(\mathbf{-10})\times(\mathbf{-10})}_{\mathbf{2}\text{ factors}}}$
$= (-10)^{\mathbf{3}}$
$= (-10)^{\mathbf{5-2}}$

3. If $a \neq 0$, then $a^9 \div a^4 = \dfrac{\overbrace{a \times a \times a \times a \times a \times a \times a \times a \times a}^{9\text{ factors}}}{\underbrace{\boldsymbol{a \times a \times a \times a}}_{\mathbf{4}\text{ factors}}}$
$= a^{\mathbf{5}}$
$= a^{\mathbf{9-4}}$

4. If $a \neq 0$ and $m > n$, then $a^m \div a^n = \dfrac{\overbrace{a \times a \times a \times \ldots \times a \times a}^{m\text{ factors}}}{\underbrace{a \times a \times \ldots \times a \times a}_{\boldsymbol{n}\text{ factors}}}$
$= \underbrace{a \times a \times \ldots \times a}_{(m-\boldsymbol{n})\text{ factors}}$
$= a^{m-\boldsymbol{n}}$

5. The result will not hold because with 0 in the denominator, it will be undefined (division by 0).

6. At this stage, students have not learnt about zero or negative indices yet. So if $m < n$, some students may say that there are not enough a's in the numerator to be divided by the a's in the denominator; and if $m = n$, some students may say that the answer is 1 because all the a's in the numerator are divided by all the a's in the denominator, which is true. Others may say that it will end up with zero or negative indices and they may also say that zero or negative indices do not make any sense. You can inform them that we will come back to this issue later in the chapter.

Practise Now 2

(a) $9^7 \div 9^3 = 9^{7-3}$
$= \mathbf{9^4}$

(b) $(-4)^8 \div (-4) = (-4)^{8-1}$
$= \mathbf{(-4)^7}$

(c) $a^{10} \div a^6 = a^{10-6}$
$= \boldsymbol{a^4}$

(d) $\dfrac{27x^9y^4}{9x^6y^3} = 3x^{9-6}y^{4-3}$
$= \boldsymbol{3x^3y}$

Investigation (Discovering Law 3 of Indices)

1. $(2^5)^2 = 2^5 \times 2^5$
$= 2^{\mathbf{5+5}}$
$= 2^{\mathbf{5\times2}}$

2. $[(-9)^4]^3 = (-9)^4 \times (-9)^4 \times (-9)^4$
$= (-9)^{\mathbf{4+4+4}}$
$= (-9)^{\mathbf{4\times3}}$

3. $(a^m)^n = \underbrace{a^m \times a^m \times \ldots \times a^m}_{n\text{ factors}}$
$= a^{\overbrace{m+m+\ldots+m}^{\boldsymbol{n}\text{ factors}}}$
$= a^{m\times\boldsymbol{n}}$

Practise Now 3

1. **(a)** $(6^3)^4 = 6^{3\times4}$
$= \mathbf{6^{12}}$

(b) $(k^5)^9 = k^{5\times9}$
$= \boldsymbol{k^{45}}$

(c) $[(-4)^p]^3 \times [(-4)^5]^p = (-4)^{3p} \times (-4)^{5p}$
$= \boldsymbol{(-4)^{8p}}$

(d) $\dfrac{(3^q)^6 \times (3^4)^q}{(3^3)^q} = \dfrac{3^{6q} \times 3^{4q}}{3^{3q}}$
$= \boldsymbol{3^{7q}}$

2. $x^8 \times (x^3)^n \div (x^n)^2 = x^{10}$
$x^8 \times x^{3n} \div x^{2n} = x^{10}$
$x^{8+3n-2n} = x^{10}$
$x^{8+n} = x^{10}$
$8 + n = 10$
$n = \mathbf{2}$

Investigation (Discovering Law 4 of Indices)

1. $2^3 \times 7^3 = \underbrace{(2 \times 2 \times 2)}_{3\text{ factors}} \times \underbrace{(\mathbf{7 \times 7 \times 7})}_{\mathbf{3}\text{ factors}}$

$= \underbrace{(2 \times 7) \times (2 \times 7) \times (2 \times \mathbf{7})}_{3\text{ factors}}$

$= (2 \times \mathbf{7})^3$

2. $(-3)^2 \times (-4)^2 = \underbrace{(-3) \times (-3)}_{2\text{ factors}} \times \underbrace{(\mathbf{-4}) \times (\mathbf{-4})}_{\mathbf{2}\text{ factors}}$

$= \underbrace{[(-3) \times (-4)] \times [(-3) \times (\mathbf{-4})]}_{2\text{ factors}}$

$= [(-3) \times (\mathbf{-4})]^{\mathbf{2}}$

3. $a^n \times b^n = \underbrace{(a \times a \times \ldots \times a)}_{m\text{ factors}} \times \underbrace{(b \times b \times \ldots \times b)}_{\boldsymbol{n}\text{ factors}}$

$= \underbrace{(a \times b) \times (a \times b) \times \ldots \times (a \times b)}_{\boldsymbol{n}\text{ factors}}$

$= (a \times b)^{\boldsymbol{n}}$

Practise Now 4

(a) $3^7 \times 8^7 = (3 \times 8)^7$

$= \mathbf{24^7}$

(b) $(5b^4)^3 = (5)^3(b^4)^3$

$= \mathbf{125}\boldsymbol{b}^{\mathbf{12}}$

(c) $(-2c^2d^5)^5 = (-2)^5(c^2)^5(d^5)^5$

$= \mathbf{-32}\boldsymbol{c}^{\mathbf{10}}\boldsymbol{d}^{\mathbf{25}}$

(d) $\dfrac{(h^4k^2)^2 \times (-5hk^5)^3}{(5h^3k)^3} = \dfrac{h^8k^4 \times (-5)^3 h^3k^{15}}{5^3h^9k^3}$

$= \dfrac{-125h^{8+3}k^{4+15}}{125h^9k^3}$

$= -\dfrac{h^{11}k^{19}}{h^9k^3}$

$= -h^{11-9}k^{19-3}$

$= \boldsymbol{-h^2k^{16}}$

Investigation (Discovering Law 5 of Indices)

1. $8^3 \div 5^3 = \dfrac{8^3}{5^3}$

$= \dfrac{\overbrace{8 \times 8 \times 8}^{3\text{ factors}}}{\underbrace{\mathbf{5 \times 5 \times 5}}_{\mathbf{3}\text{ factors}}}$

$= \underbrace{\dfrac{8}{\mathbf{5}} \times \dfrac{8}{\mathbf{5}} \times \dfrac{8}{\mathbf{5}}}_{\mathbf{3}\text{ factors}}$

$= \left(\dfrac{8}{\mathbf{5}}\right)^3$

2. $(-12)^4 \div (-7)^4 = \dfrac{(-12)^4}{(-7)^4}$

$= \dfrac{\overbrace{(-12) \times (-12) \times (-12) \times (-12)}^{4\text{ factors}}}{\underbrace{\mathbf{(-7) \times (-7) \times (-7) \times (-7)}}_{\mathbf{4}\text{ factors}}}$

$= \underbrace{\dfrac{(-12)}{\mathbf{(-7)}} \times \dfrac{(-12)}{\mathbf{(-7)}} \times \dfrac{(-12)}{\mathbf{(-7)}} \times \dfrac{(-12)}{\mathbf{(-7)}}}_{4\text{ factors}}$

$= \left[\dfrac{(-12)}{(-7)}\right]^4$

3. If $b \neq 0$, then $a^n \div b^n = \dfrac{\overbrace{a \times a \times \ldots \times a}^{n\text{ factors}}}{\underbrace{b \times b \times \ldots \times b}_{\boldsymbol{n}\text{ factors}}}$

$= \underbrace{\dfrac{a}{\boldsymbol{b}} \times \dfrac{a}{\boldsymbol{b}} \times \ldots \times \dfrac{a}{\boldsymbol{b}}}_{\boldsymbol{n}\text{ factors}}$

$= \left(\dfrac{a}{\boldsymbol{b}}\right)^{\boldsymbol{n}}$

Practise Now 5

(a) $21^3 \div 7^3 = \left(\dfrac{21}{7}\right)^3$

$= \mathbf{3^3}$

(b) $\left(\dfrac{a}{4}\right)^3 = \dfrac{\boldsymbol{a}^{\mathbf{3}}}{\mathbf{4^3}}$

(c) $\left(\dfrac{p^2}{q}\right)^3 \div \dfrac{p^5}{q^7} = \dfrac{p^{2\times 3}}{q^3} \times \dfrac{q^7}{p^5}$

$= \dfrac{p^6}{q^3} \times \dfrac{q^7}{p^5}$

$= p^{6-5}q^{7-3}$

$= \boldsymbol{pq^4}$

(d) $\left(\dfrac{4x^2}{x^3}\right)^3 \div \dfrac{64x^7}{x^{21}} = \dfrac{4^3 \times x^6}{x^9} \times \dfrac{x^{21}}{64x^7}$

$= \dfrac{64 \times x^6}{x^9} \times \dfrac{x^{21}}{64x^7}$

$= x^{6+21-9-7}$

$= \boldsymbol{x^{11}}$

Exercise 4A

1. **(a)** $2^3 \times 2^7 = 2^{3+7}$

$= \mathbf{2^{10}}$

(b) $(-4)^6 \times (-4)^5 = (-4)^{6+5}$

$= \mathbf{(-4)^{11}}$

(c) $x^8 \times x^3 = x^{8+3}$

$= \boldsymbol{x^{11}}$

(d) $(3y^2) \times (8y^7) = 24y^{2+7}$

$= \boldsymbol{24y^9}$

2. **(a)** $5^8 \div 5^5 = 5^{8-5}$

$= \mathbf{5^3}$

(b) $(-7)^{11} \div (-7)^4 = (-7)^{11-4}$

$= \mathbf{(-7)^7}$

(c) $6x^7 \div x^3 = 6x^{7-3}$

$= \boldsymbol{6x^4}$

(d) $(-15y^9) \div 5y^4 = -3y^{9-4}$

$= \boldsymbol{-3y^5}$

3. **(a)** $\dfrac{3^3 \times 3^4}{3} = 3^{3+4-1}$

$= \mathbf{3^6}$

(b) $\dfrac{2^8 \times 2^3}{2^7} = 2^{8+3-7}$

$= \mathbf{2^4}$

4. **(a)** $(9^2)^4 = 9^{2\times 4}$

$= \mathbf{9^8}$

(b) $(h^2)^5 = h^{2\times 5}$

$= \boldsymbol{h^{10}}$

(c) $2^3 \times 9^3 = (2 \times 9)^3$

$= \mathbf{18^3}$

(d) $3^{14} \times (5^2)^7 = 3^{14} \times 5^{2\times 7}$
$= 3^{14} \times 5^{14}$
$= (3 \times 5)^{14}$
$= \mathbf{15^{14}}$

(e) $(2k^6)^3 = 2^3 \times k^{6\times 3}$
$= \mathbf{8k^{18}}$

(f) $(-3x^6y^2)^4 = (-3)^4x^{6\times 4}y^{2\times 4}$
$= \mathbf{81x^{24}y^8}$

5. **(a)** $14^{13} \div 7^{13} = \left(\frac{14}{7}\right)^{13}$
$= \mathbf{2^{13}}$

(b) $(9^5)^4 \div 3^{20} = 9^{5\times 4} \div 3^{20}$
$= 9^{20} \div 3^{20}$
$= \left(\frac{9}{3}\right)^{20}$
$= \mathbf{3^{20}}$

(c) $\left(\frac{m}{2}\right)^5 = \mathbf{\frac{m^5}{2^5}}$

(d) $\left(\frac{3}{n^2}\right)^3 = \frac{3^3}{n^{2\times 3}}$
$= \mathbf{\frac{3^3}{n^6}}$

(e) $\left(\frac{p^4}{q}\right)^6 = \frac{p^{4\times 6}}{q^6}$
$= \mathbf{\frac{p^{24}}{q^6}}$

(f) $\left(-\frac{x}{y^2}\right)^4 = \frac{(-1)^4x^4}{y^{2\times 4}}$
$= \mathbf{\frac{x^4}{y^8}}$

6. **(a)** $h^2k \times h^{11}k^9 = h^{2+11}k^{1+9}$
$= \mathbf{h^{13}k^{10}}$

(b) $(-m^7n^3) \times 4m^{11}n^9 = -4m^{7+11}n^{3+9}$
$= \mathbf{-4m^{18}n^{12}}$

(c) $11p^6q^7 \times 2p^3q^{10} = 22p^{6+3}q^{7+10}$
$= \mathbf{22p^9q^{17}}$

(d) $h^9k^6 \div h^5k^4 = h^{9-5}k^{6-4}$
$= \mathbf{h^4k^2}$

(e) $15m^8n^7 \div 3m^2n = 5m^{8-2}n^{7-1}$
$= \mathbf{5m^6n^6}$

(f) $(-10x^5y^6) \div (-2xy^5) = 5x^{5-1}y^{6-5}$
$= \mathbf{5x^4y}$

7. **(a)** $(a^2)^3 \times a^5 = a^{2\times 3} \times a^5$
$= a^6 \times a^5$
$= a^{6+5}$
$= \mathbf{a^{11}}$

(b) $(b^3)^7 \times (b^4)^5 = b^{3\times 7} \times b^{4\times 5}$
$= b^{21} \times b^{20}$
$= b^{21+20}$
$= \mathbf{b^{41}}$

(c) $(c^6)^5 \div (-c^2) = c^{6\times 5} \div (-c^2)$
$= c^{30} \div (-c^2)$
$= -c^{30-2}$
$= \mathbf{-c^{28}}$

(d) $(-3d^3)^2 \div (2d)^3 = (-3)^2d^{3\times 2} \div 2^3d^3$
$= 9d^6 \div 8d^3$
$= \frac{9d^{6-3}}{8}$
$= \mathbf{\frac{9d^3}{8}}$

(e) $(e^3f^2)^5 \div (-e^2f)^4 = e^{3\times 5}f^{2\times 5} \div (-1)^4e^{2\times 4}f^4$
$= e^{15}f^{10} \div e^8f^4$
$= e^{15-8}f^{10-4}$
$= \mathbf{e^7f^6}$

(f) $(4k^6)^3 \div (-2k^3)^3 = 4^3 \times k^{6\times 3} \div (-2)^3k^{3\times 3}$
$= 64k^{18} \div (-8k^9)$
$= -8k^{18-9}$
$= \mathbf{-8k^9}$

8. $x^9 \times (x^3)^{2n} \div (-x^n)^2 = x^{13}$
$x^9 \times x^{3\times 2n} \div (-1)^2x^{2n} = x^{13}$
$x^9 \times x^{6n} \div (x^{2n}) = x^{13}$
$x^{9+6n-2n} = x^{13}$
$x^{9+4n} = x^{13}$
$9 + 4n = 13$
$4n = 4$
$n = \mathbf{1}$

9. **(a)** $(ab^2)^3 \times (2a^2b)^3 = a^3b^{2\times 3} \times 2^3 \times a^{2\times 3}b^3$
$= a^3b^6 \times 8 \times a^6b^3$
$= 8a^{3+6}b^{6+3}$
$= \mathbf{8a^9b^9}$

(b) $c^2d^2 \times (-5c^3d^3)^2 = c^2d^2 \times (-5)^2c^{3\times 2}d^{3\times 2}$
$= c^2d^2 \times 25c^6d^6$
$= 25c^{2+6}d^{2+6}$
$= \mathbf{25c^8d^8}$

(c) $(8e^5f^3)^2 \div (e^3f)^3 = 8^2 \times e^{5\times 2}f^{3\times 2} \div e^{3\times 3}f^3$
$= 64e^{10}f^6 \div e^9f^3$
$= 64e^{10-9}f^{6-3}$
$= \mathbf{64ef^3}$

(d) $16g^{10}h^7 \div (-2g^3h^2)^3 = 16g^{10}h^7 \div (-2)^3g^{3\times 3}h^{2\times 3}$
$= 16g^{10}h^7 \div (-8g^9h^6)$
$= -2g^{10-9}h^{7-6}$
$= \mathbf{-2gh}$

10. **(a)** $\frac{2a^2}{b} \times \left(\frac{a}{b^2}\right)^2 = \frac{2a^2}{b} \times \frac{a^2}{b^{2\times 2}}$
$= \frac{2a^2}{b} \times \frac{a^2}{b^4}$
$= \frac{2a^{2+2}}{b^{1+4}}$
$= \mathbf{\frac{2a^4}{b^5}}$

(b) $\left(\frac{c}{d^2}\right)^3 \times \left(\frac{c^3}{2d}\right)^3 = \frac{c^3}{d^{2\times 3}} \times \frac{c^{3\times 3}}{2^3 \times d^3}$
$= \frac{c^3}{d^6} \times \frac{c^9}{8 \times d^3}$
$= \frac{c^{3+9}}{8d^{6+3}}$
$= \mathbf{\frac{c^{12}}{8d^9}}$

(c) $\left(\dfrac{3e^3}{f^2}\right)^4 \div \dfrac{27e^9}{f^{11}} = \dfrac{3^4 \times e^{3\times4}}{f^{2\times4}} \times \dfrac{f^{11}}{27e^9}$

$= \dfrac{81e^{12}}{f^8} \times \dfrac{f^{11}}{27e^9}$

$= 3e^{12-9}f^{11-8}$

$= \mathbf{3e^3f^3}$

(d) $\left(\dfrac{-3g^5}{2h^2}\right)^3 \div \left(\dfrac{g^2}{h^3}\right)^3 = \dfrac{(-3)^3 g^{5\times3}}{2^3 \times h^{2\times3}} \div \dfrac{g^{2\times3}}{h^{3\times3}}$

$= -\dfrac{27g^{15}}{8h^6} \div \dfrac{g^6}{h^9}$

$= -\dfrac{27g^{15}}{8h^6} \times \dfrac{h^9}{g^6}$

$= -\dfrac{27g^{15-6}h^{9-6}}{8}$

$= -\mathbf{\dfrac{27g^9h^3}{8}}$

11. (a) $\dfrac{(2x^2y)^3}{(10xy^3)^2} \times \dfrac{(5xy^4)^3}{4xy} = \dfrac{2^3 \times x^{2\times3}y^3}{10^2 \times x^2y^{3\times2}} \times \dfrac{5^3 \times x^3y^{4\times3}}{4xy}$

$= \dfrac{8x^6y^3}{100x^2y^6} \times \dfrac{125x^3y^{12}}{4xy}$

$= \dfrac{5x^{6+3}y^{3+12}}{2x^{2+1}y^{6+1}}$

$= \dfrac{5x^9y^{15}}{2x^3y^7}$

$= \dfrac{5x^{9-3}y^{15-7}}{2}$

$= \mathbf{\dfrac{5x^6y^8}{2}}$

(b) $\dfrac{8x^8y^4}{(2xy^2)^2} \times \dfrac{(4x^2y^2)^2}{(3xy)^2} = \dfrac{8x^8y^4}{2^2 \times x^2y^{2\times2}} \times \dfrac{4^2 \times x^{2\times2}y^{2\times2}}{3^2 \times x^2y^2}$

$= \dfrac{8x^8y^4}{4x^2y^4} \times \dfrac{16x^4y^4}{9x^2y^2}$

$= \dfrac{32x^{8+4}y^{4+4}}{9x^{2+2}y^{4+2}}$

$= \dfrac{32x^{12}y^8}{9x^4y^6}$

$= \dfrac{32x^{12-4}y^{8-6}}{9}$

$= \mathbf{\dfrac{32x^8y^2}{9}}$

(c) $\dfrac{(2xy^2)^6}{(4x^2y)^2(xy^3)} = \dfrac{2^6 \times x^6y^{2\times6}}{4^2 \times x^{2\times2}y^2 \times xy^3}$

$= \dfrac{64x^6y^{12}}{16x^4y^2 \times xy^3}$

$= \dfrac{4x^6y^{12}}{x^{4+1}y^{2+3}}$

$= \dfrac{4x^6y^{12}}{x^5y^5}$

$= 4x^{6-5}y^{12-5}$

$= \mathbf{4xy^7}$

(d) $\dfrac{4x^2y^4 \times 8x^4y^2}{(4x^2y^2)^2} = \dfrac{4x^2y^4 \times 8x^4y^2}{4^2 \times x^{2\times2}y^{2\times2}}$

$= \dfrac{32x^2y^4 \times x^4y^2}{16x^4y^4}$

$= \dfrac{2x^{2+4}y^{4+2}}{x^4y^4}$

$= \dfrac{2x^6y^6}{x^4y^4}$

$= 2x^{6-4}y^{6-4}$

$= \mathbf{2x^2y^2}$

12. $\dfrac{(2p^3q^4)^4}{(-3q^5)^2} \div \dfrac{(4p^2q)^2}{9} = \dfrac{p^{a+b}}{q^{a-b}}$

$\dfrac{2^4 \times p^{3\times4}q^{4\times4}}{(-3)^2q^{5\times2}} \div \dfrac{4^2 \times p^{2\times2}q^2}{9} = \dfrac{p^{a+b}}{q^{a-b}}$

$\dfrac{16p^{12}q^{16}}{9q^{10}} \div \dfrac{16p^4q^2}{9} = \dfrac{p^{a+b}}{q^{a-b}}$

$\dfrac{16p^{12}q^{16}}{9q^{10}} \times \dfrac{9}{16p^4q^2} = \dfrac{p^{a+b}}{q^{a-b}}$

$\dfrac{p^{12-4}q^{16}}{q^{10+2}} = \dfrac{p^{a+b}}{q^{a-b}}$

$\dfrac{p^8q^{16}}{q^{12}} = \dfrac{p^{a+b}}{q^{a-b}}$

$\dfrac{p^8}{q^{12-16}} = \dfrac{p^{a+b}}{q^{a-b}}$

$\dfrac{p^8}{q^{-4}} = \dfrac{p^{a+b}}{q^{a-b}}$

$a + b = 8$

$a = 8 - b$ — (1)

$a - b = -4$ — (2)

Substitute (1) into (2):

$(8 - b) - b = -4$

$8 - 2b = -4$

$2b = 12$

$b = 6$

Substitute $b = 6$ into (1):

$a = 8 - 6$

$= 2$

$\therefore$ $\boldsymbol{a = 2, b = 6}$

4.3 Zero and negative indices

Investigation (Making sense of the zero index)

1.

Index form	Value
3^4	81
3^3	27
3^2	**9**
3^1	**3**
3^0	**1**

Table 4.1

2.

Index form	Value
$(-2)^4$	16
$(-2)^3$	-8
$(-2)^2$	**4**
$(-2)^1$	**-2**
$(-2)^0$	**1**

Table 4.2

3. **No**. Any number which is divided by zero is undefined.

4. If $a \neq 0$, then $a^n \div a^n = \dfrac{\overbrace{a \times a \times a \times \ldots \times a \times a}^{n \text{ factors}}}{\underbrace{a \times a \times a \times \ldots \times a \times a}_{n \text{ factors}}}$

$= 1$

But $a^n \div a^n = a^{n-n}$

$= a^0$

Therefore, if $a \neq 0$, then $a^0 = \mathbf{1}$.

Practise Now 6

1. **(a)** $2022^0 = \mathbf{1}$
   (b) $(-8)^0 = \mathbf{1}$
   (c) $3y^0 = 3 \times 1$
   $= \mathbf{3}$
   (d) $(3y)^0 = \mathbf{1}$
2. **(a)** $3^0 \times 3^3 \div 3^2 = 3^{0+3-2}$
   $= 3^1$
   $= \mathbf{3}$
   (b) $3^0 + 3^2 = 1 + 9$
   $= \mathbf{10}$

Investigation (Making sense of negative indices)

1.

Index form	Value
3^2	9
3^1	3
3^0	**1**
3^{-1}	$\mathbf{\frac{1}{3}}$
3^{-2}	$\mathbf{\frac{1}{9}}$

Table 4.3

2.

Index form	Value
$(-2)^2$	4
$(-2)^1$	-2
$(-2)^0$	**1**
$(-2)^{-1}$	$\mathbf{-\frac{1}{2}}$
$(-2)^{-2}$	$\mathbf{\frac{1}{4}}$

Table 4.4

3. **Undefined**. Any number which is divided by zero is undefined.

4. $a^4 \div a^7 = \dfrac{\mathbf{a \times a \times a \times a}}{a \times a \times a \times a \times a \times a \times a}$

$= \dfrac{1}{a \times a \times a}$

$= \dfrac{1}{a^3}$

But $a^4 \div a^7 = a^{4-7}$

$= a^{-3}$

Therefore, $a^{-3} = \mathbf{\dfrac{1}{a^3}}$.

Practise Now 7

1. **(a)** $6^{-2} = \dfrac{1}{6^2}$
   $= \mathbf{\dfrac{1}{36}}$
   (b) $(-8)^{-1} = \dfrac{1}{(-8)^1}$
   $= \mathbf{-\dfrac{1}{8}}$
   (c) $\left(\dfrac{4}{5}\right)^{-3} = \dfrac{1}{\left(\frac{4}{5}\right)^3}$
   $= 1 \div \left(\dfrac{4}{5}\right)^3$
   $= 1 \div \dfrac{4^3}{5^3}$
   $= 1 \times \dfrac{5^3}{4^3}$
   $= \mathbf{\dfrac{125}{64}}$
   (d) $9^0 \div \left(\dfrac{1}{9}\right)^{-1} = 9^0 \div \dfrac{1}{\left(\frac{1}{9}\right)^1}$
   $= 1 \div \dfrac{1}{\frac{1}{9}}$
   $= 1 \div 9$
   $= \mathbf{\dfrac{1}{9}}$
2. $(2d)^0 \div (d^2e^{-4})^{-1} = 1 \div \dfrac{1}{(d^2e^{-4})^1}$
   $= 1 \div \dfrac{1}{d^2e^{-4}}$
   $= 1 \times d^2e^{-4}$
   $= d^2e^{-4}$
   $= \mathbf{\dfrac{d^2}{e^4}}$

Thinking Time (Page 95)

1. If a and b are real numbers, and m and n are integers, then
   Law 1 of Indices: $a^m \times a^n = \mathbf{a^{m+n}}$ if $a \neq 0$
   Law 2 of Indices: $a^m \div a^n = \mathbf{a^{m-n}}$ if $a \neq 0$
   Law 3 of Indices: $(a^m)^n = \mathbf{a^{mn}}$ if $\mathbf{a \neq 0}$
   Law 4 of Indices: $a^n \times b^n = \mathbf{(a \times b)^n}$ if $a, b \neq 0$
   Law 5 of Indices: $a^n \div b^n = \mathbf{\left(\dfrac{a}{b}\right)^n}$ if $\mathbf{a, b \neq 0}$
2. **(i)** In Law 1, if index m or n is 0 or a negative integer, base $a \neq 0$ for zero or negative indices to be defined.
   (ii) In Law 4, if index n is 0 or a negative integer, bases a, $b \neq 0$ for zero or negative indices to be defined.

3. **(i)** In Law 2, if $m = n$, $a^m \div a^n = a^{m-n} = a^0$, i.e. we will get a zero index.

(ii) In Law 2, if $m < n$, $a^m \div a^n = a^{m-n}$ will involve a negative index.

(iii) Once we have defined zero and negative indices, it is no longer necessary for $m > n$ for Law 2 since it is fine to obtain a zero index (see part **3(i)**) or a negative index (see part **3(ii)**).

(iv) In Law 2, if $m = 0$, $a^m \div a^n = a^{0-n} = a^{-n} = \frac{1}{a^n}$, i.e. the definition of negative indices.

Practise Now 8

1. **(a)** $a^{-1} \times a^3 \div a^{-2} = a^{-1+3-(-2)}$
$= \boldsymbol{a^4}$

(b) $\frac{16d^{-2}e}{(2d^{-1}e)^3} = \frac{16d^{-2}e}{2^3 \times d^{-1\times 3}e^3}$
$= \frac{16d^{-2}e}{8d^{-3}e^3}$
$= 2d^{-2-(-3)}e^{1-3}$
$= 2de^{-2}$
$= \boldsymbol{\frac{2d}{e^2}}$

(c) $18g^{-6} \div 3(g^{-2})^2 = \frac{18g^{-6}}{3(g^{-2\times 2})}$
$= \frac{18g^{-6}}{3g^{-4}}$
$= 6g^{-6-(-4)}$
$= 6g^{-2}$
$= \boldsymbol{\frac{6}{g^2}}$

2. $6h^2 \div 2h^{-2} - h \times h^3 - \frac{4}{h^{-4}} = 3h^{2-(-2)} - h^{1+3} - 4h^4$
$= 3h^4 - h^4 - 4h^4$
$= \boldsymbol{-2h^4}$

4.4 Rational indices

Practise Now 9

(a) $\sqrt[4]{256} = \sqrt[4]{4 \times 4 \times 4 \times 4}$
$= \mathbf{4}$

(b) $\sqrt[5]{1024} = \sqrt[5]{4 \times 4 \times 4 \times 4 \times 4}$
$= \mathbf{4}$

(c) $\sqrt[3]{\frac{64}{125}} = \sqrt[3]{\frac{4 \times 4 \times 4}{5 \times 5 \times 5}}$
$= \sqrt[3]{\frac{4}{5} \times \frac{4}{5} \times \frac{4}{5}}$
$= \boldsymbol{\frac{4}{5}}$

(d) $\sqrt[6]{\frac{64}{729}} = \sqrt[6]{\frac{2 \times 2 \times 2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3 \times 3 \times 3}}$
$= \sqrt[6]{\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}}$
$= \boldsymbol{\frac{2}{3}}$

Investigation (Making sense of rational indices)

1. Let $p = 5^{\frac{1}{3}}$. Then $p^3 = \left(5^{\frac{1}{3}}\right)^3$
$= 5^{\frac{1}{3}\times 3}$
$= 5^1$
$= 5$
$\therefore p = \sqrt[3]{5}$
i.e. $5^{\frac{1}{3}} = \boldsymbol{\sqrt[3]{5}}$

2. Let $p = 3^{\frac{1}{2}}$. Then $p^2 = \left(3^{\frac{1}{2}}\right)^2$
$= 3^{\frac{1}{2}\times 2}$
$= 3^1$
$= 3$
$\therefore p = \pm\boldsymbol{\sqrt{3}}$
There are two values of p. We choose the positive value because we want $y = a^{\frac{1}{x}}$ to be a function, i.e. for every value of x, there should be exactly one value of y.
Hence, $p = \sqrt{3}$
i.e. $3^{\frac{1}{2}} = \boldsymbol{\sqrt{3}}$

Thinking Time (Page 97)

1. If $a < 0$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$ is undefined when n is even or $n = 0$.
2. If $a = 0$, then $0^{\frac{1}{n}} = \sqrt[n]{0} = 0$.

Practise Now 10

(a) $36^{\frac{1}{2}} = \sqrt{36}$
$= \sqrt{6 \times 6}$
$= \mathbf{6}$

(b) $625^{\frac{1}{4}} = \sqrt[4]{625}$
$= \sqrt[4]{5 \times 5 \times 5 \times 5}$
$= \mathbf{5}$

(c) $243^{-\frac{1}{5}} = \frac{1}{243^{\frac{1}{5}}}$
$= \frac{1}{\sqrt[5]{243}}$
$= \frac{1}{\sqrt[5]{3 \times 3 \times 3 \times 3 \times 3}}$
$= \boldsymbol{\frac{1}{3}}$

(d) $(-1000)^{-\frac{1}{3}} = \frac{1}{(-1000)^{\frac{1}{3}}}$
$= \frac{1}{\sqrt[3]{-1000}}$
$= \frac{1}{\sqrt[3]{(-10) \times (-10) \times (-10)}}$
$= \frac{1}{-10}$
$= -\boldsymbol{\frac{1}{10}}$

Investigation (Discovering a result involving rational indices)

(a) $5^{\frac{2}{3}} = 5^{2\times\frac{1}{3}}$

$= \left(5^2\right)^{\frac{1}{3}}$

$= \sqrt[3]{5^2}$

(b) $5^{\frac{2}{3}} = 5^{\frac{1}{3}\times 2}$

$= \left(5^{\frac{1}{3}}\right)^2$

$= \left(\sqrt[3]{5}\right)^2$

Practise Now 11

1. (a) $64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2$

$= 4^2$

$= \mathbf{16}$

(b) $32^{-\frac{3}{5}} = \dfrac{1}{32^{\frac{3}{5}}}$

$= \dfrac{1}{\left(\sqrt[5]{32}\right)^3}$

$= \dfrac{1}{2^3}$

$= \boldsymbol{\dfrac{1}{8}}$

(c) $100^{1.5} = 100^{\frac{3}{2}}$

$= \left(\sqrt{100}\right)^3$

$= 10^3$

$= \mathbf{1000}$

(d) $4^{2.5} + 27^{\frac{4}{3}} = 4^{\frac{5}{2}} + 27^{\frac{4}{3}}$

$= \left(\sqrt{4}\right)^5 + \left(\sqrt[3]{27}\right)^4$

$= 2^5 + 3^4$

$= 32 + 81$

$= \mathbf{113}$

2. (a) $\sqrt[3]{a^n} = \boldsymbol{a^{\frac{n}{3}}}$

(b) $\dfrac{1}{\sqrt[5]{x^2}} = \dfrac{1}{x^{\frac{2}{5}}}$

$= \boldsymbol{x^{-\frac{2}{5}}}$

Thinking Time (Page 99)

1. If bases a and b are real numbers, and indices m and n are rational numbers, then

Law 1 of Indices:	$a^m \times a^n = \boldsymbol{a^{m+n}}$	if $a > 0$
Law 2 of Indices:	$a^m \div a^n = \boldsymbol{a^{m-n}}$	if $a > 0$
Law 3 of Indices:	$(a^m)^n = \boldsymbol{a^{mn}}$	if $\boldsymbol{a > 0}$
Law 4 of Indices:	$a^n \times b^n = \boldsymbol{(a \times b)^n}$	if $a, b > 0$
Law 5 of Indices:	$a^n \div b^n = \boldsymbol{\left(\dfrac{a}{b}\right)^n}$	if $\boldsymbol{a, b > 0}$

2. (i) In Law 1, if index m or n is a rational number, base $a > 0$ for rational indices to be defined.

(ii) In Law 4, if index n is a rational number, bases a, $b > 0$ for rational indices to be defined.

3. For Law 4 of Indices to be defined for indices involving rational numbers, a, $b > 0$. However, in this case, $a = b = -1 < 0$, hence Law 4 of Indices cannot be applied.

Practise Now 12

1. (a) $\left(m^2\right)^{\frac{5}{6}} \times m^{\frac{1}{3}} = m^{\frac{5}{3}} \times m^{\frac{1}{3}}$

$= m^{\frac{5}{3}+\frac{1}{3}}$

$= \boldsymbol{m^2}$

(b) $\sqrt[5]{m} \div \sqrt[3]{m^2} = m^{\frac{1}{5}} \div m^{\frac{2}{3}}$

$= m^{\frac{1}{5}-\frac{2}{3}}$

$= m^{-\frac{7}{15}}$

$= \boldsymbol{\dfrac{1}{m^{\frac{7}{15}}}}$

(c) $\left(m^{-3}n^5\right)^{-\frac{1}{3}} = mn^{-\frac{5}{3}}$

$= \boldsymbol{\dfrac{m}{n^{\frac{5}{3}}}}$

(d) $\left(\dfrac{x^9y^3}{1000y^{12}}\right)^{-\frac{1}{3}} = \dfrac{1}{\left(\dfrac{x^9y^3}{1000y^{12}}\right)^{\frac{1}{3}}}$

$= \dfrac{1}{\dfrac{\left(x^9y^3\right)^{\frac{1}{3}}}{\left(1000y^{12}\right)^{\frac{1}{3}}}}$

$= \dfrac{\left(1000y^{12}\right)^{\frac{1}{3}}}{\left(x^9y^3\right)^{\frac{1}{3}}}$

$= \dfrac{\left(10^3y^{12}\right)^{\frac{1}{3}}}{\left(x^9y^3\right)^{\frac{1}{3}}}$

$= \dfrac{10^{\frac{3}{3}}y^{\frac{12}{3}}}{x^{\frac{9}{3}}y^{\frac{3}{3}}}$

$= \dfrac{10y^4}{x^3y}$

$= \dfrac{10y^{4-1}}{x^3}$

$= \boldsymbol{\dfrac{10y^3}{x^3}}$

(e) $\dfrac{h^{-\frac{1}{3}}k^{-\frac{1}{4}}}{\left(h^2k^{-\frac{1}{3}}\right)^{-2}} = \dfrac{h^{-\frac{1}{3}}k^{-\frac{1}{4}}}{h^{-4}k^{\frac{2}{3}}}$

$= h^{-\frac{1}{3}-(-4)}k^{-\frac{1}{4}-\frac{2}{3}}$

$= h^{\frac{11}{3}}k^{-\frac{11}{12}}$

$= \boldsymbol{\dfrac{h^{\frac{11}{3}}}{k^{\frac{11}{12}}}}$

(f) $\left(25p^2q^{-4}\right)^{\frac{1}{2}}\left(p^3q^{-\frac{2}{5}}\right)^2 = \left(5^2p^2q^{-4}\right)^{\frac{1}{2}}\left(p^3q^{-\frac{2}{5}}\right)^2$

$= \left(5^{\frac{2}{2}}p^{\frac{2}{2}}q^{-\frac{4}{2}}\right)\left(p^6q^{-\frac{4}{5}}\right)$

$= \left(5pq^{-2}\right)\left(p^6q^{-\frac{4}{5}}\right)$

$= 5p^{1+6}q^{-2+\left(-\frac{4}{5}\right)}$

$= 5p^7q^{-\frac{14}{5}}$

$= \boldsymbol{\dfrac{5p^7}{q^{\frac{14}{5}}}}$

2. $\left(\frac{64}{27}\right)^{-\frac{2}{3}} = \frac{1}{\left(\frac{64}{27}\right)^{\frac{2}{3}}}$

$= \frac{1}{\left(\sqrt[3]{\frac{64}{27}}\right)^2}$

$= \frac{1}{\left(\sqrt[3]{\left(\frac{4}{3}\right)^3}\right)^2}$

$= \frac{1}{\left(\frac{4}{3}\right)^2}$

$= \frac{1}{\frac{16}{9}}$

$= \mathbf{\frac{9}{16}}$

Practise Now 13

1. **(a)** $5^x = 125$

$5^x = 5^3$

$\therefore x = \mathbf{3}$

(b) $7^y = \frac{1}{49}$

$= \frac{1}{7^2}$

$= 7^{-2}$

$\therefore y = \mathbf{-2}$

(c) $8^z = 16$

$(2^3)^z = 2^4$

$2^{3z} = 2^4$

$3z = 4$

$\therefore z = \mathbf{\frac{4}{3}}$

2. $9^{-2} \times 3^k = 1$

$(3^2)^{-2} \times 3^k = 3^0$

$3^{-4} \times 3^k = 3^0$

$3^{-4+k} = 3^0$

$-4 + k = 0$

$\therefore k = \mathbf{4}$

Exercise 4B

1. **(a)** $17^0 = \mathbf{1}$

(b) $\left(-\frac{2}{7}\right)^0 = \mathbf{1}$

(c) $4a^0 = 4 \times 1$

$= \mathbf{4}$

(d) $-8b^0 = -8 \times 1$

$= \mathbf{-8}$

(e) $(72cd^2)^0 = \mathbf{1}$

(f) $7(e^8)^0 = 7 \times 1$

$= \mathbf{7}$

2. **(a)** $2^0 \times 2^4 = 1 \times 16$

$= \mathbf{16}$

(b) $7^2 \times 7^0 \div 7 = 49 \times 1 \div 7$

$= \mathbf{7}$

(c) $(-8)^0 - 8^2 = 1 - 64$

$= \mathbf{-63}$

(d) $3^3 + 3^0 - 3 = 27 + 1 - 3$

$= \mathbf{25}$

3. **(a)** $2^{-3} = \frac{1}{2^3}$

$= \mathbf{\frac{1}{8}}$

(b) $(-5)^{-1} = \frac{1}{-5}$

$= \mathbf{-\frac{1}{5}}$

(c) $\left(\frac{3}{4}\right)^{-2} = \frac{1}{\left(\frac{3}{4}\right)^2}$

$= \frac{1}{\frac{3^2}{4^2}}$

$= \frac{1}{\frac{9}{16}}$

$= \mathbf{\frac{16}{9}}$

(d) $\left(\frac{5}{3}\right)^{-1} = \frac{1}{\frac{5}{3}}$

$= \mathbf{\frac{3}{5}}$

4. **(a)** $(7^2)^{-2} \div 7^{-4} = 7^{-4} \div 7^{-4}$

$= \mathbf{1}$

(b) $5^0 - 5^{-2} = 1 - \frac{1}{5^2}$

$= 1 - \frac{1}{25}$

$= \mathbf{\frac{24}{25}}$

(c) $\left(2^{15}\right)^0 + \left(\frac{3}{5}\right)^{-1} = 1 + \frac{1}{\frac{3}{5}}$

$= 1 + \frac{5}{3}$

$= \mathbf{\frac{8}{3}}$

(d) $\left(\frac{3}{4}\right)^{-2} \times 3^2 \div 2015^0 = \frac{1}{\left(\frac{3}{4}\right)^2} \times 9 \div 1$

$= \frac{1}{\frac{3^2}{4^2}} \times 9$

$= \frac{1}{\frac{9}{16}} \times 9$

$= \frac{16}{9} \times 9$

$= \mathbf{16}$

5. **(a)** $\sqrt{196} = \sqrt{14 \times 14}$

$= \mathbf{14}$

(b) $\sqrt[3]{125} = \sqrt[3]{5 \times 5 \times 5}$

$= \mathbf{5}$

(c) $\sqrt[5]{\frac{1}{32}} = \sqrt[5]{\frac{1}{2 \times 2 \times 2 \times 2 \times 2}}$

$= \sqrt[5]{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}$

$= \mathbf{\frac{1}{2}}$

(d) $\sqrt[4]{\frac{16}{81}} = \sqrt[4]{\frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3}}$

$= \sqrt[4]{\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}}$

$= \mathbf{\frac{2}{3}}$

6. (a) $81^{\frac{1}{2}} = \sqrt{81}$

$= \sqrt{9 \times 9}$

$= \mathbf{9}$

(b) $(-27)^{\frac{1}{3}} = \sqrt[3]{-27}$

$= \sqrt[3]{(-3) \times (-3) \times (-3)}$

$= \mathbf{-3}$

(c) $32^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}}$

$= \frac{1}{\sqrt[5]{32}}$

$= \frac{1}{\sqrt[5]{2 \times 2 \times 2 \times 2 \times 2}}$

$= \mathbf{\frac{1}{2}}$

(d) $(-343)^{-\frac{1}{3}} = \frac{1}{(-343)^{\frac{1}{3}}}$

$= \frac{1}{\sqrt[3]{-343}}$

$= \frac{1}{\sqrt[3]{(-7) \times (-7) \times (-7)}}$

$= \frac{1}{-7}$

$= \mathbf{-\frac{1}{7}}$

7. (a) $216^{\frac{2}{3}} = \left(\sqrt[3]{216}\right)^2$

$= \left(\sqrt[3]{6 \times 6 \times 6}\right)^2$

$= 6^2$

$= \mathbf{36}$

(b) $8^{-\frac{5}{3}} = \frac{1}{8^{\frac{5}{3}}}$

$= \frac{1}{\left(\sqrt[3]{8}\right)^5}$

$= \frac{1}{\left(\sqrt[3]{2 \times 2 \times 2}\right)^5}$

$= \frac{1}{2^5}$

$= \mathbf{\frac{1}{32}}$

(c) $16^{1.5} = 16^{\frac{3}{2}}$

$= \left(\sqrt{16}\right)^3$

$= \left(\sqrt{4 \times 4}\right)^3$

$= 4^3$

$= \mathbf{64}$

(d) $(-1000)^{\frac{4}{3}} = \left(\sqrt[3]{-1000}\right)^4$

$= \left(\sqrt[3]{(-10) \times (-10) \times (-10)}\right)^4$

$= (-10)^4$

$= \mathbf{10\ 000}$

(e) $81^{\frac{3}{4}} - 9^{-\frac{3}{2}} = 81^{\frac{3}{4}} - \frac{1}{9^{\frac{3}{2}}}$

$= \left(\sqrt[4]{81}\right)^3 - \frac{1}{\left(\sqrt{9}\right)^3}$

$= \left(\sqrt[4]{3 \times 3 \times 3 \times 3}\right)^3 - \frac{1}{\left(\sqrt{3 \times 3}\right)^3}$

$= 3^3 - \frac{1}{3^3}$

$= 27 - \frac{1}{27}$

$= \mathbf{26\frac{26}{27}}$

(f) $625^{0.75} + 100^{2.5} = 625^{\frac{3}{4}} + 100^{\frac{5}{2}}$

$= \left(\sqrt[4]{625}\right)^3 + \left(\sqrt{100}\right)^5$

$= \left(\sqrt[4]{5 \times 5 \times 5 \times 5}\right)^3 + \left(\sqrt{10 \times 10}\right)^5$

$= 5^3 + 10^5$

$= 125 + 100\ 000$

$= \mathbf{100\ 125}$

8. (a) $\sqrt[4]{a} = \boldsymbol{a^{\frac{1}{4}}}$

(b) $\left(\sqrt[5]{b}\right)^4 = \boldsymbol{b^{\frac{4}{5}}}$

(c) $\sqrt[3]{c^{2n}} = \boldsymbol{c^{\frac{2n}{3}}}$

(d) $\frac{1}{\sqrt[n]{d}} = \frac{1}{d^{\frac{1}{n}}}$

$= \boldsymbol{d^{-\frac{1}{n}}}$

(e) $\frac{2}{\sqrt[8]{e^4}} = \frac{2}{e^{\frac{4}{8}}}$

$= \frac{2}{e^{\frac{1}{2}}}$

$= \boldsymbol{2e^{-\frac{1}{2}}}$

(f) $\frac{1}{\left(\sqrt[3]{f}\right)^5} = \frac{1}{\left(f^{\frac{1}{3}}\right)^5}$

$= \frac{1}{f^{\frac{5}{3}}}$

$= \boldsymbol{f^{-\frac{5}{3}}}$

9. $\left(\frac{1000}{27}\right)^{-\frac{4}{3}} = \frac{1}{\left(\frac{1000}{27}\right)^{\frac{4}{3}}}$

$= \frac{1}{\left(\sqrt[3]{\frac{1000}{27}}\right)^4}$

$= \frac{1}{\left(\sqrt[3]{\frac{10\times10\times10}{3\times3\times3}}\right)^4}$

$= \frac{1}{\left(\sqrt[3]{\frac{10}{3}\times\frac{10}{3}\times\frac{10}{3}}\right)^4}$

$= \frac{1}{\left(\frac{10}{3}\right)^4}$

$= \frac{1}{\frac{10^4}{3^4}}$

$= \frac{3^4}{10^4}$

$= \frac{81}{10\ 000}$

$= \mathbf{0.0081}$

10. (a) $11^a = 1331$

$11^a = 11^3$

$\therefore a = \mathbf{3}$

(b) $2^b = \frac{1}{128}$

$= \frac{1}{2^7}$

$= 2^{-7}$

$\therefore b = \mathbf{-7}$

(c) $9^c = 243$

$(3^2)^c = 3^5$

$3^{2c} = 3^5$

$2c = 5$

$\therefore c = \mathbf{\frac{5}{2}}$

(d) $10^d = 0.01$

$= \frac{1}{100}$

$= \frac{1}{10^2}$

$= 10^{-2}$

$\therefore d = \mathbf{-2}$

11. (a) $\frac{1}{256} = 4^k$

$\frac{1}{4^4} = 4^k$

$4^{-4} = 4^k$

$-4 = k$

$\therefore k = \mathbf{-4}$

(b) $32 \times 16^{\frac{3}{4}} = 2^n$

$2^5 \times \left(2^4\right)^{\frac{3}{4}} = 2^n$

$2^5 \times 2^3 = 2^n$

$2^{5+3} = 2^n$

$2^8 = 2^n$

$8 = n$

$\therefore n = \mathbf{8}$

(c) $p^{\frac{2}{3}} = 4$

$\left(p^{\frac{2}{3}}\right)^3 = 4^3$

$p^2 = 64$

$\therefore p = \sqrt{64}$

$= \sqrt{8\times8}$

$= \mathbf{8}$

12. (a) $(b^{-5}c^2)^{-1} = b^5c^{-2}$

$= \boldsymbol{\frac{b^5}{c^2}}$

(b) $5f^0 \div 3f^{-4} = \frac{5f^{0-(-4)}}{3}$

$= \boldsymbol{\frac{5f^4}{3}}$

(c) $(3c)^0 \div (c^{-3}d^5)^{-1} = \frac{1}{\left(c^{-3}d^5\right)^{-1}}$

$= c^{-3}d^5$

$= \boldsymbol{\frac{d^5}{c^3}}$

13. (a) $5a^4 \times 3a^2 \div a^{-3} = 15a^{4+2-(-3)}$

$= \boldsymbol{15a^9}$

(b) $-24b^{-6} \div (3b^{-3})^2 = -\frac{24b^{-6}}{3^2b^{-6}}$

$= -\frac{24}{9}$

$= \mathbf{-\frac{8}{3}}$

(c) $\frac{\left(4e^{-6}f^3\right)^2}{8e^{12}f^6} = \frac{4^2e^{-12}f^6}{8e^{12}f^6}$

$= \frac{16e^{-12-12}}{8}$

$= 2e^{-24}$

$= \boldsymbol{\frac{2}{e^{24}}}$

(d) $(3g^{-3}h^{-1})^2 \times (-4g^3h^{-2})^2 = 3^2g^{-6}h^{-2} \times (-4)^2g^6h^{-4}$

$= 9g^{-6}h^{-2} \times 16g^6h^{-4}$

$= 144g^{-6+6}h^{-2+(-4)}$

$= 144g^0h^{-6}$

$= \boldsymbol{\frac{144}{h^6}}$

(e) $\left(j^2k^{-1}\right)^{-3} \times \left(\frac{j^2}{k^3}\right)^{-3} = j^{-6}k^3 \times \frac{\left(j^2\right)^{-3}}{\left(k^3\right)^{-3}}$

$= j^{-6}k^3 \times \frac{j^{-6}}{k^{-9}}$

$= j^{-6+(-6)}k^{3-(-9)}$

$= j^{-12}k^{12}$

$= \boldsymbol{\frac{k^{12}}{j^{12}}}$

(f) $\frac{\left(m^5n^3\right)\times\left(m^2\right)^{-2}}{\left(m^{-1}n\right)^2} = \frac{\left(m^5n^3\right)\times m^{-4}}{m^{-2}n^2}$

$= \frac{m^{5+(-4)}n^3}{m^{-2}n^2}$

$= \frac{mn^3}{m^{-2}n^2}$

$= m^{1-(-2)}n^{3-2}$

$= \boldsymbol{m^3n}$

14. **(a)** $3a \div a^{-2} + a^2 \times a - \dfrac{6a^{-1}}{2a^{-4}} = 3a^{1-(-2)} + a^{2+1} - 3a^{-1-(-4)}$
$= 3a^3 + a^3 - 3a^3$
$= \boldsymbol{a^3}$

(b) $(5p)^3 - 10p \times 7p^2 + \dfrac{6}{p^{-3}} = 5^3p^3 - 70p^{1+2} + 6p^3$
$= 125p^3 - 70p^3 + 6p^3$
$= \mathbf{61}\boldsymbol{p^3}$

15. **(a)** $\sqrt[3]{a^2} \times \sqrt[6]{a} = a^{\frac{2}{3}} \times a^{\frac{1}{6}}$
$= a^{\frac{2}{3}+\frac{1}{6}}$
$= \boldsymbol{a^{\frac{5}{6}}}$

(b) $b^{\frac{4}{5}} \times b^{\frac{1}{2}} \div b^{-\frac{2}{5}} = b^{\frac{4}{5}+\frac{1}{2}-\left(-\frac{2}{5}\right)}$
$= \boldsymbol{b^{\frac{17}{10}}}$

(c) $c^{\frac{1}{10}} \div c^{-\frac{1}{5}} \times c^{-\frac{3}{2}} = c^{\frac{1}{10}-\left(-\frac{1}{5}\right)+\left(-\frac{3}{2}\right)}$
$= c^{-\frac{6}{5}}$
$= \dfrac{\mathbf{1}}{\boldsymbol{c^{\frac{6}{5}}}}$

(d) $\left(m^{\frac{1}{3}}n^{-10}\right)^{\frac{3}{5}} = m^{\frac{1}{5}}n^{-6}$
$= \dfrac{\boldsymbol{m^{\frac{1}{5}}}}{\boldsymbol{n^6}}$

(e) $\left(p^{\frac{2}{3}}q^{-\frac{4}{5}}\right)^{\frac{3}{2}} = pq^{-\frac{6}{5}}$
$= \dfrac{\boldsymbol{p}}{\boldsymbol{q^{\frac{6}{5}}}}$

(f) $\left(\dfrac{81x^{20}}{16x^8y^8}\right)^{-\frac{1}{4}} = \dfrac{1}{\left(\dfrac{81x^{20}}{16x^8y^8}\right)^{\frac{1}{4}}}$
$= \dfrac{1}{\dfrac{\left(81x^{20}\right)^{\frac{1}{4}}}{\left(16x^8y^8\right)^{\frac{1}{4}}}}$
$= \dfrac{\left(16x^8y^8\right)^{\frac{1}{4}}}{\left(81x^{20}\right)^{\frac{1}{4}}}$
$= \dfrac{16^{\frac{1}{4}}x^2y^2}{81^{\frac{1}{4}}x^5}$
$= \dfrac{\left(2^4\right)^{\frac{1}{4}}x^2y^2}{\left(3^4\right)^{\frac{1}{4}}x^5}$
$= \dfrac{2x^2y^2}{3x^5}$
$= \dfrac{2x^{2-5}y^2}{3}$
$= \dfrac{2x^{-3}y^2}{3}$
$= \dfrac{\boldsymbol{2y^2}}{\boldsymbol{3x^3}}$

16. **(a)** $\left(a^{-2}b^3\right)^{\frac{1}{3}} \times \left(a^4b^{-5}\right)^{\frac{1}{2}} = a^{-\frac{2}{3}}b \times a^2b^{-\frac{5}{2}}$
$= a^{-\frac{2}{3}+2}b^{1+\left(-\frac{5}{2}\right)}$
$= a^{\frac{4}{3}}b^{-\frac{3}{2}}$
$= \dfrac{\boldsymbol{a^{\frac{4}{3}}}}{\boldsymbol{b^{\frac{3}{2}}}}$

(b) $\left(c^{-3}d^{\frac{3}{5}}\right)^{-2} \times \left(c^{\frac{4}{5}}d^{-\frac{2}{5}}\right)^5 = c^6d^{-\frac{6}{5}} \times c^4d^{-2}$
$= c^{6+4}d^{-\frac{6}{5}+(-2)}$
$= c^{10}d^{-\frac{16}{5}}$
$= \dfrac{\boldsymbol{c^{10}}}{\boldsymbol{d^{\frac{16}{5}}}}$

(c) $\dfrac{e^{-\frac{1}{3}}f^{-\frac{1}{4}}}{\left(e^2f^{-\frac{1}{3}}\right)^{-2}} = \dfrac{e^{-\frac{1}{3}}f^{-\frac{1}{4}}}{e^{-4}f^{\frac{2}{3}}}$
$= e^{-\frac{1}{3}-(-4)}f^{-\frac{1}{4}-\frac{2}{3}}$
$= e^{\frac{11}{3}}f^{-\frac{11}{12}}$
$= \dfrac{\boldsymbol{e^{\frac{11}{3}}}}{\boldsymbol{f^{\frac{11}{12}}}}$

(d) $\left(\dfrac{g^{-2}h^2}{25}\right)^{-\frac{1}{2}} = \dfrac{1}{\left(\dfrac{g^{-2}h^2}{25}\right)^{\frac{1}{2}}}$
$= \dfrac{1}{\dfrac{\left(g^{-2}h^2\right)^{\frac{1}{2}}}{25^{\frac{1}{2}}}}$
$= \dfrac{25^{\frac{1}{2}}}{\left(g^{-2}h^2\right)^{\frac{1}{2}}}$
$= \dfrac{\left(5^2\right)^{\frac{1}{2}}}{g^{-1}h}$
$= \dfrac{5}{g^{-1}h}$
$= \dfrac{\boldsymbol{5g}}{\boldsymbol{h}}$

(e) $\left(4j^4k\right)^{\frac{1}{2}} \div 2h^3k^{-\frac{1}{2}} = \dfrac{4^{\frac{1}{2}}j^2k^{\frac{1}{2}}}{2h^3k^{-\frac{1}{2}}}$
$= \dfrac{(2 \times 2)^{\frac{1}{2}}j^2k^{\frac{1}{2}}}{2h^3k^{-\frac{1}{2}}}$
$= \dfrac{2j^2k^{\frac{1}{2}}}{2h^3k^{-\frac{1}{2}}}$
$= \dfrac{j^2k^{\frac{1}{2}-\left(-\frac{1}{2}\right)}}{h^3}$
$= \dfrac{\boldsymbol{j^2k}}{\boldsymbol{h^3}}$

(f) $\left(m^3n^{-\frac{1}{4}}\right)^4 \div \sqrt[5]{32m^4n^{-8}} = \dfrac{\left(m^3n^{-\frac{1}{4}}\right)^4}{\left(32m^4n^{-8}\right)^{\frac{1}{5}}}$
$= \dfrac{m^{12}n^{-1}}{32^{\frac{1}{5}}m^{\frac{4}{5}}n^{-\frac{8}{5}}}$
$= \dfrac{m^{12}n^{-1}}{\left(2^5\right)^{\frac{1}{5}}m^{\frac{4}{5}}n^{-\frac{8}{5}}}$
$= \dfrac{m^{12}n^{-1}}{2m^{\frac{4}{5}}n^{-\frac{8}{5}}}$
$= \dfrac{m^{12-\frac{4}{5}}n^{-1-\left(-\frac{8}{5}\right)}}{2}$
$= \dfrac{\boldsymbol{m^{\frac{56}{5}}n^{\frac{3}{5}}}}{\mathbf{2}}$

17. (a) $\left(\dfrac{x^{-4}y^7z^{-6}}{x^3y^{-1}z^3}\right)^3 \times \left(\dfrac{x^5y^2z^{-6}}{x^{-3}y^{-5}z^4}\right)^{-4} = \dfrac{\left(x^{-4}y^7z^{-6}\right)^3}{\left(x^3y^{-1}z^3\right)^3} \times \dfrac{\left(x^5y^2z^{-6}\right)^{-4}}{\left(x^{-3}y^{-5}z^4\right)^{-4}}$

$= \dfrac{x^{-12}y^{21}z^{-18}}{x^9y^{-3}z^9} \times \dfrac{x^{-20}y^{-8}z^{24}}{x^{12}y^{20}z^{-16}}$

$= \dfrac{x^{-12+(-20)}y^{21+(-8)}z^{-18+24}}{x^{9+12}y^{-3+20}z^{9+(-16)}}$

$= \dfrac{x^{-32}y^{13}z^6}{x^{21}y^{17}z^{-7}}$

$= x^{-32-21}y^{13-17}z^{6-(-7)}$

$= x^{-53}y^{-4}z^{13}$

$= \boldsymbol{\dfrac{z^{13}}{x^{53}y^4}}$

(b) $\left(\dfrac{x^3y^{-4}z^7}{x^{-5}y^2}\right)^3 \div \left(\dfrac{x^{-4}yz^{-5}}{x^7y^{-3}}\right)^{-2} = \left(\dfrac{x^3y^{-4}z^7}{x^{-5}y^2}\right)^3 \times \dfrac{1}{\left(\dfrac{x^{-4}yz^{-5}}{x^7y^{-3}}\right)^{-2}}$

$= \left(\dfrac{x^3y^{-4}z^7}{x^{-5}y^2}\right)^3 \times \left(\dfrac{x^{-4}yz^{-5}}{x^7y^{-3}}\right)^2$

$= \dfrac{\left(x^3y^{-4}z^7\right)^3}{\left(x^{-5}y^2\right)^3} \times \dfrac{\left(x^{-4}yz^{-5}\right)^2}{\left(x^7y^{-3}\right)^2}$

$= \dfrac{x^9y^{-12}z^{21}}{x^{-15}y^6} \times \dfrac{x^{-8}y^2z^{-10}}{x^{14}y^{-6}}$

$= \dfrac{x^{9+(-8)}y^{-12+2}z^{21+(-10)}}{x^{-15+14}y^{6+(-6)}}$

$= \dfrac{xy^{-10}z^{11}}{x^{-1}y^0}$

$= x^{1-(-1)}y^{-10}z^{11}$

$= x^2y^{-10}z^{11}$

$= \boldsymbol{\dfrac{x^2z^{11}}{y^{10}}}$

(c) $\dfrac{ab^n}{bc} \times \dfrac{c^nd}{cd} \div \dfrac{b^{n+2}}{c^{n+3}} = \dfrac{ab^n}{bc} \times \dfrac{c^nd}{cd} \times \dfrac{c^{n+3}}{b^{n+2}}$

$= \dfrac{ab^nc^{n+n+3}d}{b^{1+n+2}c^{1+1}d}$

$= \dfrac{ab^nc^{2n+3}d}{b^{n+3}c^2d}$

$= ab^{n-(n+3)}c^{2n+3-2}$

$= ab^{-3}c^{2n+1}$

$= \boldsymbol{\dfrac{ac^{2n+1}}{b^3}}$

(d) $\dfrac{(a+b)^n}{bc^2} \div \dfrac{(a+b)^{n+3}}{abc} = \dfrac{(a+b)^n}{bc^2} \times \dfrac{abc}{(a+b)^{n+3}}$

$= ac^{1-2}(a+b)^{n-(n+3)}$

$= ac^{-1}(a+b)^{-3}$

$= \boldsymbol{\dfrac{a}{c(a+b)^3}}$

4.5 Surds

Investigation (Simplifying surds)

1. (a) **No.**
(b) **No.**
(c) **Yes.**
(d) **Yes.**

2. (c) Using the Laws of Indices,

$(ab)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}$ if bases $a, b > 0$

$\therefore \sqrt{ab} = \sqrt{a} \times \sqrt{b}$ (proven)

(d) Using the Laws of Indices,

$\left(\dfrac{a}{b}\right)^{\frac{1}{2}} = \dfrac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ if bases $a, b > 0$

$\therefore \sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ (proven)

3. $\sqrt{a} \times \sqrt{a} = a^{\frac{1}{2}} \times a^{\frac{1}{2}}$

$= (a \times a)^{\frac{1}{2}}$

$= (a^2)^{\frac{1}{2}}$

$= \boldsymbol{a}$

Practise Now 14

(a) $\sqrt{27} \times \sqrt{3} = \sqrt{81}$
$= \mathbf{9}$

(b) $\dfrac{\sqrt{125}}{\sqrt{5}} = \sqrt{\dfrac{125}{5}}$
$= \sqrt{25}$
$= \mathbf{5}$

(c) $\sqrt{12} = \sqrt{4 \times 3}$
$= \mathbf{2\sqrt{3}}$

(d) $\dfrac{\sqrt{80} \times \sqrt{12}}{\left(\sqrt{16}\right)^2} = \dfrac{4\sqrt{5} \times 2\sqrt{3}}{16}$
$= \boldsymbol{\dfrac{\sqrt{15}}{2}}$

Class Discussion (Mathematical fallacy: 1 = –1?)

$\sqrt{-1}$ is undefined.

Practise Now 15

(a) $\sqrt{75} + \sqrt{108} = 5\sqrt{3} + 6\sqrt{3}$
$= \mathbf{11\sqrt{3}}$

(b) $\sqrt{80} - \sqrt{20} = 4\sqrt{5} - 2\sqrt{5}$
$= \mathbf{2\sqrt{5}}$

(c) $\sqrt{24} + \sqrt{54} - \sqrt{216} = 2\sqrt{6} + 3\sqrt{6} - 6\sqrt{6}$
$= \mathbf{-\sqrt{6}}$

Practise Now 16

(a) $\left(7+2\sqrt{3}\right)\left(5-\sqrt{3}\right) = 35 - 7\sqrt{3} + 10\sqrt{3} - 6$

$= \mathbf{29 + 3\sqrt{3}}$

(b) $\left(4-3\sqrt{2}\right)^2 = 16 - 24\sqrt{2} + 18$

$= \mathbf{34 - 24\sqrt{2}}$

(c) $\left(3+2\sqrt{5}\right)\left(3-2\sqrt{5}\right) = 9 - 20$

$= \mathbf{-11}$

(d) $\left(3\sqrt{6}+4\sqrt{2}\right)^2 = 54 + 48\sqrt{3} + 32$

$= \mathbf{86 + 48\sqrt{3}}$

Class Discussion (Product of two irrational numbers)

1. $3 + 2\sqrt{5}$: Plus the irrational part

$3 - 2\sqrt{5}$: Minus the same irrational part

So the product of these two irrational numbers will be a rational number because

$\left(a+b\sqrt{c}\right)\left(a-b\sqrt{c}\right) = a^2 - \left(b\sqrt{c}\right)^2$

$= a^2 - b^2c$ is rational.

2. $\mathbf{1 + \sqrt{2}}$ **and** $\mathbf{1 - \sqrt{2}}$

3. $\sqrt{\mathbf{5}}$

Thinking Time (Page 107)

1. $\left(p+q\sqrt{a}\right)\left(p-q\sqrt{a}\right) = p^2 - \left(q\sqrt{a}\right)^2$

$= p^2 - aq^2$

2. Let $p = 0$ and $q = 1$.

$\left(p+q\sqrt{a}\right)\left(p-q\sqrt{a}\right) = 0^2 - a(1)^2$

$= \boldsymbol{-a}$

3. **Yes**, if a, p and q are rational.

$\left(q\sqrt{a}+p\right)\left(q\sqrt{a}-p\right) = \left(q\sqrt{a}\right)^2 - p^2$

$= aq^2 - p^2$

4. Let $p = 0$ and $q = 1$.

$\left(q\sqrt{a}+p\right)\left(q\sqrt{a}-p\right) = a(1)^2 - 0^2$

$= \boldsymbol{a}$

5. $\boldsymbol{p\sqrt{a} - q\sqrt{b}}$.

Using the algebraic identity $(x + y)(x - y) = x^2 - y^2$,

$\left(q\sqrt{a}+q\sqrt{b}\right)\left(p\sqrt{a}-q\sqrt{a}\right) = \left(p\sqrt{a}\right)^2 - \left(q\sqrt{b}\right)^2$

$= ap^2 - bq^2$

Practise Now 17

1. **(a)** $\dfrac{12}{\sqrt{3}} = \dfrac{12}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \mathbf{4\sqrt{3}}$

(b) $\dfrac{22}{4+\sqrt{5}} = \dfrac{22}{4+\sqrt{5}} \times \dfrac{4-\sqrt{5}}{4-\sqrt{5}}$

$= \dfrac{88-22\sqrt{5}}{16-5}$

$= \mathbf{8 - 2\sqrt{5}}$

(c) $\dfrac{5}{2\sqrt{6}-3} = \dfrac{5}{2\sqrt{6}-3} \times \dfrac{2\sqrt{6}+3}{2\sqrt{6}+3}$

$= \dfrac{10\sqrt{6}+15}{24-9}$

$= \dfrac{\mathbf{2\sqrt{6}+3}}{\mathbf{3}}$

(d) $\dfrac{5}{4-3\sqrt{3}} + \dfrac{7}{3\sqrt{3}+4} = \dfrac{5\left(3\sqrt{3}+4\right)+7\left(4-3\sqrt{3}\right)}{16-27}$

$= \dfrac{15\sqrt{3}+20+28-21\sqrt{3}}{-11}$

$= \dfrac{\mathbf{6\sqrt{3}-48}}{\mathbf{11}}$

2. $\left(4-\sqrt{6}\right)^2 - \dfrac{6}{3-\sqrt{6}} = 16 - 8\sqrt{6} + 6 - \dfrac{6}{3-\sqrt{6}} \times \dfrac{3+\sqrt{6}}{3+\sqrt{6}}$

$= 22 - 8\sqrt{6} - \dfrac{18+6\sqrt{6}}{9-6}$

$= 22 - 8\sqrt{6} - 6 - 2\sqrt{6}$

$= \mathbf{16 - 10\sqrt{6}}$

3. $\dfrac{4}{\left(3-\sqrt{5}\right)^2} = \dfrac{4}{\left(3-\sqrt{5}\right)^2} \times \dfrac{\left(3+\sqrt{5}\right)^2}{\left(3+\sqrt{5}\right)^2}$

$= \dfrac{4\left(3+\sqrt{5}\right)^2}{(9-5)^2}$

$= \dfrac{\left(3+\sqrt{5}\right)^2}{4}$

$= \sqrt{h+k\sqrt{5}}$

$\left[\dfrac{\left(3+\sqrt{5}\right)^2}{4}\right]^2 = h + k\sqrt{5}$

$\dfrac{\left(14+6\sqrt{5}\right)^2}{16} = h + k\sqrt{5}$

$23.5 + 10.5\sqrt{5} = h + k\sqrt{5}$

$\therefore \boldsymbol{h} = \dfrac{\mathbf{47}}{\mathbf{2}}$ **and** $\boldsymbol{k} = \dfrac{\mathbf{21}}{\mathbf{2}}$

Investigation (Rational and irrational roots of quadratic equations)

1. **(a)** $2x^2 + 3x - 2 = 0$

$(2x - 1)(x + 2) = 0$

$x = \dfrac{\mathbf{1}}{\mathbf{2}}$ **or** $\mathbf{-2}$

(b) $x^2 + 2x - 1 = 0$

Using the Quadratic Formula,

$x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$

$= \dfrac{-2 \pm \sqrt{8}}{2}$

$= \dfrac{-2 \pm 2\sqrt{2}}{2}$

$= \mathbf{-1 \pm \sqrt{2}}$

2. (i) The equation $2x^2 + 3x - 2 = 0$ can be solved by the factorisation method. Its roots are rational.
(ii) The equation $x^2 + 2x - 1 = 0$ has irrational roots.
(iii) For a quadratic equation $ax^2 + bx + c = 0$, the formula for the general solution is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The roots are rational when the value of $b^2 - 4ac$ is a perfect square, and irrational when the value of $b^2 - 4ac$ is not a perfect square.

3. The roots for a quadratic equation can be expressed as $\dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$.
Let $X = \dfrac{-b}{2a}$, $Y = \dfrac{1}{2a}$ and $Z = b^2 - 4ac$.
For roots that are irrational, $\sqrt{Z}$ is an irrational number.
The product of roots $= \left(X + Y\sqrt{Z}\right)\left(X - Y\sqrt{Z}\right)$
$= X^2 + Y^2Z$ is rational.
$\therefore$ if the roots of $ax^2 + bx + c = 0$ are irrational, then they must be congugate surds, provided that a, b and c are rational.

Practise Now 18

1. $$\text{Breadth of rectangle} = \frac{\text{Area}}{\text{Length}} = \frac{7-\sqrt{3}}{5+\sqrt{3}} = \frac{7-\sqrt{3}}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{38-12\sqrt{3}}{25-3} = \left(\mathbf{\frac{19}{11} - \frac{6}{11}\sqrt{3}}\right)\ \textbf{cm}$$

2. $$\text{Height of cuboid} = \frac{\text{Volume}}{\text{Area of base}} = \frac{15+6\sqrt{3}}{\left(2+\sqrt{3}\right)^2} = \frac{15+6\sqrt{3}}{\left(2+\sqrt{3}\right)^2} \times \frac{\left(2-\sqrt{3}\right)^2}{\left(2-\sqrt{3}\right)^2} = \frac{33-18\sqrt{3}}{(4-3)^2} = \left(\mathbf{33 - 18\sqrt{3}}\right)\ \textbf{cm}$$

Exercise 4C

1. (a) $\sqrt{2} \times \sqrt{32} = \sqrt{64} = \mathbf{8}$
(b) $\dfrac{\sqrt{343}}{\sqrt{7}} = \sqrt{49} = \mathbf{7}$
(c) $\sqrt{63} = \mathbf{3\sqrt{7}}$
(d) $\dfrac{\sqrt{75} \times \sqrt{72}}{\sqrt{24}} = \dfrac{5\sqrt{3} \times 6\sqrt{2}}{2\sqrt{6}} = \mathbf{15}$

2. (a) $\sqrt{112} + \sqrt{28} = 4\sqrt{7} + 2\sqrt{7} = \mathbf{6\sqrt{7}}$
(b) $\sqrt{180} - \sqrt{125} = 6\sqrt{5} - 5\sqrt{5} = \mathbf{\sqrt{5}}$
(c) $\sqrt{48} + \sqrt{12} - \sqrt{27} = 4\sqrt{3} + 2\sqrt{3} - 3\sqrt{3} = \mathbf{3\sqrt{3}}$
(d) $-\sqrt{11} - \sqrt{99} + \sqrt{44} = -\sqrt{11} - 3\sqrt{11} + 2\sqrt{11} = \mathbf{-2\sqrt{11}}$
(e) $\sqrt{240} - \sqrt{12} \times \sqrt{45} = 4\sqrt{15} - 2\sqrt{3} \times 3\sqrt{5} = \mathbf{-2\sqrt{15}}$
(f) $\dfrac{\sqrt{245} - \sqrt{20}}{\sqrt{500}} = \dfrac{7\sqrt{5} - 2\sqrt{5}}{10\sqrt{5}} = \mathbf{\dfrac{1}{2}}$

3. (a) $\left(5+\sqrt{2}\right)\left(6-3\sqrt{2}\right) = 30 - 15\sqrt{2} + 6\sqrt{2} - 6 = \mathbf{24 - 9\sqrt{2}}$
(b) $\left(3+2\sqrt{6}\right)^2 = 9 + 12\sqrt{6} + 24 = \mathbf{33 + 12\sqrt{6}}$
(c) $\left(3\sqrt{11}-4\right)^2 = 99 - 24\sqrt{11} + 16 = \mathbf{115 - 24\sqrt{11}}$
(d) $\left(7\sqrt{3}-3\sqrt{7}\right)^2 = 147 - 42\sqrt{21} + 63 = \mathbf{210 - 42\sqrt{21}}$
(e) $\left(9-2\sqrt{5}\right)\left(9+2\sqrt{5}\right) = 81 - 20 = \mathbf{61}$
(f) $\left(2\sqrt{7}+3\sqrt{5}\right)\left(2\sqrt{7}-3\sqrt{5}\right) = 28 - 45 = \mathbf{-17}$

4. (a) $\dfrac{10}{\sqrt{5}} = \dfrac{10}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} = \mathbf{2\sqrt{5}}$
(b) $\dfrac{1}{3\sqrt{10}} = \dfrac{1}{3\sqrt{10}} \times \dfrac{3\sqrt{10}}{3\sqrt{10}} = \mathbf{\dfrac{\sqrt{10}}{30}}$
(c) $\dfrac{7}{2+\sqrt{3}} = \dfrac{7}{2+\sqrt{3}} \times \dfrac{2-\sqrt{3}}{2-\sqrt{3}} = \mathbf{14 - 7\sqrt{3}}$
(d) $\dfrac{9}{8-\sqrt{6}} = \dfrac{9}{8-\sqrt{6}} \times \dfrac{8+\sqrt{6}}{8+\sqrt{6}} = \mathbf{\dfrac{72 + 9\sqrt{6}}{58}}$
(e) $\dfrac{5}{4\sqrt{3}-2} = \dfrac{5}{4\sqrt{3}-2} \times \dfrac{4\sqrt{3}+2}{4\sqrt{3}+2} = \mathbf{\dfrac{5 + 10\sqrt{3}}{22}}$
(f) $\dfrac{4}{2\sqrt{7}+3} = \dfrac{4}{2\sqrt{7}+3} \times \dfrac{2\sqrt{7}-3}{2\sqrt{7}-3} = \mathbf{\dfrac{8\sqrt{7} - 12}{19}}$

5. **(a)** $\dfrac{\sqrt{3}}{2\sqrt{5}+8} = \dfrac{\sqrt{3}}{2\sqrt{5}+8} \times \dfrac{2\sqrt{5}-8}{2\sqrt{5}-8}$

$= \dfrac{\mathbf{4\sqrt{3}-\sqrt{15}}}{\mathbf{22}}$

(b) $\dfrac{8}{2\sqrt{5}+3} - \dfrac{4}{2\sqrt{5}-3} = \dfrac{8\sqrt{5}-36}{20-9}$

$= \dfrac{\mathbf{8\sqrt{5}-36}}{\mathbf{11}}$

6. **(a)** $\dfrac{3}{\sqrt{8}} + \dfrac{5}{\sqrt{2}} - \dfrac{\sqrt{32}}{3} = \dfrac{3(3)+5(6)-\sqrt{32}\left(2\sqrt{2}\right)}{6\sqrt{2}}$

$= \dfrac{39-16}{6\sqrt{2}}$

$= \dfrac{23}{6\sqrt{2}} \times \dfrac{6\sqrt{2}}{6\sqrt{2}}$

$= \dfrac{\mathbf{23\sqrt{2}}}{\mathbf{12}}$

(b) $\dfrac{4}{\sqrt{27}} - \dfrac{\sqrt{18}}{4} + \dfrac{4}{\sqrt{3}} = \dfrac{4(4)-\left(3\sqrt{2}\right)\left(3\sqrt{3}\right)+4(12)}{12\sqrt{3}}$

$= \dfrac{64-9\sqrt{6}}{12\sqrt{3}} \times \dfrac{12\sqrt{3}}{12\sqrt{3}}$

$= \dfrac{\mathbf{64\sqrt{3}-27\sqrt{2}}}{\mathbf{36}}$

(c) $\dfrac{2}{\sqrt{3}}\left(\dfrac{4}{\sqrt{12}} + \dfrac{\sqrt{27}}{3}\right) = \dfrac{8}{6} + 2$

$= \dfrac{\mathbf{10}}{\mathbf{3}}$

(d) $\dfrac{6}{\sqrt{2}}\left(\dfrac{3}{\sqrt{8}} - \dfrac{\sqrt{128}}{3}\right) = \dfrac{18}{4} - 16$

$= -\dfrac{\mathbf{23}}{\mathbf{2}}$

7. **(a)** $\dfrac{13}{\left(\sqrt{3}+4\right)^2} = \dfrac{13}{\left(\sqrt{3}+4\right)^2} \times \dfrac{\left(\sqrt{3}-4\right)^2}{\left(\sqrt{3}-4\right)^2}$

$= \dfrac{13\left(\sqrt{3}-4\right)^2}{(3-16)^2}$

$= \dfrac{\mathbf{19-8\sqrt{3}}}{\mathbf{13}}$

(b) $\dfrac{3}{\left(\sqrt{2}+6\right)^2} + \dfrac{5}{\left(\sqrt{2}-6\right)^2}$

$= \dfrac{3\left(\sqrt{2}-6\right)^2 + 5\left(\sqrt{2}+6\right)^2}{(2-36)^2}$

$= \dfrac{3\left(2-12\sqrt{2}+36\right)+5\left(2+12\sqrt{2}+36\right)}{(-34)^2}$

$= \dfrac{\mathbf{76+6\sqrt{2}}}{\mathbf{289}}$

(c) $\dfrac{3\sqrt{2}-6}{6+3\sqrt{2}} = \dfrac{3\sqrt{2}-6}{6+3\sqrt{2}} \times \dfrac{3\sqrt{2}-6}{3\sqrt{2}-6}$

$= \dfrac{18-36\sqrt{2}+36}{(18-36)}$

$= \mathbf{2\sqrt{2}-3}$

(d) $\dfrac{\sqrt{48}-\sqrt{50}}{\sqrt{27}-\sqrt{8}} = \dfrac{4\sqrt{3}-5\sqrt{2}}{3\sqrt{3}-2\sqrt{2}} \times \dfrac{3\sqrt{3}+2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}$

$= \dfrac{36+8\sqrt{6}-15\sqrt{6}-20}{27-8}$

$= \dfrac{\mathbf{16-7\sqrt{6}}}{\mathbf{19}}$

8. $\left(9-\sqrt{3}\right)^2 - \dfrac{78}{\sqrt{3}+9} = \left(81-18\sqrt{3}+3\right) - \dfrac{78}{\sqrt{3}+9} \times \dfrac{9-\sqrt{3}}{9-\sqrt{3}}$

$= 84 - 18\sqrt{3} - 9 + \sqrt{3}$

$= \mathbf{75 - 17\sqrt{3}}$

9. $\dfrac{\left(3+\sqrt{2}\right)^2+1}{3+\sqrt{2}-2} = \dfrac{12+6\sqrt{2}}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}}$

$= \dfrac{-6\sqrt{2}}{-1}$

$= \mathbf{6\sqrt{2}}$

10. Length of rectangle $= \dfrac{\text{Area}}{\text{Breadth}}$

$= \dfrac{2\sqrt{6}}{3-\sqrt{6}}$

$= \dfrac{2\sqrt{6}}{3-\sqrt{6}} \times \dfrac{3+\sqrt{6}}{3+\sqrt{6}}$

$= \dfrac{6\sqrt{6}+12}{9-6}$

$= \left(\mathbf{4+2\sqrt{6}}\right)$ **cm**

11. Height of right circular cylinder $= \dfrac{\text{Volume}}{\text{Base area}}$

$= \dfrac{\pi\left(6+2\sqrt{3}\right)}{\pi\left(1+\sqrt{3}\right)^2}$

$= \dfrac{6+2\sqrt{3}}{\left(1+\sqrt{3}\right)^2} \times \dfrac{\left(1-\sqrt{3}\right)^2}{\left(1-\sqrt{3}\right)^2}$

$= \dfrac{\left(6+2\sqrt{3}\right)\left(1-2\sqrt{3}+3\right)}{(1-3)^2}$

$= \dfrac{12-4\sqrt{3}}{4}$

$= \left(\mathbf{3-\sqrt{3}}\right)$ **cm**

12. **(i)** Length $AC = \dfrac{2 \times \text{Area of triangle}}{\text{Length } AB \times \sin 60^\circ}$

$= \dfrac{2\left(\dfrac{1}{4}\left(12+9\sqrt{3}\right)\right)}{\left(\sqrt{3}+2\right)\dfrac{\sqrt{3}}{2}}$

$= \dfrac{12+9\sqrt{3}}{3+2\sqrt{3}} \times \dfrac{3-2\sqrt{3}}{3-2\sqrt{3}}$

$= \dfrac{36-24\sqrt{3}+27\sqrt{3}-54}{9-12}$

$= \left(\mathbf{6-\sqrt{3}}\right)$ **cm**

(ii) $BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos 60^\circ$

$= \left(\sqrt{3}+2\right)^2 + \left(6-\sqrt{3}\right)^2 - 2\left(\sqrt{3}+2\right)^2\left(6-\sqrt{3}\right)\left(\dfrac{1}{2}\right)$

$= 3 + 4\sqrt{3} + 4 + 36 - 12\sqrt{3} + 3 - 6\sqrt{3} + 3 - 12 + 2\sqrt{3}$

$= \left(\mathbf{37-12\sqrt{3}}\right)$ $\mathbf{cm^2}$

13. (i) $b = \dfrac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$

$= \dfrac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} \times \dfrac{1+\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$

$= \dfrac{1+\frac{2}{\sqrt{2}}+\frac{1}{2}}{\frac{1}{2}}$

$= 2\left(\dfrac{3}{2}+\dfrac{2}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}\right)$

$= \mathbf{3 + 2\sqrt{2}}$

(ii) $b - \dfrac{1}{b} = 3 + 2\sqrt{2} - \dfrac{1}{3+2\sqrt{2}} \times \dfrac{3-2\sqrt{2}}{3-2\sqrt{2}}$

$= 3 + 2\sqrt{2} - 3 + 2\sqrt{2}$

$= \mathbf{4\sqrt{2}}$

14. $\sqrt{a+b\sqrt{11}} = \dfrac{c}{\left(\sqrt{11}-3\right)^2}$

$a + b\sqrt{11} = \dfrac{c^2}{\left(\sqrt{11}-3\right)^4} \times \dfrac{\left(\sqrt{11}+3\right)^4}{\left(\sqrt{11}+3\right)^4}$

$a + b\sqrt{11} = \dfrac{c^2\left(20+6\sqrt{11}\right)^2}{16}$

$16a + 16b\sqrt{11} = 796c^2 + 240c^2\sqrt{11}$

$16a = 796c^2$

$16b = 240c^2$

$c^2 = \dfrac{4}{199}a = \dfrac{1}{15}b$

$a = \dfrac{199}{60}b$

Additional conditions:

$a + b\sqrt{11} \geqslant 0 \rightarrow b\left(\dfrac{199}{60}+\sqrt{11}\right) \geqslant 0 \rightarrow b \geqslant 0$

$c \geqslant 0$

A possible solution:

$\mathbf{b = 15, a = \dfrac{199}{4}, c = 1}$

15. Based on Waseem's answer, the quadratic equation

$= a\left[x-\left(2-\sqrt{3}\right)\right]\left[x-\left(1+\sqrt{3}\right)\right]$

$= a\left[x^2-\left(1+\sqrt{3}+2-\sqrt{3}\right)x+2+2\sqrt{3}-\sqrt{3}-3\right]$

$= a\left(x^2-x+\sqrt{3}-1\right)$, where a is rational.

Since the quadratic equation has surds, it is different from the initial equation and hence Waseem's answer is wrong.

4.6 Exponential growth and decay

Practise Now 19

Since the value increases by 3.5% each year, the value is 103.5% = 1.035 times of that in the preceding year.

Value of house after 10 years = $\$100\ 000 \times 1.035^{10}$

= **\$141 000** (to 3 s.f.)

Practise Now 20

Since the value depreciates by 7.5% each year, the value is 92.5% = 0.925 times of that in the preceding year.

Value of car after 6 years = $\$20\ 000 \times 0.925^6$

= **\$12 500** (to 3 s.f.)

Exercise 4D

1. Since the number of bacteria doubles every hour, the number of bacteria is 200% = 2 times of that at the preceding hour.

Number of bacteria after 6 hours = 50×2^6

= **3200**

2. Since the number of tourists declines by 12% each year, the number of tourists is 88% = 0.88 times of that in the preceding year.

After 12 years,

number of tourists in 2030 = $22\ 000 \times 0.88^{12}$

= **4740** (to 3 s.f.)

3. (i) Since the population increase by 1% daily, the population is 101% = 1.01 times of that on the preceding day.

Population, P, after d days = 1000×1.01^d

$\mathbf{P = 1000(1.01)^d}$

(ii) Population after 30 days = $1000(1.01)^{30}$

= **1350** (to 3 s.f.)

4. (i) Initial value of painting, $\$P = 2000 \times 1.1^0$

= **\$2000**

(ii) After 10 years,

value of painting in 2030, $\$P = 2000 \times 1.1^{10}$

= **\$5190** (to 3 s.f.)

5. (i) Amount Yasir spent on car, $\$P = 25\ 000 \times 0.86^0$

= **\$25 000**

(ii) Since the value of the car is 0.86 times = 86% of that in the preceding year, percentage decrease = 100% – 86%

= **14%**

6. (i) Initial number of customers, $N = 30 \times 1.2^0$

= **30**

(ii) Number of customers, N, after 3 years = 30×1.2^3

= **52** (to the nearest whole number)

(iii)

Date	t	N
1 January 2023	1	36
1 January 2024	2	43 (to the nearest whole number)
1 January 2025	3	52 (to the nearest whole number)
1 January 2026	4	62 (to the nearest whole number)
1 January 2027	5	75 (to the nearest whole number)
1 January 2028	6	90 (to the nearest whole number)
1 January 2029	7	107 (to the nearest whole number)

From 1 January 2028 to 1 January 2029, the number of customers ranges from 90 to 107.

$\therefore$ the company will have 100 customers in the year **2028**.

7. **(i)**

Number of hours after start of experiment	Mass of radioactive sample (g)
6	300
4.5	$300 \div \frac{1}{2} = 600$
3	$600 \div \frac{1}{2} = 1200$
1.5	$1200 \div \frac{1}{2} = 2400$
0	$2400 \div \frac{1}{2} = 4800$

$\therefore$ the initial mass of the radioactive sample was **4800 g**.

(ii) Let the mass of the radioactive sample after t intervals of 1.5 hours each be m g.

Since the mass of the sample halves every 1.5 hours, the mass of the sample is 50% = 0.5 times of that in the preceding 1.5-hour interval.

$\therefore m = 4800 \times 0.5^t$

Number of 1.5-hour intervals in 24 hours $= \frac{24}{1.5}$

$= 16$

Mass of sample, m g, after 24 hours $= 4800 \times 0.5^{16}$

$=$ **0.0732 g** (to 3 s.f.)

(iii)

Number of hours after start of experiment	Mass of radioactive sample (g)
6	300
7.5	$300 \times \frac{1}{2} = 150$
9	$150 \times \frac{1}{2} = 75$

$\therefore$ the experiment ends after **9 hours**.

8. Let the population of the town after t intervals of 2 years each be P.

Since the population increases by 10% in the 1st year followed by a decrease of 10% in the 2nd year,

$P = 50\ 000 \times (1.1 \times 0.9)^t$

$= 50\ 000 \times 0.99^t$

Number of 2-year intervals in 40 years $= \frac{40}{2}$

$= 20$

Population, P, after 40 years $= 50\ 000 \times 0.99^{20}$

$=$ **40 895** (to the nearest whole number)

4.7 Standard form

Class Discussion (Exploring standard form)

1. The powers of 10 are all positive integers.

2. The powers of 10 are all negative integers.

3. **(v)** 3×10^8 m/s

(vi) 3.8×10^{-5} cm

(vii) 2.99×10^{-23} g

4. For a number in the form $A \times 10^n$ to be considered a standard form, $1 \leqslant A < 10$ and n must be an integer. If the number is greater than 1, $n \geqslant 0$; if the number is between 0 and 1, $n < 0$. By doing so, very big and very small numbers can be expressed in a precise and concise manner.

Practise Now 21

1. **(a)** $5\ 300\ 000 = 5.3 \times 1\ 000\ 000$

$= \mathbf{5.3 \times 10^6}$

(b) $600\ 000\ 000 = 6 \times 100\ 000\ 000$

$= \mathbf{6 \times 10^8}$

(c) $0.000\ 048 = 4.8 \times 0.000\ 01$

$= \mathbf{4.8 \times 10^{-5}}$

(d) $0.000\ 000\ 000\ 167 = 1.67 \times 0.000\ 000\ 0001$

$= \mathbf{1.67 \times 10^{-10}}$

2. **(a)** $1.325 \times 10^6 = 1.325 \times 1\ 000\ 000$

$= \mathbf{1\ 325\ 000}$

(b) $4.4 \times 10^{-3} = 4.4 \times 0.001$

$= \mathbf{0.0044}$

Practise Now 22

(a) 4.0 terabytes = $\mathbf{4.0 \times 10^{12}}$ **bytes**

(b) 25.4 micrometres $= 25.4 \times 10^{-6}$ m

$= 2.54 \times 10 \times 10^{-6}$ m

$= 2.54 \times 10^{1 + (-6)}$ mv

$= \mathbf{2.54 \times 10^{-5}}$ **m**

(c) 494 mm $= 494 \times 10^{-3}$ m

$= 4.94 \times 10^2 \times 10{-}3$ m

$= 4.94 \times 10^{2 + (-3)}$ m

$= \mathbf{4.94 \times 10^{-1}}$ **m**

Performance Task (Page 116)

1 GB = 1 073 741 824 bytes

All computer data is stored in a binary format as either a one or a zero, and units of measurement are defined in terms of powers of 2. By convention, the kilobyte is used to represent 2^{10} bytes, the megabyte is used to represent $(2^{10})^2$ bytes, the gigabyte is used to represent $(2^{10})^3$ bytes, and so on. As such, 1 GB = 2^{30} bytes.

It is not possible to produce a thumbdrive with exactly 1 billion bytes since 10^9 cannot be expressed in the form 2^x, where x is an integer.

This is also the reason why computer storage systems always come in the form of 128 MB (= 2^7 MB), 256 MB (= 2^8 MB), 512 MB (2^9 MB), and so on.

Practise Now 23

(a) Sequence of calculator keys:

(1 . 1 4 $\times 10^x$ 5)
$\times$ (4 . 5 6 $\times 10^x$ 4) =

$(1.14 \times 10^5) \times (4.56 \times 10^4) = 5\ 198\ 400\ 000$
$= 5.1984 \times 10^9$
$= \mathbf{5.20 \times 10^9}$ (to 3 s.f.)

(b) Sequence of calculator keys:

(4 . 2 $\times 10^x$ (–) 4)
$\times$ (2 . 6 $\times 10^x$ 2) =

$(4.2 \times 10^{-4}) \times (2.6 \times 10^2) = 0.1092$
$= 1.092 \times 10^{-1}$
$= \mathbf{1.09 \times 10^{-1}}$ (to 3 s.f.)

(c) Sequence of calculator keys:

(2 . 4 $\times 10^x$ 8)
$\div$ (6 $\times 10^x$ 4) =

$(2.4 \times 10^8) \div (6 \times 10^4) = 4000$
$= \mathbf{4 \times 10^3}$

(d) Sequence of calculator keys:

$\frac{\square}{\square}$ 3 . 5 $\times 10^x$ (–) 5 ▶
1 . 4 $\times 10^x$ 8 =

$\dfrac{3.5 \times 10^{-5}}{1.4 \times 10^8} = \mathbf{2.5 \times 10^{-13}}$

(e) Sequence of calculator keys:

1 . 1 4 $\times 10^x$ 5
+ 4 . 5 6 $\times 10^x$ 4 =

$1.14 \times 10^5 + 4.56 \times 10^4 = 159\ 600$
$= 1.596 \times 10^5$
$= \mathbf{1.60 \times 10^5}$ (to 3 s.f.)

(f) Sequence of calculator keys:

2 . 6 $\times 10^x$ 6 – 4 $\times 10^x$ 4 =

$2.6 \times 10^6 - 4 \times 10^4 = 2\ 560\ 000$
$= \mathbf{2.56 \times 10^6}$

(g) Sequence of calculator keys:

$\frac{\square}{\square}$ 2 . 3 7 $\times 10^x$ (–) 3 + 3 . 2 5
$\times 10^x$ (–) 4 ▶ 4 . 1 $\times 10^x$ 5 =

$\dfrac{2.37 \times 10^{-3} + 3.25 \times 10^{-4}}{4.1 \times 10^5} = 0.000\ 000\ 006\ 57$
$= \mathbf{6.57 \times 10^{-9}}$ (to 3 s.f.)

(h) Sequence of calculator keys:

$\frac{\square}{\square}$ 6 . 3 $\times 10^x$ 6 ▶ 1 . 5
$\times 10^x$ 2 – 3 $\times 10^x$ (–) 1 =

$\dfrac{6.3 \times 10^6}{1.5 \times 10^2 - 3 \times 10^{-1}} = 42\ 084.168\ 34$
$= 4.208\ 416\ 834 \times 10^4$
$= \mathbf{4.21 \times 10^4}$ (to 3 s.f.)

Practise Now 24

1 MB $= 10^6$ bytes
512 MB $= 512 \times 10^6$ bytes
$= 5.12 \times 10^2 \times 10^6$ bytes
$= 5.12 \times 10^{2+6}$ bytes
$= 5.12 \times 10^8$ bytes
1 kB $= 10^3$ bytes
640 kB $= 640 \times 10^3$ bytes
$= 6.4 \times 10^2 \times 10^3$ bytes
$= 6.4 \times 10^{2+3}$ bytes
$= 6.4 \times 10^5$ bytes

Number of photographs that can be stored $= \dfrac{5.12 \times 10^8}{6.4 \times 10^5}$
$= \mathbf{800}$

Exercise 4E

1. **(a)** $85\ 300 = 8.53 \times 10\ 000$
$= \mathbf{8.53 \times 10^4}$

(b) $52\ 700\ 000 = 5.27 \times 10\ 000\ 000$
$= \mathbf{5.27 \times 10^7}$

(c) $0.000\ 23 = 2.3 \times 0.0001$
$= \mathbf{2.3 \times 10^{-4}}$

(d) $0.000\ 000\ 0904 = 9.04 \times 0.000\ 000\ 01$
$= \mathbf{9.04 \times 10^{-8}}$

2. **(a)** $9.6 \times 10^3 = 9.6 \times 1000$
$= \mathbf{9600}$

(b) $4 \times 10^5 = 4 \times 100\ 000$
$= \mathbf{400\ 000}$

(c) $2.8 \times 10^{-4} = 2.8 \times 0.0001$
$= \mathbf{0.000\ 28}$

(d) $1 \times 10^{-6} = 1 \times 0.000\ 001$
$= \mathbf{0.000\ 001}$

3. 7000 kg $= 7000 \times 10^3$ g
$= 7 \times 10^3 \times 10^3$ g
$= 7 \times 10^{3+3}$ g
$= \mathbf{7 \times 10^6}$ **g**

4. **(i)** 300 000 000 Hz $= 3 \times 10^8$ Hz
$= 3 \times 10^{2+6}$ Hz
$= 3 \times 10^2 \times 10^6$ Hz
$= \mathbf{3 \times 10^2}$ **MHz**

(ii) 300 GHz $= 300 \times 10^9$ Hz
$= 3 \times 10^2 \times 10^9$ Hz
$= 3 \times 10^{2+9}$ Hz
$= 3 \times 10^{11}$ Hz
$= 3 \times 10^{5+6}$ Hz
$= 3 \times 10^5 \times 10^6$ Hz
$= \mathbf{3 \times 10^5}$ **MHz**

5. **(i)** a pm $= 70 \times 10^{-12}$ m
$= 7 \times 10 \times 10^{-12}$ m
$= 7 \times 10^{1+(-12)}$ m
$= \mathbf{7 \times 10^{-11}}$ **m**

(ii) b nm $= 0.074 \times 10^{-9}$ m
$= 7.4 \times 10^{-2} \times 10^{-9}$ m
$= 7.4 \times 10^{-2+(-9)}$ m
$= \mathbf{7.4 \times 10^{-11}}$ **m**

(iii) $a : b = 10 \times 7^{-11} : 7.4 \times 10^{-11}$
$= 7 : 7.4$
$= \mathbf{35 : 37}$

6. c Gm $= 150 \times 10^9$ m
$= 1.5 \times 10^2 \times 10^9$ m
$= 1.5 \times 10^{2+9}$ m
$= 1.5 \times 10^{11}$ m

d Tm $= 5.91 \times 10^{12}$ m

$$\frac{d}{c} \times 100\% = \frac{5.91 \times 10^{12}}{1.5 \times 10^{11}} \times 100\%$$
$= 3940\%$
$= \mathbf{3.94 \times 10^3\%}$

7. **(a)** Sequence of calculator keys:

(2 . 3 4 ×10^x 5)
× (7 . 1 2 ×10^x (–) 4) =

$(2.34 \times 10^5) \times (7.12 \times 10^{-4}) = 166.608$
$= 1.666\,08 \times 10^2$
$= \mathbf{1.67 \times 10^2}$ (to 3 s.f.)

(b) Sequence of calculator keys:

(5 . 1 ×10^x (–) 6)
× (2 . 7 6 ×10^x (–) 3) =

$(5.1 \times 10^{-6}) \times (2.76 \times 10^{-3}) = 0.000\,000\,014\,07$
$= 1.407 \times 10^{-8}$
$= \mathbf{1.41 \times 10^{-8}}$ (to 3 s.f.)

(c) Sequence of calculator keys:

(1 3 . 4 ×10^x 4)
÷ (4 ×10^x 5) =

$(13.4 \times 10^4) \div (4 \times 10^5) = 0.335$
$= \mathbf{3.35 \times 10^{-1}}$

(d) Sequence of calculator keys:

[fraction] 3 ×10^x (–) 4 ▶
9 ×10^x (–) 8 =

$$\frac{3 \times 10^{-4}}{9 \times 10^{-8}} = 3333.\dot{3}$$
$= 3.333\,\dot{3} \times 10^3$
$= \mathbf{3.33 \times 10^3}$ (to 3 s.f.)

(e) Sequence of calculator keys:

2 . 5 4 ×10^x 3
+ 3 . 1 1 ×10^x 4 =

$2.54 \times 10^3 + 3.11 \times 10^4 = 33\,640$
$= 3.364 \times 10^4$
$= \mathbf{3.36 \times 10^4}$ (to 3 s.f.)

(f) Sequence of calculator keys:

3 . 1 ×10^x 7 – 6 ×10^x 5 =

$3.1 \times 10^7 - 6 \times 10^5 = 30\,400\,000$
$= \mathbf{3.04 \times 10^7}$

(g) Sequence of calculator keys:

[fraction] 4 . 3 7 ×10^x (–) 4
+ 2 . 1 6 ×10^x (–) 5 ▶
3 ×10^x (–) 3 =

$$\frac{4.37 \times 10^{-4} + 2.16 \times 10^{-5}}{3 \times 10^{-3}} = 0.152\,866\,6667$$
$= 1.528\,666\,667 \times 10^{-1}$
$= \mathbf{1.53 \times 10^{-1}}$ (to 3 s.f.)

(h) Sequence of calculator keys:

[fraction] 2 . 4 ×10^x (–) 1 0 ▶ 7 . 2 ×10^x (–) 6
– 3 . 5 ×10^x (–) 8 =

$$\frac{2.4 \times 10^{-10}}{7.2 \times 10^{-6} - 3.5 \times 10^{-8}} = 0.000\,033\,496\,16$$
$= 3.349\,616 \times 10^{-5}$
$= \mathbf{3.35 \times 10^{-5}}$ (to 3 s.f.)

8. **(a)** Sequence of calculator keys:

(1 . 3 5 ×10^x (–) 4) $x^{\blacksquare}$ 3 =

$(1.35 \times 10^{-4})^3 = 2.460\,375 \times 10^{-12}$
$= \mathbf{2.46 \times 10^{-12}}$ (to 3 s.f.)

(b) Sequence of calculator keys:

6 × (3 . 4 ×10^x 3) $x^{\blacksquare}$ 2 =

$6(3.4 \times 10^3)^2 = 69\,360\,000$
$= 6.936 \times 10^7$
$= \mathbf{6.94 \times 10^7}$ (to 3 s.f.)

(c) Sequence of calculator keys:

$\sqrt{\blacksquare}$ (1 . 2 1 ×10^x 8) =

$\sqrt{1.21 \times 10^8} = 11\,000$
$= \mathbf{1.1 \times 10^4}$

(d) Sequence of calculator keys:

shift $\sqrt{\blacksquare}$ ($\sqrt[3]{\blacksquare}$) (9 . 2 6 1 ×10^x 6) =

$\sqrt[3]{9.261 \times 10^6} = 210$
$= \mathbf{2.1 \times 10^2}$

(e) Sequence of calculator keys:

[fraction] 2 . 3 ×10^x (–) 2 × 4 . 7
×10^x 3 ▶ 2 ×10^x 3 =

$$\frac{2.3 \times 10^{-2} \times 4.7 \times 10^3}{2 \times 10^3} = 0.054\,05$$
$= 5.405 \times 10^{-2}$
$= \mathbf{5.41 \times 10^{-2}}$ (to 3 s.f.)

(f) Sequence of calculator keys:

[fraction] 8 ×10^x 2 + 2 . 5 ×10^x 3 ▶
2 ×10^x (–) 2 – 3 . 4 ×10^x (–) 3 =

$$\frac{8 \times 10^2 + 2.5 \times 10^3}{2 \times 10^{-2} - 3.4 \times 10^{-3}} = 198\,795.1807$$
$= 1.987\,951\,807 \times 10^5$
$= \mathbf{1.99 \times 10^5}$ (to 3 s.f.)

9. **(a)** $2P \times 4Q = 2(7.5 \times 10^3) \times 4(5.25 \times 10^4)$
$= \mathbf{3.15 \times 10^9}$

(b) $Q - P = 5.25 \times 10^4 - 7.5 \times 10^3$
$= \mathbf{4.5 \times 10^4}$

10. $x + 8y = 2 \times 10^{-3} + 8(7 \times 10^{-4})$
$= \mathbf{7.6 \times 10^{-3}}$

11. **(a)** $MN = (3.2 \times 10^6) \times (5.0 \times 10^7)$
$= \mathbf{1.6 \times 10^{14}}$

(b) $$\frac{M}{N} = \frac{3.2 \times 10^6}{5.0 \times 10^7}$$
$= 0.064$
$= \mathbf{6.4 \times 10^{-2}}$

12. $R = \dfrac{M}{EI}$

$= \dfrac{6 \times 10^4}{\left(4.5 \times 10^8\right) \times \left(4 \times 10^2\right)}$

$= \mathbf{3.33 \times 10^{-7}}$ (to 3 s.f.)

13. (i) 300 000 000 m/s = $\mathbf{3 \times 10^8}$ **m/s**

(ii) 778.5 million km $= 778.5 \times 10^6$ km

$= 778.5 \times 10^6 \times 10^3$ m

$= \mathbf{7.785 \times 10^{11}}$ **m**

Time taken $= \dfrac{\text{Distance}}{\text{Speed}}$

$= \dfrac{7.785 \times 10^{11}}{3 \times 10^8}$

= 2595 seconds

= **43 minutes 15 seconds**

14. (i) Distance travelled by rocket in 4 days $= 4.8 \times 10^5$ km

Distance travelled by rocket in 12 days $= \dfrac{4.8 \times 10^5}{4} \times 12$

$= \mathbf{1.44 \times 10^6}$ **km**

(ii) Speed $= \dfrac{\text{Distance}}{\text{Time}}$

$= \dfrac{4.8 \times 10^5}{4}$

$= 1.2 \times 10^5$ km/day

Time taken $= \dfrac{4.8 \times 10^7}{1.2 \times 10^5}$

= **400 days**

15. (i) Increase in population $= 5.45 \times 10^8 - 4.20 \times 10^8$

$= \mathbf{1.25 \times 10^8}$

(ii) $\dfrac{1.17 \times 10^9}{5.45 \times 10^8} = \mathbf{2.15}$ (to 3 s.f.)

(iii) $\dfrac{1.44 \times 10^9}{7.28 \times 10^8} = \mathbf{1.98}$ (to 3 s.f.)

Chapter 5 Coordinate Geometry

TEACHING NOTES

Suggested Approach

Teachers should revise with the students on what they have learnt about functions and linear graphs in Book 1 before teaching this chapter. As students may have difficulty distinguishing between equations parallel to the x- and y- axes, or may incorrectly refer to the equation of the x-axis as $x = 0$ and the equation of the y-axis as $y = 0$, teachers should highlight these common mistakes to students and give them ample practice so that they can better grasp the concepts in coordinate geometry.

Section 5.1: Length of a line segment

Teachers should recall Pythagoras' Theorem with the students before showing them how the formula for the length of any line segment PQ is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is proven. The formula is used to evaluate the distance between two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$. While students might ask whether the length of any line segment PQ can also be given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, teachers can get the students to check if that is true (see Thinking Time on page 124 of the textbook).

Section 5.2: Gradient of a straight line

Students have learnt in Book 2 that the gradient of a straight line is the ratio of the vertical change to the horizontal change. Teachers can get students to recall this and build upon what they have learnt to find the formula for gradient given two points on a line (see Class Discussion: Finding the gradient of a straight line).

In this section, the students will learn that the gradient of a straight line can be positive, negative, zero or undefined (see Investigation: Gradient of a straight line). Teachers should not only teach students how to find the gradient given two points, but also how to use the gradient to determine the coordinates of a point on the line.

Section 5.3: Equation of a straight line

Students have learnt in Book 2 how the graph of a straight line in the form $y = mx + c$ changes when either m or c varies. Teachers should recap the equations of a vertical line and a horizontal line and introduce the gradient-intercept form of the equation of a straight line, $y = mx + c$. Teachers can guide the students along and show them the different cases and ways of finding the equation of a straight line (see Journal Writing on page 136 of the textbook).

Section 5.4: Midpoint of a line segment

Teachers may begin this section by associating the midpoint of two points as the 'average' of both points. It is also important to highlight the geometrical significance of the midpoint of a line segment or of two points (see Thinking Time on page 142 of the textbook), as well as the usefulness of the midpoint in solving geometrical problems involving polygons (see Worked Example 10).

Section 5.5: Parallel and perpendicular lines

Teachers should demonstrate to students how the gradient of a line is related to the angle of inclination (see Investigation: Angle of inclination), and how gradients of parallel and perpendicular lines are related (see Class Discussion: Parallel lines and Class Discussion: Perpendicular lines).

A common mistake which students might make is to omit the negative sign when finding the gradient of a perpendicular line given that of another line. Teachers may guide students to understand this by observing a pair of perpendicular lines which are neither vertical nor horizontal. Students should be able to see that if one line slopes upwards then the perpendicular line must slope downwards.

Section 5.6: Equation of a straight line involving parallel and perpendicular lines

In this section, students will extend what they have learnt in the previous section to obtain the equation of a line when given the coordinates of a point on the line and the gradient of another line which is either parallel or perpendicular to it. As some students might have difficulties identifying related parallel or perpendicular lines, teachers should guide them along as they do so.

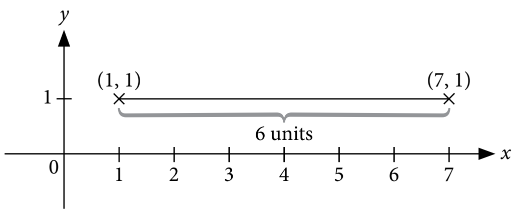
Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 7).

5.1 Length of a line segment

Class Discussion (Determining the length of a line segment in a Cartesian plane)

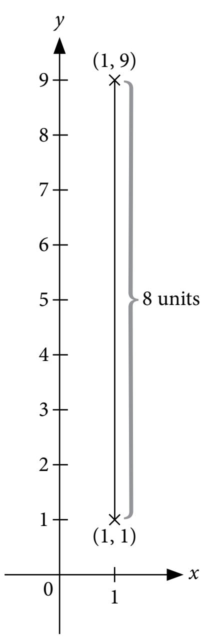
1. **(a)** Since the y-coordinates of both points are the same, the line segment formed is a horizontal line segment.



Length of line segment = 7 – 1
= 6 units

(b) The length of a horizontal line segment can be found by taking the difference between the x-coordinates of its two endpoints.

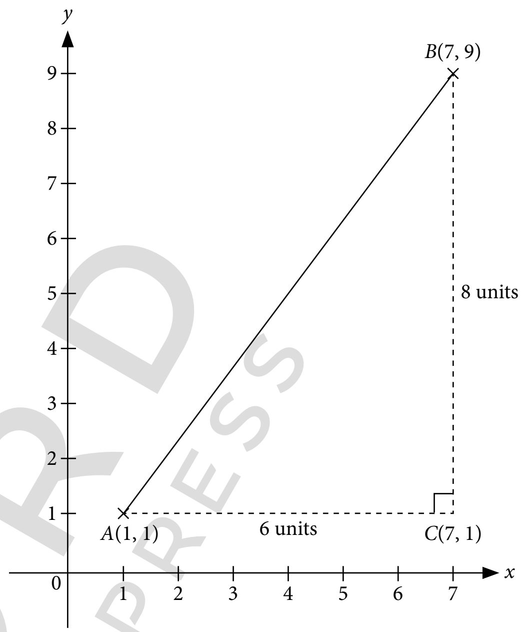
2. **(a)** Since the x-coordinates of both points are the same, the line segment formed is a vertical line segment.



Length of line segment = 9 – 1
= 8 units

(b) The length of a vertical line segment can be found by taking the difference between the y-coordinates of its two endpoints.

3. This line segment is a slanted line as opposed to the horizontal line in Question 1 and the vertical line in Question 2.
Neither the x-coordinates nor the y-coordinates of the two endpoints of this line segment are the same.



By forming a right-angled triangle ABC, where the x-coordinate of C = x-coordinate of B = 7 and the y-coordinate of C = y-coordinate of A = 1, we can use Pythagoras' Theorem to find the length of the line segment AB.
Using Pythagoras' Theorem,

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= 6^2 + 8^2 \\ &= 100 \\ AB &= \sqrt{100} \\ &= 10 \text{ units} \end{aligned}$$

Thinking Time (Page 124)

$$\begin{aligned} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{[-(-x_2 + x_1)]^2 + [-(-y_2 + y_1)]^2} \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \end{aligned}$$

Yes, the two algebraic expressions are equal.

Practise Now 1

(a) Length of line segment $CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$\begin{aligned} &= \sqrt{(3-6)^2 + (-2-2)^2} \\ &= \sqrt{(-3)^2 + (-4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= \mathbf{5 \text{ units}} \end{aligned}$$

(b) Length of line segment $MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$
\begin{aligned}
&= \sqrt{[6 - (-1)]^2 + (-4 - 5)^2} \\
&= \sqrt{7^2 + (-9)^2} \\
&= \sqrt{49 + 81} \\
&= \sqrt{130} \\
&= \mathbf{11.4\ units} \text{ (to 3 s.f.)}
\end{aligned}
$$

(c) Length of line segment $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$
\begin{aligned}
&= \sqrt{(8 - 2)^2 + (7 - 7)^2} \\
&= \sqrt{6^2 + 0^2} \\
&= \sqrt{36 + 0} \\
&= \sqrt{36} \\
&= \mathbf{6\ units}
\end{aligned}
$$

Practise Now 2

(a) Let the coordinates of E be $(0, k)$.

$$
\begin{aligned}
CE &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(0 - 4)^2 + [k - (-1)]^2} \\
&= \sqrt{(-4)^2 + (k + 1)^2} \\
&= \sqrt{16 + (k + 1)^2} \text{ units}
\end{aligned}
$$

$$
\begin{aligned}
DE &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{[0 - (-2)]^2 + (k - 7)^2} \\
&= \sqrt{2^2 + (k - 7)^2} \\
&= \sqrt{4 + (k - 7)^2} \text{ units}
\end{aligned}
$$

Since $CE = DE$,

$$
\begin{aligned}
\sqrt{16 + (k + 1)^2} &= \sqrt{4 + (k - 7)^2} \\
\left[\sqrt{16 + (k + 1)^2}\right]^2 &= \left[\sqrt{4 + (k - 7)^2}\right]^2 \\
16 + (k + 1)^2 &= 4 + (k - 7)^2 \\
16 + k^2 + 2k + 1 &= 4 + k^2 - 14k + 49 \\
k^2 + 2k + 17 &= k^2 - 14k + 53 \\
2k + 14k &= 53 - 17 \\
16k &= 36 \\
k &= \frac{36}{16} \\
&= 2\frac{1}{4}
\end{aligned}
$$

$\therefore$ the coordinates of E are $\left(\mathbf{0, 2\frac{1}{4}}\right)$.

(b) Let the coordinates of F be $(p, 0)$.

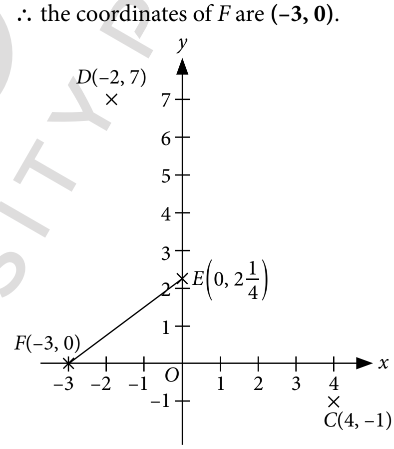
$$
\begin{aligned}
CF &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(p - 4)^2 + [0 - (-1)]^2} \\
&= \sqrt{(p - 4)^2 + 1^2} \\
&= \sqrt{(p - 4)^2 + 1} \text{ units}
\end{aligned}
$$

$$
\begin{aligned}
DF &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{[p - (-2)]^2 + (0 - 7)^2} \\
&= \sqrt{(p + 2)^2 + (-7)^2} \\
&= \sqrt{(p + 2)^2 + 49} \text{ units}
\end{aligned}
$$

Since $CF = DF$,

$$
\begin{aligned}
\sqrt{(p - 4)^2 + 1} &= \sqrt{(p + 2)^2 + 49} \\
\left(\sqrt{(p - 4)^2 + 1}\right)^2 &= \left(\sqrt{(p + 2)^2 + 49}\right)^2 \\
(p - 4)^2 + 1 &= (p + 2)^2 + 49 \\
p^2 - 8p + 16 + 1 &= p^2 + 4p + 4 + 49 \\
p^2 - 8p + 17 &= p^2 + 4p + 53 \\
-12p + 17 &= 53 \\
-12p &= 36 \\
p &= -3
\end{aligned}
$$

$\therefore$ the coordinates of F are $\mathbf{(-3, 0)}$.



$$
\begin{aligned}
\text{Area of } \triangle OEF &= \frac{1}{2} \times \text{base} \times \text{height} \\
&= \frac{1}{2} \times OF \times OE \\
&= \frac{1}{2} \times 3 \times 2\frac{1}{4} \\
&= \mathbf{3\frac{3}{8}\ units^2}
\end{aligned}
$$

Practise Now 3

1. $DE^2 = (2 - 6)^2 + (3 - 1)^2$
$= (-4)^2 + 2^2$
$= 16 + 4$
$= 20 \text{ units}^2$

$EF^2 = (-1 - 2)^2 + (-3 - 3)^2$
$= (-3)^2 + (-6)^2$
$= 9 + 36$
$= 45 \text{ units}^2$

$DF^2 = (-1 - 6)^2 + (-3 - 1)^2$
$= (-7)^2 + (-4)^2$
$= 49 + 16$
$= 65 \text{ units}^2$

Since $DE^2 + EF^2 = 20 + 45$
$= 65$
$= DF^2$,
the triangle is a right-angled triangle with $\angle \boldsymbol{DEF} = °90$. (converse of Pythagoras' Theorem)

2. $PQ^2 = [6 - (-3)]^2 + (3 - 1)^2$
$= 9^2 + 2^2$
$= 81 + 4$
$= 85 \text{ units}^2$

$QR^2 = (1 - 6)^2 + (8 - 3)^2$
$= (-5)^2 + 5^2$
$= 25 + 25$
$= 50 \text{ units}^2$

$PR^2 = [1 - (-3)]^2 + (8 - 1)^2$
$= 4^2 + 7^2$
$= 16 + 49$
$= 65 \text{ units}^2$

Since $QR^2 + PR^2 = 50 + 65$
$= 115$
$\neq PQ^2$,
the triangle is **not** a right-angled triangle.

Exercise 5A

1. **(a)** $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(9 - 2)^2 + (7 - 3)^2}$
$= \sqrt{7^2 + 4^2}$
$= \sqrt{49 + 16}$
$= \sqrt{65}$
$=$ **8.06 units** (to 3 s.f.)

(b) $CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(-5 - 3)^2 + (9 - 6)^2}$
$= \sqrt{(-8)^2 + 3^2}$
$= \sqrt{64 + 9}$
$= \sqrt{73}$
$=$ **8.54 units** (to 3 s.f.)

(c) $EF = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{[8 - (-1)]^2 + (-3 - 4)^2}$
$= \sqrt{9^2 + (-7)^2}$
$= \sqrt{81 + 49}$
$= \sqrt{130}$
$=$ **11.4 units** (to 3 s.f.)

(d) $GH = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{[-4 - (-10)]^2 + (-7 - 2)^2}$
$= \sqrt{6^2 + (-9)^2}$
$= \sqrt{36 + 81}$
$= \sqrt{117}$
$=$ **10.8 units** (to 3 s.f.)

2. $AB = 10$ units

$$\begin{aligned}
\sqrt{(p - 0)^2 + (0 - p)^2} &= 10 \\
\sqrt{p^2 + (-p)^2} &= 10 \\
\sqrt{2p^2} &= 10 \\
\left(\sqrt{2p^2}\right)^2 &= 10^2 \\
2p^2 &= 100 \\
p^2 &= 50 \\
p &= \pm\sqrt{50} \\
&= \pm\mathbf{7.07} \text{ (to 3 s.f.)}
\end{aligned}$$

3. **(a)** Let the coordinates of R be $(0, y)$.

$PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{[0 - (-2)]^2 + (y - 6)^2}$
$= \sqrt{2^2 + (y - 6)^2}$
$= \sqrt{4 + (y - 6)^2}$ units

$QR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(0 - 9)^2 + (y - 3)^2}$
$= \sqrt{(-9)^2 + (y - 3)^2}$
$= \sqrt{81 + (y - 3)^2}$ units

Since $PR = QR$,

$$\begin{aligned}
\sqrt{4 + (y - 6)^2} &= \sqrt{81 + (y - 3)^2} \\
\left[\sqrt{4 + (y - 6)^2}\right]^2 &= \left[\sqrt{81 + (y - 3)^2}\right]^2 \\
4 + (y - 6)^2 &= 81 + (y - 3)^2 \\
4 + y^2 - 12y + 36 &= 81 + y^2 - 6y + 9 \\
y^2 - 12y + 40 &= y^2 - 6y + 90 \\
-6y + 40 &= 90 \\
-6y &= 50 \\
y &= -8\frac{1}{3}
\end{aligned}$$

$\therefore$ the coordinates of R are $\left(\mathbf{0, -8\frac{1}{3}}\right)$.

(b) Let the coordinates of S be $(x, 0)$.

$$PS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[x - (-2)]^2 + (0 - 6)^2}$$
$$= \sqrt{(x + 2)^2 + (-6)^2}$$
$$= \sqrt{(x + 2)^2 + 36} \text{ units}$$

$$QS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(x - 9)^2 + (0 - 3)^2}$$
$$= \sqrt{(x - 9)^2 + (-3)^2}$$
$$= \sqrt{(x - 9)^2 + 9} \text{ units}$$

Since $PS = QS$,

$$\sqrt{(x + 2)^2 + 36} = \sqrt{(x - 9)^2 + 9}$$
$$\left[\sqrt{(x + 2)^2 + 36}\right]^2 = \left[\sqrt{(x - 9)^2 + 9}\right]^2$$
$$(x + 2)^2 + 36 = (x - 9)^2 + 9$$
$$x^2 + 4x + 4 + 36 = x^2 - 18x + 81 + 9$$
$$x^2 + 4x + 40 = x^2 - 18x + 90$$
$$22x + 40 = 90$$
$$22x = 50$$
$$x = 2\frac{3}{11}$$

$\therefore$ the coordinates of S are $\left(\mathbf{2\frac{3}{11}, 0}\right)$.

4. Let the coordinates of W be $(0, y)$.

$$MW = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(0 - 3)^2 + (y - 7)^2}$$
$$= \sqrt{(-3)^2 + (y - 7)^2}$$
$$= \sqrt{9 + (y - 7)^2} \text{ units}$$

$$NW = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(0 - 11)^2 + [y - (-6)]^2}$$
$$= \sqrt{(-11)^2 + (y + 6)^2}$$
$$= \sqrt{121 + (y + 6)^2} \text{ units}$$

Since W is equidistant from M and from N,

$$\sqrt{9 + (y - 7)^2} = \sqrt{121 + (y + 6)^2}$$
$$\left[\sqrt{9 + (y - 7)^2}\right]^2 = \left[\sqrt{121 + (y + 6)^2}\right]^2$$
$$9 + (y - 7)^2 = 121 + (y + 6)^2$$
$$9 + y^2 - 14y + 49 = 121 + y^2 + 12y + 36$$
$$y^2 - 14y + 58 = y^2 + 12y + 157$$
$$-26y + 58 = 157$$
$$-26y = 99$$
$$y = -3\frac{21}{26}$$

$\therefore$ the coordinates of W are $\left(\mathbf{0, -3\frac{21}{26}}\right)$.

5. (i) $AB = 12$ units

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[2 - (-4)]^2 + [6 - (-2)]^2}$$
$$= \sqrt{6^2 + 8^2}$$
$$= \sqrt{36 + 64}$$
$$= \sqrt{100}$$
$$= 10 \text{ units}$$

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(2 - 8)^2 + [6 - (-2)]^2}$$
$$= \sqrt{(-6)^2 + 8^2}$$
$$= \sqrt{36 + 64}$$
$$= \sqrt{100}$$
$$= 10 \text{ units}$$

Perimeter of $\triangle ABC = 12 + 10 + 10$
$= \mathbf{32\ units}$

Area of $\triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times AB \times 8$
$= \frac{1}{2} \times 12 \times 8$
$= \mathbf{48\ units^2}$

(ii) Let the length of the perpendicular from A to BC be x units.

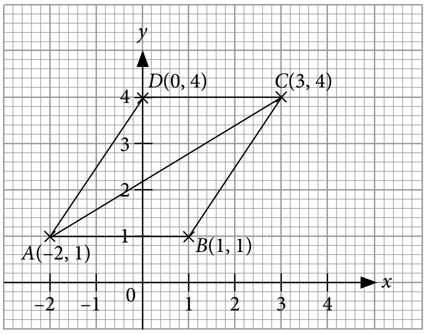
Area of $\triangle ABC = 48 \text{ units}^2$

$$\frac{1}{2} \times \text{base} \times \text{height} = 48$$
$$\frac{1}{2} \times BC \times x = 48$$
$$\frac{1}{2} \times 10 \times x = 48$$
$$5x = 48$$
$$x = 9.6$$

$\therefore$ the length of the perpendicular from A to BC is **9.6 units**.

6. (i) Area of $\triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times 3 \times 3$
$= \mathbf{4.5\ units^2}$

(ii) $AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$$= \sqrt{[3 - (-2)]^2 + (4 - 1)^2}$$
$$= \sqrt{5^2 + 3^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34}$$
$= \mathbf{5.83\ units}$ (to 2 d.p.)

(iii) 

$\therefore$ from the graph, the coordinates of D are **(0, 4)**.

(iv) Length of the perpendicular from B to CK = 3 units

Area of $\triangle BCK$ = 12 units^2

$$\frac{1}{2} \times \text{base} \times \text{height} = 12$$

$$\frac{1}{2} \times CK \times 3 = 12$$

$$\frac{1}{2} \times (t-3) \times 3 = 12 \quad \text{or} \quad \frac{1}{2} \times (3-t) \times 3 = 12$$

$$3(t-3) = 24 \qquad 3(3-t) = 24$$

$$3t - 9 = 24 \qquad 9 - 3t = 24$$

$$3t = 33 \qquad 3t = -15$$

$$t = 11 \qquad t = -5$$

$\therefore$ t = **–5 or 11**

7. **(i)** AB = **3 units**

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[3-(-1)]^2 + [-1-(-2)]^2}$$
$$= \sqrt{4^2 + 1^2}$$
$$= \sqrt{16+1}$$
$$= \sqrt{17}$$

= **4.12 units** (to 3 s.f.)

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[3-(-1)]^2 + (-1-1)^2}$$
$$= \sqrt{4^2 + (-2)^2}$$
$$= \sqrt{16+4}$$
$$= \sqrt{20}$$

= **4.47 units** (to 3 s.f.)

(ii) Area of $\triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times AB \times 4$$
$$= \frac{1}{2} \times 3 \times 4$$

= **6 units²**

(iii) Length of the perpendicular from B to CE = 4 units

Area of $\triangle BCE$ = 14 units^2

$$\frac{1}{2} \times \text{base} \times \text{height} = 14$$

$$\frac{1}{2} \times CE \times 4 = 14$$

$$2CE = 14$$

$$CE = 7 \text{ units}$$

$\therefore$ possible values of k = **–8 or 6**

8. $$\sqrt{[1-(1-t)]^2 + (2t-1)^2} = \sqrt{11-9t}$$

$$\left(\sqrt{[1-(1-t)]^2 + (2t-1)^2}\right)^2 = \left(\sqrt{11-9t}\right)^2$$

$$[1-(1-t)]^2 + (2t-1)^2 = 11 - 9t$$
$$t^2 + 4t^2 - 4t + 1 = 11 - 9t$$
$$5t^2 - 4t + 1 = 11 - 9t$$
$$5t^2 + 5t - 10 = 0$$
$$t^2 + t - 2 = 0$$
$$(t+2)(t-1) = 0$$
$$t + 2 = 0 \quad \text{or} \quad t - 1 = 0$$
$$t = -2 \qquad t = 1$$

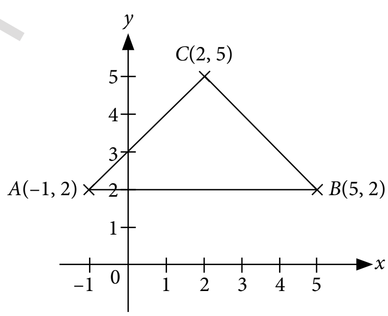
$\therefore$ t = **–2 or 1**

9. **(i)** AB = 6 units

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(2-5)^2 + (5-2)^2}$$
$$= \sqrt{(-3)^2 + 3^2}$$
$$= \sqrt{9+9}$$
$$= \sqrt{18} \text{ units}$$

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[2-(-1)]^2 + (5-2)^2}$$
$$= \sqrt{3^2 + 3^2}$$
$$= \sqrt{9+9}$$
$$= \sqrt{18} \text{ units}$$

Since $BC = AC$, the points A, B and C are the vertices of an isosceles triangle.

(ii) 

Length of the perpendicular from C to AB = 3 units

Area of $\triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times AB \times 3$$
$$= \frac{1}{2} \times 6 \times 3$$

= **9 units²**

10. $PQ^2 = (3 - 3)^2 + (1 - 4)^2$
$= 0^2 + (-3)^2$
$= 9 \text{ units}^2$
$QR^2 = (8 - 3)^2 + (4 - 1)^2$
$= 5^2 + 3^2$
$= 25 + 9$
$= 34 \text{ units}^2$
$PR^2 = (8 - 3)^2 + (4 - 4)^2$
$= 5^2 + 0^2$
$= 25 \text{ units}^2$
Since $PQ^2 + PR^2 = 9 + 25$
$= 34$
$= QR^2$,
the triangle is a right-angled triangle with $\angle QPR = °90$. (converse of Pythagoras' Theorem) (shown)

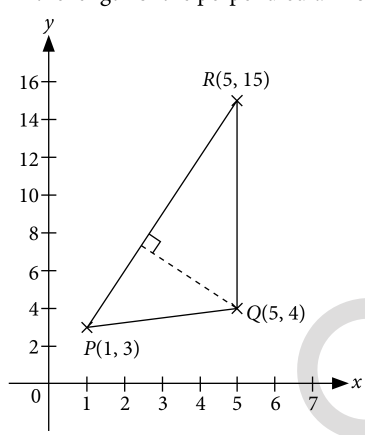
$$\text{Area of } \triangle PQR = \frac{1}{2} \times PQ \times PR$$
$$= \frac{1}{2} \times \sqrt{9} \times \sqrt{25}$$
$$= \frac{1}{2} \times 3 \times 5$$
$$= 7.5 \text{ units}^2$$

Let the length of the perpendicular from P to QR be x units.
Area of $\triangle PQR = 7.5 \text{ units}^2$

$$\frac{1}{2} \times \text{base} \times \text{height} = 7.5$$
$$\frac{1}{2} \times QR \times x = 7.5$$
$$\frac{1}{2} \times \sqrt{34} \times x = 7.5$$
$$x = 2.57 \text{ (to 3 s.f.)}$$

$\therefore$ the length of the perpendicular from P to QR is **2.57 units**.

11.



$$PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5 - 1)^2 + (15 - 3)^2}$$
$$= \sqrt{4^2 + 12^2}$$
$$= \sqrt{16 + 144}$$
$$= \sqrt{160} \text{ units}$$

Length of the perpendicular from P to QR = 4 units

$$\text{Area of } \triangle PQR = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times QR \times 4$$
$$= \frac{1}{2} \times 11 \times 4$$
$$= 22 \text{ units}^2$$

Let the length of the perpendicular from Q to PR be h units.
Area of $\triangle PQR = 22 \text{ units}^2$

$$\frac{1}{2} \times \text{base} \times \text{height} = 22$$
$$\frac{1}{2} \times PR \times h = 22$$
$$\frac{1}{2} \times \sqrt{160} \times h = 22$$
$$h = 3.48 \text{ (to 3 s.f.)}$$

$\therefore$ the length of the perpendicular from Q to PR is **3.48 units**.

5.2 Gradient of a straight line

Class Discussion (Finding the gradient of a straight line)

1. **(i)** In Fig. 5.5**(a)**,
vertical change from point A to point B = **6 units**
In Fig. 5.5**(b)**,
vertical change from point A to point B = **–3 units**

(ii) In Fig. 5.5**(a)**,
horizontal change from point A to point B = **3 units**
In Fig. 5.5**(b)**,
horizontal change from point A to point B = **1 unit**

(iii) In Fig. 5.5**(a)**,

$$\text{Gradient of } AB = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{6}{3}$$
$$= \mathbf{2}$$

In Fig. 5.5**(b)**,

$$\text{Gradient of } AB = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{-3}{1}$$
$$= \mathbf{-3}$$

(iv) In Fig. 5.5**(a)**, let the coordinates of C and D be (1, 3) and (2, 5),

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{2}{1}$$
$$= \mathbf{2}$$

In Fig. 5.5**(b)**, let the coordinates of C and D be (0, 6) and (1.5, 1.5),

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{-4.5}{1.5}$$
$$= \mathbf{-3}$$

For each of the two figures, the gradient of line segments CD and AB are the same. This is because the points lie on the same line.

2. Gradient of $AB = \boldsymbol{\dfrac{y_2 - y_1}{x_2 - x_1}}$ **or** $\boldsymbol{\dfrac{y_1 - y_2}{x_1 - x_2}}$

3. (a) Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7 - 4}{3 - (-1)} = \boldsymbol{\dfrac{3}{4}}$

(b) Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-11 - (-3)}{2 - (-4)} = \dfrac{-8}{6} = \boldsymbol{-\dfrac{4}{3}}$

(c) Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - 3}{-4 - 6} = \dfrac{0}{-10} = \mathbf{0}$

(d) Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8 - (-1)}{2 - 2} = \dfrac{9}{0} =$ **undefined**

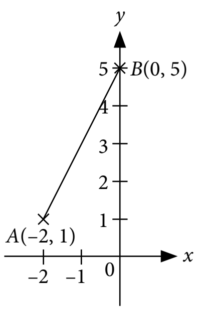
Thinking Time (Page 130)

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(-y_2 + y_1)}{-(-x_2 + x_1)} = \frac{y_1 - y_2}{x_1 - x_2}$$

Yes, the two algebraic expressions are equal.

Investigation (Gradient of a straight line)

1.



4.

	Coordinates of endpoints	Gradient of line segment	Sign of gradient	$y_2 - y_1$	$x_2 - x_1$
(a)	$A(-2, 1)$ and $B(0, 5)$	$\frac{4}{2} = \mathbf{2}$	positive	$5 - 1 = \mathbf{4}$	$0 - (-2) = \mathbf{2}$
(b)	$C(7, 5)$ and $D(4, 8)$	$\frac{3}{-3} = \mathbf{-1}$	**negative**	$\mathbf{8 - 5 = 3}$	$\mathbf{4 - 7 = -3}$
(c)	$E(-2, 6)$ and $F(-4, 3)$	$\frac{-3}{-2} = \boldsymbol{\frac{3}{2}}$	**positive**	$\mathbf{3 - 6 = -3}$	$\mathbf{-4 - (-2) = -2}$
(d)	$G(1, -2)$ and $H(2, -3)$	$\frac{-1}{1} = \mathbf{-1}$	**negative**	$\mathbf{-3 - (-2) = -1}$	$\mathbf{2 - 1 = 1}$
(e)	$I(1, 1)$ and $J(3, 1)$	$\frac{0}{2} = \mathbf{0}$	**zero**	$\mathbf{1 - 1 = 0}$	$\mathbf{3 - 1 = 2}$
(f)	$K(-4, 3)$ and $L(-4, 6)$	$\frac{3}{0}$ = **undefined**	**undefined**	$\mathbf{6 - 3 = 3}$	$\mathbf{-4 - (-4) = 0}$

Table 5.1

5. (a) When $y_2 - y_1 > 0$ and $x_2 - x_1 < 0$, the sign of the gradient is **negative**.
(b) When $y_2 - y_1 < 0$ and $x_2 - x_1 > 0$, the sign of the gradient is **negative**.
(c) When the signs of $y_2 - y_1$ and $x_2 - x_1$ are the same, the sign of the gradient is **positive**.
(d) When $y_2 - y_1 = 0$, i.e. $y_1 = y_2$, the gradient of the line is **zero**.
(e) When $x_2 - x_1 = 0$, i.e. $x_1 = x_2$, the gradient of the line is **undefined**.

Practise Now 4

(a) Gradient of $CD = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{3 - 1}{6 - 3}$

$= \boldsymbol{\dfrac{2}{3}}$

(b) Gradient of $HK = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-2 - (-7)}{0 - 5}$

$= \dfrac{5}{-5}$

$= \mathbf{-1}$

(c) Gradient of $MN = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{1 - 1}{16 - (-4)}$

$= \dfrac{0}{20}$

$= \mathbf{0}$

Practise Now 5

1. Gradient of line, $\frac{y_2 - y_1}{x_2 - x_1} = -3$

$$\frac{h-(-9)}{-3-4} = -3$$
$$\frac{h+9}{-7} = -3$$
$$h + 9 = 21$$
$$\therefore h = \mathbf{12}$$

2. Since the points A, B and C are collinear,
Gradient of AB = Gradient of BC

$$\frac{-18-(-9)}{m-2} = \frac{m-(-9)}{4-2}$$
$$\frac{-9}{m-2} = \frac{m+9}{2}$$
$$-18 = (m+9)(m-2)$$
$$= m^2 + 7m - 18$$
$$m^2 + 7m = 0$$
$$m(m+7) = 0$$
$$m = 0 \quad \text{or} \quad m + 7 = 0$$
$$m = -7$$

$\therefore m = \mathbf{0 \text{ or } -7}$

Exercise 5B

1. (a) Gradient of $AB = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{1-0}{-2-0}$$
$$= \frac{1}{-2}$$
$$= -\mathbf{\frac{1}{2}}$$

(b) Gradient of $CD = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{7-(-3)}{1-2}$$
$$= \frac{10}{-1}$$
$$= \mathbf{-10}$$

(c) Gradient of $EF = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{8-4}{-5-(-2)}$$
$$= \frac{4}{-3}$$
$$= -\mathbf{\frac{4}{3}}$$

(d) Gradient of $GH = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{-8-7}{1-(-4)}$$
$$= \frac{-15}{5}$$
$$= \mathbf{-3}$$

(e) Gradient of $IJ = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{6-(-5)}{2-(-2)}$$
$$= \mathbf{\frac{11}{4}}$$

(f) Gradient of $KL = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{9-9}{6-(-7)}$$
$$= \frac{0}{13}$$
$$= \mathbf{0}$$

2. Gradient of $AB = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{1-1}{7-0}$$
$$= \frac{0}{7}$$
$$= \mathbf{0}$$

Gradient of $AE = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{4-1}{6-0}$$
$$= \frac{3}{6}$$
$$= \mathbf{\frac{1}{2}}$$

Gradient of $DC = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{0-5}{6-0}$$
$$= \frac{-5}{6}$$
$$= -\mathbf{\frac{5}{6}}$$

Gradient of $DE = \frac{y_2 - y_1}{x_2 - x_1}$
$$= \frac{4-5}{6-0}$$
$$= \frac{-1}{6}$$
$$= -\mathbf{\frac{1}{6}}$$

3. Gradient of line, $\frac{y_2 - y_1}{x_2 - x_1} = \frac{3}{5}$

$$\frac{p-(-7)}{4-(-3)} = \frac{3}{5}$$
$$\frac{p+7}{7} = \frac{3}{5}$$
$$5(p+7) = 21$$
$$5p + 35 = 21$$
$$5p = -14$$
$$\therefore p = \mathbf{-2\frac{4}{5}}$$

4. Gradient of AB, $\frac{y_2 - y_1}{x_2 - x_1} = 3$

$$\frac{8-(-3)}{3k-k} = 3$$
$$\frac{11}{2k} = 3$$
$$11 = 6k$$
$$\therefore k = \mathbf{1\frac{5}{6}}$$

5. Gradient of line, $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2}{a}$

$$\frac{1-a}{2a-9} = \frac{2}{a}$$
$$a(1-a) = 2(2a-9)$$
$$a - a^2 = 4a - 18$$
$$-a^2 - 3a + 18 = 0$$
$$a^2 + 3a - 18 = 0$$
$$(a+6)(a-3) = 0$$
$$a + 6 = 0 \quad \text{or} \quad a - 3 = 0$$
$$a = -6 \qquad a = 3$$

$\therefore a =$ **−6 or 3**

6. Gradient of $PQ = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-9-(-11)}{k-6}$$
$$= \frac{2}{k-6}$$

Gradient of $PR = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-3-(-11)}{2k-6}$$
$$= \frac{8}{2k-6}$$

Since the gradient of PQ is equal to the gradient of PR,
Gradient of PQ = Gradient of PR

$$\frac{2}{k-6} = \frac{8}{2k-6}$$
$$2(2k-6) = 8(k-6)$$
$$4k - 12 = 8k - 48$$
$$36 = 4k$$

$\therefore k =$ **9**

7. Gradient of $PQ = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-2-(-3)}{3-2}$$
$$= \frac{1}{1}$$
$$= 1$$

Gradient of $PR = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{z-(-3)}{8-2}$$
$$= \frac{z+3}{6}$$

Since the points P, Q and R are collinear,
Gradient of PR = Gradient of PQ

$$\frac{z+3}{6} = 1$$
$$z + 3 = 6$$

$\therefore z =$ **3**

8. Gradient of line, $\frac{y_2 - y_1}{x_2 - x_1} = 2$

$$\frac{2t^2 + 7 - t}{7-2} = 2$$
$$\frac{2t^2 - t + 7}{5} = 2$$
$$2t^2 - t + 7 = 10$$
$$2t^2 - t - 3 = 0$$
$$(2t-3)(t+1) = 0$$
$$2t - 3 = 0 \quad \text{or} \quad t + 1 = 0$$
$$2t = 3 \qquad t = -1$$
$$t = 1\tfrac{1}{2}$$

$\therefore t =$ **−1 or $1\frac{1}{2}$**

9. (i) Gradient of $AB = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{1-6}{2-0}$$
$$= \frac{-5}{2}$$
$$= -\frac{5}{2}$$

Gradient of $BC = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{3-1}{7-2}$$
$$= \frac{2}{5}$$

Gradient of $CD = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{8-3}{5-7}$$
$$= \frac{5}{-2}$$
$$= -\frac{5}{2}$$

Gradient of $AD = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{8-6}{5-0}$$
$$= \frac{2}{5}$$

(ii) Gradient of AB = Gradient of $CD = -\frac{5}{2}$

Gradient of BC = Gradient of $AD = \frac{2}{5}$

The gradients of the opposite sides of a square are **equal**.

5.3 Equation of a straight line

Practise Now 6

1. Since (–1, 2) lies on the line $y = 5x + a$, the coordinates (–1, 2) must satisfy the equation,
   i.e. $2 = 5(-1) + a$
   $= -5 + a$
   $\therefore a = \mathbf{7}$
2. Since (6, 8) lies on the line $y = -4x + b$, the coordinates (6, 8) must satisfy the equation,
   i.e. $8 = -4(6) + b$
   $= -24 + b$
   $\therefore b = \mathbf{32}$

Practise Now 7

(a) Gradient of $AB = \dfrac{3-1}{5-(-2)}$
$= \dfrac{2}{7}$

Equation of AB is in the form $y = \dfrac{2}{7}x + c$

Since (–2, 1) lies on the line,

$1 = \dfrac{2}{7}(-2) + c$

$c = \dfrac{11}{7}$

$\therefore$ equation of AB is $\boldsymbol{y = \dfrac{2}{7}x + \dfrac{11}{7}}$.

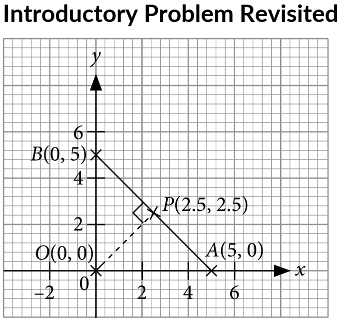
(b) $C(6, 4)$ and $D(-4, 4)$ have the same y-coordinate of value 4.
$\therefore$ CD is a horizontal line with equation $\boldsymbol{y = 4}$.

(c) $E(-3, 5)$ and $F(-3, 8)$ have the same x-coordinate of value –3.
$\therefore$ EF is a vertical line with equation $\boldsymbol{x = -3}$.

Journal Writing (Page 136)

Case 1: Given the gradient m and the y-intercept c, substitute m and c into $y = mx + c$ to find the equation of the line.
Case 2: Given the gradient m and the coordinates of a point (a, b), substitute the coordinates of the point and the gradient into $y = mx + c$ to find the y-intercept, c. Then, substitute m and c into $y = mx + c$ to find the equation of the line.
Case 3: Given the coordinates of two points (a, b) and (c, d), find the gradient of the line, m, using the coordinates of the two points. Substitute the coordinates of the point and the gradient into $y = mx + c$ to find the y-intercept, c. Then, substitute m and c into $y = mx + c$ to find the equation of the line.

Introductory Problem Revisited



(i) OP is shortest when $\angle OPA = \angle OPB = 90°$. Since $OA = OB = 5$ units, this occurs when P lies exactly in between A and B.

$\therefore$ coordinates of $P = \left(\dfrac{5}{2}, \dfrac{5}{2}\right)$
$= (2.5, 2.5)$

(ii) $OP = \sqrt{(2.5-0)^2 + (2.5-0)^2}$
$= \sqrt{\dfrac{25}{2}}$
$= \dfrac{5}{\sqrt{2}}$
$= \dfrac{5}{2}\sqrt{2}$ units (shown)

Exercise 5C

1. Since (1, 2) lies on the line,
   $2 = -1 + c$
   $c = \mathbf{3}$
2. Since (–3, 3) lies on the line,
   $3 = 4(-3) + k$
   $k = \mathbf{15}$
3. **(a)** Gradient of $AB = \dfrac{-1-0}{1-0}$
   $= \dfrac{-1}{1}$
   $= -1$
   Equation of AB is in the form $y = -x + c$
   Since (0, 0) lies on the line,
   $0 = -0 + c$
   $c = 0$
   $\therefore$ equation of AB is $\boldsymbol{y = -x}$.

   (b) Gradient of $CD = \dfrac{5-3}{2-1}$
   $= \dfrac{2}{1}$
   $= 2$
   Equation of CD is in the form $y = 2x + c$
   Since (1, 3) lies on the line,
   $3 = 2(1) + c$
   $c = 1$
   $\therefore$ equation of CD is $\boldsymbol{y = 2x + 1}$.

   (c) Gradient of $EF = \dfrac{3-4}{-2-2}$
   $= \dfrac{-1}{-4}$
   $= \dfrac{1}{4}$
   Equation of EF is in the form $y = \dfrac{1}{4}x + c$
   Since (2, 4) lies on the line,
   $4 = \dfrac{1}{4}(2) + c$
   $c = \dfrac{7}{2}$
   $\therefore$ equation of EF is $\boldsymbol{y = \dfrac{1}{4}x + \dfrac{7}{2}}$.

(d) Gradient of $GH = \frac{4-(-5)}{4-(-6)}$

$= \frac{9}{10}$

Equation of GH is in the form $y = \frac{9}{10}x + c$

Since (4, 4) lies on the line,

$4 = \frac{9}{10}(4) + c$

$c = \frac{2}{5}$

$\therefore$ equation of GH is $\boldsymbol{y = \frac{9}{10}x + \frac{2}{5}}$.

(e) Gradient of $IJ = \frac{-7-(-4)}{1-(-2)}$

$= \frac{-3}{3}$

$= -1$

Equation of IJ is in the form $y = -x + c$

Since (1, –7) lies on the line,

$-7 = -1 + c$

$c = -6$

$\therefore$ equation of IJ is $\boldsymbol{y = -x - 6}$.

(f) Gradient of $KL = \frac{-1-(-5)}{-1-(-7)}$

$= \frac{4}{6}$

$= \frac{2}{3}$

Equation of KL is in the form $y = \frac{2}{3}x + c$

Since (–1, –1) lies on the line,

$-1 = \frac{2}{3}(-1) + c$

$c = -\frac{1}{3}$

$\therefore$ equation of KL is $\boldsymbol{y = \frac{2}{3}x - \frac{1}{3}}$.

(g) $M(8, 0)$ and $N(-9, 0)$ have the same y-coordinate of value 0.

$\therefore$ MN is a horizontal line with equation $\boldsymbol{y = 0}$.

(h) $O(0, 0)$ and $P(0, 7)$ have the same x-coordinate of value 0.

$\therefore$ OP is a vertical line with equation $\boldsymbol{x = 0}$.

4. (a) $y = \frac{1}{3}x + c$

Since (0, 0) lies on the line,

$0 = \frac{1}{3}(0) + c$

$c = 0$

$\therefore$ equation of line is $\boldsymbol{y = \frac{1}{3}x}$.

(b) $y = 3x + c$

Since (1, 1) lies on the line,

$1 = 3(1) + c$

$c = -2$

$\therefore$ equation of line is $\boldsymbol{y = 3x - 2}$.

(c) $y = -3x + c$

Since (2, –5) lies on the line,

$-5 = -3(2) + c$

$c = 1$

$\therefore$ equation of line is $\boldsymbol{y = -3x + 1}$.

(d) $y = -\frac{1}{2}x + c$

Since (5, 7) lies on the line,

$7 = -\frac{1}{2}(5) + c$

$c = \frac{19}{2}$

$\therefore$ equation of line is $\boldsymbol{y = -\frac{1}{2}x + \frac{19}{2}}$.

(e) $y = c$

Since (5, 4) lies on the line,

$4 = c$

$c = 4$

$\therefore$ equation of line is $\boldsymbol{y = 4}$.

(f) $y = ax + c$

Since $(0, a)$ lies on the line,

$a = a(0) + c$

$c = a$

$\therefore$ equation of line is $\boldsymbol{y = ax + a}$.

5. $y = 2x + c$

Since (0, 0) lies on the line,

$0 = 2(0) + c$

$c = 0$

$\therefore$ equation of line is $\boldsymbol{y = 2x}$.

6. (a) Since the line is a horizontal line, the gradient is **0** and the y-intercept = **1**.

$\therefore$ equation of line is $\boldsymbol{y = 1}$.

(b) Since the line is a vertical line, the gradient is **undefined** and there is **no *y*-intercept**.

$\therefore$ equation of line is $\boldsymbol{x = 1.5}$.

(c) The line passes through the points (0, –1) and (1, 0).

Gradient of line $= \frac{0-(-1)}{1-0}$

$= \frac{1}{1}$

$= \mathbf{1}$

Equation of line is in the form $y = x + c$

Since (0, –1) lies on the line,

$-1 = 0 + c$

$c = \mathbf{-1}$

$\therefore$ equation of line is $\boldsymbol{y = x - 1}$.

(d) The line passes through the points (0, 1) and (2, 0).

Gradient of line $= \frac{0-1}{2-0}$

$= \frac{-1}{2}$

$= \boldsymbol{-\frac{1}{2}}$

Equation of line is in the form $y = -\frac{1}{2}x + c$

Since (0, 1) lies on the line,

$1 = -\frac{1}{2}(0) + c$

$c = \mathbf{1}$

$\therefore$ equation of line is $\boldsymbol{y = -\frac{1}{2}x + 1}$.

7. **(i)** Length of the perpendicular from C to $AB = 4$ units

$$\text{Area of } \triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times AB \times 4$$
$$= \frac{1}{2} \times 2 \times 4$$
$$= \mathbf{4\ units^2}$$

(ii) $$\text{Gradient of } BC = \frac{5-3}{5-1}$$
$$= \frac{2}{4}$$
$$= \mathbf{\frac{1}{2}}$$

(iii) $$\text{Gradient of } AC = \frac{5-1}{5-1}$$
$$= \frac{4}{4}$$
$$= 1$$

Equation of AC is in the form $y = x + c$
Since (1, 1) lies on the line,
$1 = 1 + c$
$c = 0$
$\therefore$ equation of AC is $\boldsymbol{y = x}$.

8. For the line $2x - 5 = ky$,
$ky = 2x - 5$
$$y = \frac{2}{k}x - \frac{5}{k}$$
Gradient of the line $2x - 5 = ky$ is $\frac{2}{k}$.
For the line $(k + 1)x = 6y - 3$,
$6y = (k + 1)x + 3$
$$y = \frac{k+1}{6}x + \frac{1}{2}$$
Gradient of the line $(k + 1)x = 6y - 3$ is $\frac{k+1}{6}$.
Since both lines have the same gradient,
$$\frac{2}{k} = \frac{k+1}{6}$$
$$12 = k(k + 1)$$
$$= k^2 + k$$
$$k^2 + k - 12 = 0$$
$$(k + 4)(k - 3) = 0$$
$k + 4 = 0$ or $k - 3 = 0$
$k = -4$ $\quad$ $k = 3$
$\therefore$ $k = \mathbf{-4\ or\ 3}$

9. **(i)** $$\frac{x}{3} + \frac{y}{2} = 1$$
$$6\left(\frac{x}{3} + \frac{y}{2}\right) = 6(1)$$
$$2x + 3y = 6$$
$$3y = -2x + 6$$
$$\boldsymbol{y = -\frac{2}{3}x + 2}$$

(ii) $y = -\frac{2}{3}x + 2$

Gradient of the line $= \mathbf{-\frac{2}{3}}$

(iii) $y = 0$ at the point where the line cuts the x-axis.
$$0 = -\frac{2}{3}x + 2$$
$$\frac{2}{3}x = 2$$
$$2x = 6$$
$$x = 3$$
$\therefore$ the coordinates of the point are **(3, 0)**.

10. **(i)** $y = -\frac{2}{3}x + c$
Since (–3, 5) lies on the line,
$$5 = -\frac{2}{3}(-3) + c$$
$c = 3$
$\therefore$ equation of line is $\boldsymbol{y = -\frac{2}{3}x + 3}$.

(ii) Since $(p, 3)$ lies on the line,
$$3 = -\frac{2}{3}p + 3$$
$$\frac{2}{3}p = 0$$
$$p = \mathbf{0}$$

11. $2y = 5x + 7$
$$y = \frac{5}{2}x + \frac{7}{2}$$
Gradient of the line $= \frac{5}{2}$
Equation of line is in the form $y = \frac{5}{2}x + c$
Since (3, –2) lies on the line,
$$-2 = \frac{5}{2}(3) + c$$
$$c = -\frac{19}{2}$$
$\therefore$ equation of line is $\boldsymbol{y = \frac{5}{2}x - \frac{19}{2}}$.

12. **(i)** $y = 3x + c$
Since (3, 1) lies on the line,
$1 = 3(3) + c$
$c = -8$
$\therefore$ equation of line is $\boldsymbol{y = 3x - 8}$.

(ii) $y = 3x - 8$ — (1)
$y = x$ — (2)
Substitute (1) into (2):
$$3x - 8 = x$$
$$2x = 8$$
$$x = 4$$
Substitute $x = 4$ into (2):
$y = 4$
$\therefore$ the coordinates of the point of intersection are **(4, 4)**.

13. **(i)** $y = 0$ at the point where l crosses the x-axis.
$$5x + 6(0) + 30 = 0$$
$$5x + 30 = 0$$
$$5x = -30$$
$$x = -6$$
$\therefore$ the coordinates of the point are **(–6, 0)**.

(ii) When $x = 2$,

$5(2) + 6y + 30 = 0$

$10 + 6y + 30 = 0$

$6y + 40 = 0$

$6y = -40$

$y = -6\frac{2}{3}$

$\therefore$ the coordinates of the point are $\left(\mathbf{2, -6\frac{2}{3}}\right)$.

(iii) $5x + 6y + 30 = 0$

$6y = -5x - 30$

$y = -\frac{5}{6}x - 5$

Gradient of line $= -\frac{5}{6}$

Equation of line is in the form $y = -\frac{5}{6}x + c$

Since $(3, -1)$ lies on the line,

$-1 = -\frac{5}{6}(3) + c$

$c = \frac{3}{2}$

$\therefore$ equation of line is $\boldsymbol{y = -\frac{5}{6}x + \frac{3}{2}}$.

(iv) Equation of line is in the form $y = c$

Since $(3, -1)$ lies on the line,

$-1 = c$

$c = -1$

$\therefore$ equation of line is $\boldsymbol{y = -1}$.

14. (a) (i) Gradient of $l = \frac{12 - 3}{3 - 0}$

$= \frac{9}{3}$

$= \mathbf{3}$

(ii) Equation of line is in the form $y = 3x + c$

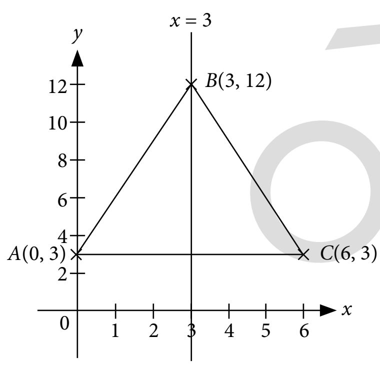
Since $(0, 3)$ lies on the line,

$3 = 3(0) + c$

$c = 3$

$\therefore$ equation of line is $\boldsymbol{y = 3x + 3}$.

(b)



$\therefore$ the coordinates of C are **(6, 3)**.

15. $mx = ny + 2$

$ny = mx - 2$

$y = \frac{m}{n}x - \frac{2}{n}$

Since the gradient of the x-axis is 0,

$\frac{m}{n} = 0$

$m = \mathbf{0}$

For the line to be parallel to the y-axis, $\boldsymbol{n = 0}$.

16. (i) $y = 0$ at the point where the line crosses the x-axis.

$3x + 4(0) = 24$

$3x = 24$

$x = 8$

$\therefore$ the coordinates of A are **(8, 0)**.

$x = 0$ at the point where the line crosses the y-axis.

$3(0) + 4y = 24$

$4y = 24$

$y = 6$

$\therefore$ the coordinates of B are **(0, 6)**.

(ii) $AB = \sqrt{(8 - 0)^2 + (0 - 6)^2}$

$= \sqrt{8^2 + (-6)^2}$

$= \sqrt{64 + 36}$

$= \sqrt{100}$

$=$ **10 units**

(iii) Let the coordinates of C be (k, k).

Gradient of $OC = \frac{k - 0}{k - 0}$

$= \frac{k}{k}$

$= 1$

Equation of OC is in the form $y = x + c$

Since $(0, 0)$ lies on the line,

$0 = 0 + c$

$c = 0$

$\therefore$ equation of OC is $\boldsymbol{y = x}$.

(iv) $y = x$ — (1)

$3x + 4y = 24$ — (2)

Substitute (1) into (2):

$3x + 4x = 24$

$7x = 24$

$x = 3\frac{3}{7}$

Substitute $x = 3\frac{3}{7}$ into (1):

$y = 3\frac{3}{7}$

$\therefore$ the coordinates of C are $\left(\mathbf{3\frac{3}{7}, 3\frac{3}{7}}\right)$.

17. (i) Gradient of $PQ = \frac{5 - 3}{9 - 2}$

$= \frac{2}{7}$

Equation of PQ is in the form $y = \frac{2}{7}x + c$

Since $(2, 3)$ lies on the line,

$3 = \frac{2}{7}(2) + c$

$c = \frac{17}{7}$

$\therefore$ equation of PQ is $y = \frac{2}{7}x + \frac{17}{7}$.

$y = 0$ at the point where the line passing through P and Q intersects the x-axis.

$0 = \frac{2}{7}x + \frac{17}{7}$

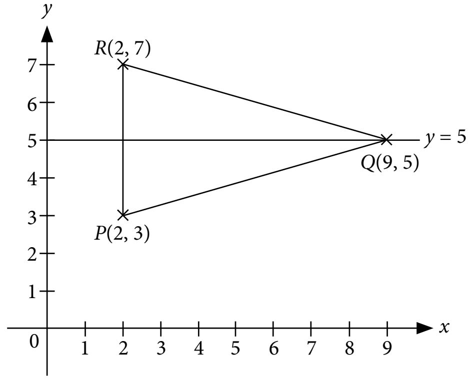
$\frac{2}{7}x = -\frac{17}{7}$

$2x = -17$

$x = -8.5$

$\therefore$ the coordinates of the point are **(–8.5, 0)**.

(ii)



$\therefore$ the coordinates of R are **(2, 7)**.

(iii) $PQ = \sqrt{(9-2)^2 + (5-3)^2}$
$= \sqrt{7^2 + 2^2}$
$= \sqrt{53}$
$=$ **7.28 units** (to 3 s.f.)

(iv) Length of the perpendicular from Q to RP = 7 units

Area of $\triangle PQR = \frac{1}{2} \times$ base $\times$ height
$= \frac{1}{2} \times 4 \times 7$
$= 14$ units^2

Let the length of the perpendicular from R to PQ be h units.
Area of $\triangle PQR$ = 14 units^2

$\frac{1}{2} \times PQ \times h = 14$

$\frac{1}{2} \times \sqrt{53} \times h = 14$

$h = 3.85$ (to 3 s.f.)

$\therefore$ the length of the perpendicular from R to PQ is **3.85 units**.

5.4 Midpoint of a line segment

Practise Now 8

(a) **(i)** Midpoint of $AB = \left(\frac{5+(-1)}{2}, \frac{3+7}{2}\right)$
$=$ **(2, 5)**

(ii) Midpoint of $CD = \left(\frac{2+6}{2}, \frac{3+3}{2}\right)$
$=$ **(4, 3)**

(iii) Midpoint of $EF = \left(\frac{1+1}{2}, \frac{4+(-2)}{2}\right)$
$=$ **(1, 1)**

(b) $\left(\frac{8+p}{2}, \frac{-3+q}{2}\right) = (2, 0)$

$\frac{8+p}{2} = 2$

$\boldsymbol{p} = \mathbf{-4}$

$\frac{-3+q}{2} = 0$

$\boldsymbol{q} = \mathbf{3}$

Practise Now 9

1. y-coordinate of midpoint of $AB = \frac{6+6}{2}$
$= 6$

Since the y-coordinate of Q is 9 ($\neq 6$), $Q(2, 9)$ is not the midpoint of AB.

2. x-coordinate of midpoint of $CD = \frac{-5+(-5)}{2}$
$= -5$

Since the x-coordinate of R is -3 ($\neq -5$), $R(-3, 7)$ is not the midpoint of CD.

Thinking Time (Page 142)

(a) The statement is always true. The x-coordinate of M would be the mean of that of A and B; the y-coordinate of M would be the mean of that of A and B. By Pythagoras' Theorem, M must then be equidistant from points A and B.

(b) Sometimes true. Any point on the line that contains the midpoint of AB and is perpendicular to the line segment AB is equidistant from points A and B. Thus M may not be the midpoint of AB.

Practise Now 10

1. **(i)** Let the coordinates of D be (x, y).

Midpoint of $AC = \left(\frac{-3+4}{2}, \frac{5+6}{2}\right)$
$= (0.5, 5.5)$

Since midpoint of AC = midpoint of BD

$(0.5, 5.5) = \left(\frac{2+x}{2}, \frac{7+y}{2}\right)$

$\frac{2+x}{2} = 0.5$ and $\frac{7+y}{2} = 5.5$

$x = -1$ $\quad$ $y = 4$

$\therefore$ ***D*(−1, 4)**

(ii) Length of $BD = \sqrt{(2-(-1))^2 + (7-4)^2}$
$=$ **4.24 units** (to 3 s.f.)

2. **(i)** $5x + y = 17$

$y = 17 - 5x$ — (1)

$5x^2 + y^2 = 49$ — (2)

Substitute (1) into (2):

$5x^2 + (17-5x)^2 = 49$

$10(3x-8)(x-3) = 0$

$x = \frac{8}{3}$ or 3

When $x = \frac{8}{3}$, $y = \frac{11}{3}$.

When $x = 3$, $y = 2$.

Midpoint of $PQ = \left(\frac{\frac{8}{3}+3}{2}, \frac{\frac{11}{3}+2}{2}\right)$
$= \left(\mathbf{2\frac{5}{6}}, \mathbf{2\frac{5}{6}}\right)$

(ii) Length of $PQ = \sqrt{\left(3-\frac{8}{3}\right)^2 + \left(2-\frac{11}{3}\right)^2}$
$=$ **1.70 units** (to 3 s.f.)

Class Discussion (Finding 4th vertex of quadrilateral)

1. Gradient of $BC = \frac{1-5}{7-(-1)}$

$= -\frac{1}{2}$

$=$ gradient of AD

Equation of AD: $y = -\frac{1}{2}x + c$ — (1)

Substitute $(2, -4)$ into (1):

$-4 = -\frac{1}{2}(2) + c$

$c = -3$

Equation of AD: $y = -\frac{1}{2}x - 3$ — (2)

Gradient of $AB = \frac{-4-1}{2-7}$

$= 1$

$=$ gradient of CD

Equation of CD: $y = x + d$ — (3)

Substitute $(-1, 5)$ into (3):

$5 = -1 + d$

$d = 6$

Equation of CD: $y = x + 6$ — (4)

Substitute (2) into (4):

$-\frac{1}{2}x - 3 = x + 6$

$\frac{3}{2}x = -9$

$x = -6$

Substitute $x = -6$ into (4):

$y = -6 + 6$

$= 0$

$\therefore$ $\boldsymbol{D(-6, 0)}$

2. This rule can be applied to a square, a rectangle and a rhombus, but not a trapezium or a kite. This is because the diagonals of a trapezium or kite do not bisect each other.

Exercise 5D

1. **(a)** Midpoint $= \left(\frac{1+7}{2}, \frac{1+3}{2}\right)$

$= \mathbf{(4, 2)}$

(b) Midpoint $= \left(\frac{3+(-2)}{2}, \frac{-2+7}{2}\right)$

$= \left(\boldsymbol{\frac{1}{2}, 2\frac{1}{2}}\right)$

(c) Midpoint $= \left(\frac{4+8}{2}, \frac{4+4}{2}\right)$

$= \mathbf{(6, 4)}$

(d) Midpoint $= \left(\frac{0+0}{2}, \frac{-2+6}{2}\right)$

$= \mathbf{(0, 2)}$

(e) Midpoint $= \left(\frac{2a+b+b-a}{2}, \frac{3b-a+2a-b}{2}\right)$

$= \left(\boldsymbol{\frac{1}{2}a + b, \frac{1}{2}a + b}\right)$

(f) Midpoint $= \left(\frac{ah^2+ak^2}{2}, \frac{2ah+4ak}{2}\right)$

$= \left(\boldsymbol{\frac{ah^2+ak^2}{2}, ah + 2ak}\right)$

2. **(a)** Let the coordinates of Q be (x, y).

$(5, 0) = \left(\frac{-5+x}{2}, \frac{0+y}{2}\right)$

$\frac{-5+x}{2} = 5$ and $\frac{0+y}{2} = 0$

$x = 15$ $y = 0$

$\therefore$ $\mathbf{Q(15, 0)}$

(b) Let the coordinates of Q be (x, y).

$(3, -7) = \left(\frac{3+x}{2}, \frac{-10+y}{2}\right)$

$\frac{3+x}{2} = 3$ and $\frac{-10+y}{2} = -7$

$x = 3$ $y = -4$

$\therefore$ $\mathbf{Q(3, -4)}$

(c) Let the coordinates of Q be (x, y).

$(6, 2) = \left(\frac{1+x}{2}, \frac{1+y}{2}\right)$

$\frac{1+x}{2} = 6$ and $\frac{1+y}{2} = 2$

$x = 11$ $y = 3$

$\therefore$ $\mathbf{Q(11, 3)}$

(d) Let the coordinates of Q be (x, y).

$(0, -3) = \left(\frac{4+x}{2}, \frac{-1+y}{2}\right)$

$\frac{4+x}{2} = 0$ and $\frac{-1+y}{2} = -3$

$x = -4$ $y = -5$

$\therefore$ $\mathbf{Q(-4, -5)}$

3. **(i)** Midpoint of $AC = \left(\frac{-3+11}{2}, \frac{1+(-3)}{2}\right)$

$= \mathbf{(4, -1)}$

(ii) Let the coordinates of D be (x, y).

midpoint of AC = midpoint of BD

$(4, -1) = \left(\frac{4+x}{2}, \frac{9+y}{2}\right)$

$\frac{4+x}{2} = 4$ and $\frac{9+y}{2} = -1$

$x = 4$ $y = -11$

$\therefore$ $\mathbf{D(4, -11)}$

4. **(a)** Midpoint of $PQ = \left(\frac{k+k}{2}, \frac{5.5+7}{2}\right)$

$= (k, 6.25)$

Since the y-coordinate of $E(k, 4)$ is 4 ($\neq 6.25$), $E(k, 4)$ is not the midpoint of PQ.

Since E, P and Q have the same x-coordinate, they are collinear.

(b) **Method 1:**

The y-coordinate of the midpoint of PQ must be equal to $\frac{-1+5}{2} = 2$.

Since the y-coordinate of $F(-6, 1)$ is 1 ($\neq 2$), then $F(-6, 1)$ is not the midpoint of PQ.

Method 2:

y-coordinate of midpoint of $PQ = \frac{-1+5}{2}$

$= 2$

Since the y-coordinate of $F(-6, 1)$ is 1 ($\neq 2$), then $F(-6, 1)$ is not the midpoint of PQ.

5. Let the coordinates of S be (x, y).

midpoint of PQ = midpoint of RS

$$\left(\frac{1+7}{2}, \frac{-2+4}{2}\right) = \left(\frac{5+x}{2}, \frac{0+y}{2}\right)$$

$$\frac{1+7}{2} = \frac{5+x}{2} \quad \text{and} \quad \frac{-2+4}{2} = \frac{0+y}{2}$$

$$x = 3 \qquad\qquad y = 2$$

$\therefore$ **$S(3, 2)$**

6. $y = x + 2$ — (1)

$y = x^2 + 5x - 3$ — (2)

Substitute (1) into (2):

$x + 2 = x^2 + 5x - 3$

$x^2 + 4x - 5 = 0$

$(x + 5)(x - 1) = 0$

$x = -5$ or 1

When $x = -5$, $y = -3$.

When $x = 1$, $y = 3$.

Midpoint of $PQ = \left(\frac{-5+1}{2}, \frac{-3+3}{2}\right)$

$= \mathbf{(-2, 0)}$

7. **(i)** $x + 2y = 5$

$x = 5 - 2y$ — (1)

$5x^2 + 4y^2 = 29 - 12x$ — (2)

Substitute (1) into (2):

$5(5 - 2y)^2 + 4y^2 = 29 - 12(5 - 2y)$

$24y^2 - 124y + 156 = 0$

$4(6y - 13)(y - 3) = 0$

$y = \frac{13}{6}$ or 3

When $y = \frac{13}{6}$, $x = \frac{2}{3}$.

When $y = 3$, $x = -1$.

Midpoint of $AB = \left(\frac{\frac{2}{3}+(-1)}{2}, \frac{\frac{13}{6}+3}{2}\right)$

$= \left(\mathbf{-\frac{1}{6}, 2\frac{7}{12}}\right)$

(ii) Length of $AB = \sqrt{\left(-1-\frac{2}{3}\right)^2 + \left(3-\frac{13}{6}\right)^2}$

$=$ **1.86 units** (to 3 s.f.)

8. **(a)** Let the initial midpoint of AC be (h, k).

After moving 4 units in the positive direction of the x–axis, the midpoint of AC becomes $(h + 4, k)$.

(b) Initial midpoint of $AC = \left(\frac{h+m}{2}, \frac{k+n}{2}\right)$

Final midpoint of $AC = \left(\frac{h+m}{2} - k, \frac{k+n}{2} - k\right)$

$= \left(\mathbf{\frac{h+m-2k}{2}, \frac{n-k}{2}}\right)$

9. **(i)** Midpoint of AB, $P = \left(\frac{-2+8}{2}, \frac{2+6}{2}\right)$

$= (3, 4)$

Midpoint of AC, $Q = \left(\frac{-2+10}{2}, \frac{2+1}{2}\right)$

$= (4, 1.5)$

Let the coordinates of R be $(x, 0)$.

$\frac{0-4}{x-3} = \frac{0-1.5}{x-4}$

$\frac{5}{2}x = \frac{23}{2}$

$x = 4\frac{3}{5}$

$\therefore$ **$R\left(4\frac{3}{5}, 0\right)$**

(ii) y-coordinate of midpoint of $PR = \frac{0+4}{2}$

$= 2$

Since the y-coordinate of $Q(4, 1.5)$ is 1.5 ($\neq 2$), then $Q(4, 1.5)$ is not the midpoint of PR.

10. Let the coordinates of P be (a, b).

Let the coordinates of Q be (c, d).

Let the coordinates of R be (e, f).

$(-2, 3) = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

$\frac{a+c}{2} = -2$

$a = -c - 4$ — (1)

$\frac{b+d}{2} = 3$

$b = -d + 6$ — (2)

$(5, -1) = \left(\frac{c+e}{2}, \frac{d+f}{2}\right)$

$\frac{c+e}{2} = 5$

$c = -e + 10$ — (3)

$\frac{d+f}{2} = -1$

$d = -f - 2$ — (4)

$(-4, -7) = \left(\frac{a+e}{2}, \frac{b+f}{2}\right)$

$\frac{a+e}{2} = -4$

$a + e = -8$ — (5)

$\frac{b+f}{2} = -7$

$b + f = -14$ — (6)

Substitute (1) into (5):

$-c + e = -4$ — (7)

Substitute (3) into (7):

$e = 3$

Substitute (2) into (6):

$-d + f = -20$ — (8)

Substitute (4) into (8):

$f = -11$

Substitute $e = 3$ into (3):

$c = 7$

Substitute $c = 7$ into (1):
$a = -11$
Substitute $f = -11$ into (4):
$d = 9$
Substitute $d = 9$ into (2):
$b = -3$
$\therefore$ **$P(-11, -3)$, $Q(7, 9)$ and $R(3, -11)$**

5.5 Parallel and perpendicular lines

Investigation (Angle of inclination)

1. (a) **63.0°**
   (b) **2.0** (to 2 s.f.)
2. $m_1 = \dfrac{6-0}{0-(-3)}$
   $= \mathbf{2}$
   $= \tan\theta_1$
   Since θ_1 is acute, $\tan\theta_1 = \dfrac{\text{opposite side}}{\text{adjacent side}}$
   $= \dfrac{\text{rise}}{\text{run}}$ of l_1.
3. θ_2 is **obtuse**.
4. (a) **161.5°**
   (b) **−0.33** (to 2 s.f.). $\tan\theta_2$ is **negative**.
5. $m_2 = \dfrac{2-0}{0-6}$
   $= \mathbf{-0.33}$ (to 2 s.f.)
   $= \tan\theta_2$
   Since θ_2 is obtuse, $\tan\theta_2 = -\dfrac{\text{opposite side}}{\text{adjacent side}}$
   $= \dfrac{\text{rise}}{\text{run}}$ of l_2.
6. $\boldsymbol{m = \tan\theta}$.
   For horizontal lines ($m = 0$), $\tan 180° = 0$. Thus the relationship holds for horizontal lines.
   Both m and $\tan\theta$ are undefined for vertical lines.

Class Discussion (Parallel lines)

1. **Yes.**
2. $\theta_1 = \mathbf{63.0°}$
   $\theta_2 = \mathbf{63.0°}$
   Since $\theta_1 = \theta_2$, the two lines must be parallel as they behave like corresponding angles between two parallel lines.
3. The gradient of both lines = **2**.
5. (a) If two lines are parallel to each other, then they have the same **gradient**.
   (b) If two lines have the same gradient, then they are **parallel** to each other.

Thinking Time (Page 146)

1. Gradient of $AB = \dfrac{5-3}{3-2}$
   $= \mathbf{2}$
   Gradient of $BC = \dfrac{9-5}{5-3}$
   $= \mathbf{2}$
   Gradient of $AC = \dfrac{9-3}{5-2}$
   $= \mathbf{2}$
   They are parallel. In fact, they are collinear.
2. Since the gradients are equal and Y is a common point, they are collinear.

Practise Now 11

(i) $\dfrac{k+4-9}{k+1-0} = \dfrac{k+6-(k+3)}{2k+2-2k}$
$\dfrac{k-5}{k+1} = \dfrac{3}{2}$
$k = \mathbf{-13}$

(ii) $\dfrac{k+4-9}{k+1-0} = \dfrac{k+3-9}{2k-0}$
$\dfrac{k-5}{k+1} = \dfrac{k-6}{2k}$
$(k-3)(k-2) = 0$
$k = \mathbf{3}$ **or** $\mathbf{2}$

Class Discussion (Perpendicular lines)

1. **Yes**, the two lines are perpendicular.
2. The gradients of the two lines are **2** and $\mathbf{-\frac{1}{2}}$ respectively.
3. The product of the gradients of the two perpendicular lines
   $= 2 \times -\dfrac{1}{2}$
   $= \mathbf{-1}$
4. The product of the gradients of two perpendicular lines is −1, except for the case where one of them is a vertical line.
5. The relationship does not hold if one of them is a vertical line. This is because the gradient of a vertical line is undefined and hence we will not be able to calculate the product of their gradients.

Practise Now 12

1. $AB = \sqrt{(-1-0)^2 + (-2-1)^2}$
   $= \sqrt{10}$
   $BC = \sqrt{(-1-2)^2 + (-2-2)^2}$
   $= \sqrt{25}$
   $AC = \sqrt{(2-0)^2 + (-2-1)^2}$
   $= \sqrt{13}$
   No. Since none of the combinations in the form $X^2 + Y^2 = Z^2$ is true, $\triangle ABC$ is not a right-angled triangle.

2. (i) Midpoint of AB, $M = \left(\frac{1+5}{2}, \frac{3+1}{2}\right)$

$= \mathbf{(3, 2)}$

(ii) Gradient of $AB = \frac{1-3}{5-1}$

$= -0.5$

Gradient of $MC = -\frac{1}{-0.5}$

$= 2$

$2 = \frac{-1-2}{k-3}$

$k = \mathbf{\frac{3}{2}}$

Practise Now 13

Gradient of $OA = \frac{1-0}{5-0}$

$= \frac{1}{5}$

Gradient of $OB = -\frac{1}{\frac{1}{5}}$

$= \frac{q-0}{p-0}$

$q = -5p$

$\therefore$ $\mathbf{p = 2}$ **and** $\mathbf{q = -10}$

Exercise 5E

1. (i) Gradient of $OA = \frac{3k-0}{2-0}$

$= \frac{3k}{2}$

Gradient of $BC = \frac{7-6}{10k-4k}$

$= \frac{1}{6k}$

$\frac{3k}{2} = \frac{1}{6k}$

$k^2 = \frac{1}{9}$

$k = \pm\mathbf{\frac{1}{3}}$

(ii) Gradient of $OB = \frac{6-0}{4k-0}$

$= \frac{6}{4k}$

$\frac{3k}{2} = \frac{6}{4k}$

$k^2 = 1$

$k = \mathbf{1\pm}$

2. Gradient of $AB = \frac{4-1}{-1-1}$

$= -\frac{3}{2}$

Gradient of $BC = -\frac{1}{-\frac{3}{2}}$

$= \frac{2}{3}$

$\frac{2}{3} = \frac{k-4}{6-(-1)}$

$k = \mathbf{\frac{26}{3}}$

3. Gradient of $AB = \frac{3-(-3)}{2-(-1)}$

$= 2$

Gradient of $BC = -\frac{1}{2}$

$-\frac{1}{2} = \frac{k-3}{k+5-2}$

$-k-3 = 2k-6$

$k = \mathbf{1}$

4. (i) Gradient of $AB = \frac{-12-(-6)}{3-(-1)}$

$= -\frac{3}{2}$

Gradient of $AC = -\frac{1}{-\frac{3}{2}}$

$= \frac{2}{3}$

$\frac{2}{3} = \frac{6-(-6)}{k-(-1)}$

$2k+2 = 36$

$k = \mathbf{17}$

(ii) Gradient of $AC = -\frac{3}{2}$

$-\frac{3}{2} = \frac{6-(-6)}{k-(-1)}$

$-3k-3 = 24$

$k = \mathbf{-9}$

5. (a) Let $f(x) = 7x + 14$.
Let $g(x) = 7x + 16$.
Since the gradient of $f(x) = g(x) = 7$, the two lines are **parallel**.

(b) Let $f(x) = 4x - 7$.

Let $g(x) = -\frac{1}{4}x + 5$.

Since the product of the gradients $= 4\left(-\frac{1}{4}\right) = -1$, the two lines are **perpendicular**.

(c) Let $f(x) = -2x + 2$.

$2y = x + 6$

$y = \frac{1}{2}x + 3$

Let $g(x) = \frac{1}{2}x + 3$.

Since the product of the gradients $= -2\left(\frac{1}{2}\right) = -1$, the two lines are **perpendicular**.

(d) $x + 2y = 6$

$$y = -\frac{1}{2}x + 3$$

Let $f(x) = -\frac{1}{2}x + 3$.

Let $g(x) = 3x - 8$.

Since the two gradients are different and the product of the gradients $= -\frac{1}{2}(3) \neq -1$, the lines are **neither parallel nor perpendicular**.

6. Gradient of $AB = \frac{6-n}{4-m}$

Gradient of $BC = -\frac{1}{\frac{6-n}{4-m}}$

$$= \frac{m-4}{6-n}$$

$$\frac{m-4}{6-n} = \frac{3-6}{5-4}$$

$$m - 4 = -18 + 3n$$

$$m = 3n - 14$$

$\therefore$ **$n = 5$ and $m = 1$**

7. (i) Let the coordinates of P be $(2h, h)$.

$$PR = \sqrt{[2h-(-2)]^2 + (h-4)^2}$$
$$= \sqrt{5h^2 + 20}$$
$$PS = \sqrt{(2h-6)^2 + [h-(-4)]^2}$$
$$= \sqrt{5h^2 - 16h + 52}$$
$$\sqrt{5h^2+20} = \sqrt{5h^2 - 16h + 52}$$
$$5h^2 + 20 = 5h^2 - 16h + 52$$
$$h = 2$$

$\therefore$ **$P(4, 2)$**

(ii) Gradient of $PS = \frac{-4-2}{6-4}$

$$= -3$$

Gradient of $PR = \frac{4-2}{-2-4}$

$$= -\frac{1}{3}$$

Since the gradient of $PS \neq$ gradient of PR, the 3 points are not collinear.

8. $3y - 2x = 4$

$$y = \frac{2}{3}x + \frac{4}{3}$$

Let $f(x) = \frac{2}{3}x + \frac{4}{3}$.

$4x = 6y - 8$

$$y = \frac{2}{3}x + \frac{8}{6}$$

Let $g(x) = \frac{2}{3}x + \frac{8}{6}$.

Since the gradient of $f(x)$ = gradient of $g(x) = \frac{2}{3}$, the two lines are parallel.

9. $PQ = \sqrt{(-1-6)^2 + (3-8)^2}$

$$= \sqrt{74}$$

$PR = \sqrt{(-1-11)^2 + (3-1)^2}$

$$= \sqrt{148}$$

$QR = \sqrt{(6-11)^2 + (8-1)^2}$

$$= \sqrt{74}$$

Since $PQ = QR$, PQR is an isosceles triangle. (shown)

As $PQ^2 + QR^2 = PR^2$, by the converse of Pythagoras' Theorem, $\triangle PQR$ is a right-angled triangle.

10. (a) Let the coordinates of C be $(x, 0)$.

$$AB = \sqrt{(6-0)^2 + (6-4)^2}$$
$$= \sqrt{40}$$
$$BC = \sqrt{(6-x)^2 + (6-0)^2}$$
$$= \sqrt{72 - 12x + x^2}$$
$$\sqrt{40} = \sqrt{72 - 12x + x^2}$$
$$40 = 72 - 12x + x^2$$
$$(x-8)(x-4) = 0$$
$$x = 8 \text{ or } x = 4$$

$\therefore$ **$C(8, 0)$ or $C(4, 0)$**

(b) Let the coordinates of D be $(0, y)$.

Gradient of $AB = \frac{6-4}{6-0}$

$$= \frac{1}{3}$$

Gradient of $BD = -3$

$$= \frac{y-6}{0-6}$$

$y = 24$

$\therefore$ **$D(0, 24)$**

11. $x + 3y = 1$

$x = 1 - 3y$ — (1)

$5y = 20 - 3x - x^2$ — (2)

Substitute (1) into (2):

$$5y = 20 - 3(1-3y) - (1-3y)^2$$
$$9y^2 - 10y - 16 = 0$$
$$(9y+8)(y-2) = 0$$
$$y = 2 \text{ or } -\frac{8}{9}$$

From (1), when $y = 2$, $x = -5$.

From (1), when $y = -\frac{8}{9}$, $x = \frac{11}{3}$.

Gradient of $PQ = \frac{2 - \left(-\frac{8}{9}\right)}{-5 - \frac{11}{3}}$

$$= -\frac{1}{3}$$

Equation of AB: $y - 3x = 4$

$$y = 3x + 4$$

Equation of CD: $3y + x = 4$

$$y = -\frac{1}{3}x + \frac{4}{3}$$

(a) Since the product of the gradients = $3\left(-\frac{1}{3}\right) = -1$, AB is **perpendicular** to PQ.

(b) Since gradient of CD = gradient of $PQ = -\frac{1}{3}$, the two lines are **parallel**.

12. $3x + y = 8$

$y = -3x + 8$

Let $f(x) = -3x + 8$.

Let the line connecting $(-4, 1)$ and $(8, 7)$ be A.

$$\text{Gradient of } A = \frac{7-1}{8-(-4)} = \frac{1}{2}$$

Since the product of the gradients $= -3\left(\frac{1}{2}\right) \neq -1$, A is not perpendicular to $f(x)$.

Let the equation of A be $y = \frac{1}{2}x + c$. — (1)

Substitute $(-4, 1)$ into (1):

$c = 3$

Equation of A: $y = \frac{1}{2}x + 3$

$$\text{Midpoint of line segment} = \left(\frac{-4+8}{2}, \frac{1+7}{2}\right) = (2, 4)$$

$$-3x + 8 = \frac{1}{2}x + 3$$

$$x = \frac{10}{7}$$

Since the x-coordinate of the intersection point between $f(x)$ and A is $\frac{10}{7}$ ($\neq 2$), $f(x)$ is not the perpendicular bisector of the line segment.

13. (i) $$\text{Gradient of } AB = \frac{-3-3}{-1-2} = 2$$

$$\text{Gradient of } BC = -\frac{1}{2}$$

$$-\frac{1}{2} = \frac{k-3}{6-2}$$

$k = \mathbf{1}$

(ii) $$\text{Gradient of } AC = \frac{-3-1}{-1-6} = \mathbf{\frac{4}{7}}$$

(iii) Acute angle between AC and x-axis $= \tan^{-1}\frac{4}{7}$

$= \mathbf{29.7°}$

14. $$\text{Gradient of } OP = \frac{b-0}{a-0} = \frac{b}{a}$$

$$\text{Gradient of } OQ = -\frac{1}{\frac{b}{a}} = -\frac{a}{b}$$

$$-\frac{a}{b} = \frac{d-0}{c-0}$$

$bd + ac = 0$

A possible set of values would be: $\mathbf{a = 2, b = 3, c = 3, d = -2}$

15. Let $A = (2, 1)$, $B = (-1, -5)$, $C = (-2, -1)$ and $D = (1, 5)$.

Method 1:

$$\text{Gradient of } AD = \frac{1-5}{2-1} = -4$$

$$\text{Gradient of } BC = \frac{-5-(-1)}{-1-(-2)} = -4$$

$$\text{Gradient of } CD = \frac{-1-5}{-2-1} = 2$$

$$\text{Gradient of } AB = \frac{-5-1}{-1-2} = 2$$

Since gradient of AD = gradient of $BC = -4$, $AD \parallel BC$.

Since gradient of CD = gradient of $AB = 2$, $CD \parallel AB$.

Therefore, $ABCD$ is a parallelogram.

Method 2:

$$\text{Midpoint of } AC = \left(\frac{2-2}{2}, \frac{1-1}{2}\right) = (0, 0)$$

$$\text{Midpoint of } BD = \left(\frac{-1+1}{2}, \frac{-5+5}{2}\right) = (0, 0)$$

Since the midpoint of AC = midpoint of BD, $ABCD$ is a parallelogram.

16. Let $A = (5, 8)$, $B = (7, 5)$, $C = (5, 2)$ and $D = (3, 5)$.

Method 1:

$$AB = \sqrt{(8-5)^2 + (5-7)^2} = \sqrt{13}$$

$$BC = \sqrt{(7-5)^2 + (5-2)^2} = \sqrt{13}$$

$$CD = \sqrt{(5-3)^2 + (5-2)^2} = \sqrt{13}$$

$$AD = \sqrt{(5-3)^2 + (8-5)^2} = \sqrt{13}$$

Since $AB = BC = CD = AD$, $ABCD$ is a rhombus.

Method 2:

Midpoint of $AC = \left(\frac{5+5}{2}, \frac{8+2}{2}\right)$

$= (5, 5)$

Midpoint of $BD = \left(\frac{7+3}{2}, \frac{5+5}{2}\right)$

$= (5, 5)$

Gradient of $BD = \frac{5-5}{7-3}$

$= 0$ (Horizontal line)

AC is a vertical line since the x-coordinates of A and C are the same.

Thus AC and BD are perpendicular bisectors of each other, making $ABCD$ a rhombus.

5.6 Equation of a straight line involving parallel and perpendicular lines

Practise Now 14

1. Equation of line l is $y - 4 = 2(x - 1)$.

$\mathbf{y = 2x + 2}$

OPEN

A line that is perpendicular to l may have a possible equation $\mathbf{y = -\frac{1}{2}x + 3}$.

2. (a) Gradient of required line $= -\frac{1}{1}$

$= -1$

Equation of required line: $y - 7 = -(x - 2)$

$\mathbf{y = -x + 9}$

(b) $2y + x = 2$

$y = -\frac{1}{2}x + 1$

Gradient of required line $= -\frac{1}{-\frac{1}{2}}$

$= 2$

Equation of required line: $y - 3 = 2[x - (-2)]$

$\mathbf{y = 2x + 7}$

(c) $3y + 2x + 5 = 0$

$y = -\frac{2}{3}x - \frac{5}{3}$

Gradient of required line $= -\frac{2}{3}$

Equation of required line: $y - (-9) = -\frac{2}{3}(x - 4)$

$y = -\frac{2}{3}x - \frac{19}{3}$

$\mathbf{3y = -2x - 19}$

(d) $6x = 2y - 7$

$y = 3x + \frac{7}{2}$

Gradient of required line $= 3$

Equation of required line: $y - (-2) = 3[x - (-5)]$

$\mathbf{y = 3x + 13}$

Practise Now 15

Midpoint of $CD = \left(\frac{5-7}{2}, \frac{7+1}{2}\right)$

$= (-1, 4)$

Gradient of $CD = \frac{1-7}{-7-5}$

$= \frac{1}{2}$

Gradient of perpendicular bisector of $CD = -2$

Equation of perpendicular bisector of CD:

$y - 4 = -2(x + 1)$

$\mathbf{y = -2x + 2}$

Practise Now 16

(i) Gradient of $AB = \frac{8-2}{16-4}$

$= \frac{1}{2}$

Gradient of $DC = \frac{1}{2}$

Equation of DC: $y - 10 = \frac{1}{2}(x - 5)$

$\mathbf{y = \frac{1}{2}x + \frac{15}{2}}$ — (1)

Gradient of $BC = -2$

Equation of BC: $y - 8 = -2(x - 16)$

$\mathbf{y = -2x + 40}$ — (2)

(ii) $\frac{1}{2}x + \frac{15}{2} = -2x + 40$

$x = 13$

Substitute $x = 13$ into (2):

$y = 14$

$\therefore$ **C(13, 14)**

(iii) $CD = \sqrt{(13-5)^2 + (14-10)^2}$

$=$ **8.94 units** (to 3 s.f.)

Exercise 5F

1. (a) Equation of l: $y - 8 = 3(x + 2)$

$\mathbf{y = 3x + 14}$

OPEN

A line that is perpendicular to l may have a possible equation $\mathbf{y = -\frac{1}{3}x + 5}$

(b) Equation of l: $y + 3 = -\frac{1}{2}(x - 7)$

$\mathbf{y = -\frac{1}{2}x + \frac{1}{2}}$

A line that is parallel to l may have a possible equation $\mathbf{y = -\frac{1}{2}x - 4}$

2. (a) Gradient of $AB = \frac{-1-5}{2-5}$

$= 2$

Equation of line: $y - 5 = 2(x - 5)$

$\mathbf{y = 2x - 5}$

(b) Gradient of $CD = \frac{-4-6}{1-0}$

$= -10$

Equation of line: $y - 6 = -10(x - 0)$

$\mathbf{y = -10x + 6}$

(c) Gradient of $EF = \frac{3-3}{-3-7}$

$= 0$

Equation of line: $y - 3 = 0(x - 7)$

$\boldsymbol{y = 3}$

(d) Gradient of $GH = \frac{-2-8}{-9-(-9)}$ (undefined)

Equation of line: $\boldsymbol{x = -9}$

3. (a) $3y + 7 = 29$

$y = \frac{22}{3}$

Gradient of required line = 0

Equation of required line: $y - 5 = 0[x - (-2)]$

$\boldsymbol{y = 5}$

(b) $42 - 7y = 5$

$y = \frac{37}{7}$

Gradient of required line $= -\frac{1}{0}$ (undefined)

Equation of required line: $\boldsymbol{x = -1}$

(c) $3x + y = 17$

$y = -3x + 17$

Gradient of required line = -3

Equation of required line: $y - 8 = -3(x - 4)$

$\boldsymbol{y = -3x + 20}$

(d) $y + 2x = 13$

$y = -2x + 13$

Gradient of required line $= -\frac{1}{-2}$

$= \frac{1}{2}$

Equation of required line: $y - (-3) = \frac{1}{2}(x - 2)$

$\boldsymbol{y = \frac{1}{2}x - 4}$

4. $y = 2x^2 - 3$

When $x = -1$, $y = -1$.

When $x = 1$, $y = -1$.

Gradient of line $= \frac{-1-(-1)}{1-(-1)}$

$= 0$

Equation of line: $\boldsymbol{y = -1}$

5. (a) Let A be $(0, -2)$ and B be $(2, 0)$.

Gradient of $AB = \frac{-2-0}{0-2}$

$= 1$

Midpoint of $AB = \left(\frac{0+2}{2}, \frac{-2+0}{2}\right)$

$= (1, -1)$

Equation of perpendicular bisector of AB:

$y + 1 = -1(x - 1)$

$\boldsymbol{y = -x}$

(b) Let A be $(1, 8)$ and B be $(2, 3)$.

Gradient of $AB = \frac{8-3}{1-2}$

$= -5$

Midpoint of $AB = \left(\frac{1+2}{2}, \frac{8+3}{2}\right)$

$= \left(\frac{3}{2}, \frac{11}{2}\right)$

Equation of perpendicular bisector of AB:

$y - \frac{11}{2} = \frac{1}{5}\left(x - \frac{3}{2}\right)$

$\boldsymbol{y = \frac{1}{5}x + \frac{26}{5}}$

(c) Let A be $(0, -7)$ and B be $(6, -7)$.

Gradient of $AB = \frac{-7-(-7)}{0-6}$

$= 0$

Midpoint of $AB = \left(\frac{0+6}{2}, \frac{-7+(-7)}{2}\right)$

$= (3, -7)$

Equation of perpendicular bisector of AB:

$\boldsymbol{x = 3}$

(d) Let A be $(5, -4)$ and B be $(5, 9)$.

Gradient of $AB = \frac{-4-9}{5-5}$ (undefined)

Midpoint of $AB = \left(\frac{5+5}{2}, \frac{-4+9}{2}\right)$

$= \left(5, 2\frac{1}{2}\right)$

Equation of perpendicular bisector of AB:

$\boldsymbol{y = \frac{5}{2}}$

6. $2y - x = 7$

$y = \frac{1}{2}x + \frac{7}{2}$

Let A be $(3, 1)$ and B be $(1, -5)$.

Midpoint of $AB = \left(\frac{3+1}{2}, \frac{1-5}{2}\right)$

$= (2, -2)$

Equation of line:

$y + 2 = \frac{1}{2}(x - 2)$

$\boldsymbol{y = \frac{1}{2}x - 3}$

7. $3y + x = 3$

$x = 3 - 3y$ — (1)

$4y - 3x = 5$ — (2)

Substitute (1) into (2):

$4y - 3(3 - 3y) = 5$

$y = \frac{14}{13}$

Substitute $y = \frac{14}{13}$ into (1):

$x = -\frac{3}{13}$

$$\text{Gradient of line} = \frac{y-0}{0-x} = \frac{y}{-2y} = -\frac{1}{2}$$

Equation of line:

$$y - \frac{14}{13} = -\frac{1}{2}\left(x + \frac{3}{13}\right)$$

$\mathbf{26y + 13x - 25 = 0}$

8. $$\text{Gradient of } AC = \frac{8-2}{12-0} = \frac{1}{2}$$

$$\text{Midpoint of } AC = \left(\frac{0+12}{2}, \frac{2+8}{2}\right) = (6, 5)$$

Equation of perpendicular bisector of AC:

$y - 5 = -2(x - 6)$

$\mathbf{y = -2x + 17}$

Gradient of $BD = -2$

Equation of perpendicular bisector of BD:

$y - 5 = \frac{1}{2}(x - 6)$

$\mathbf{y = \frac{1}{2}x + 2}$

9. (i) $$\text{Coordinates of } M = \left(\frac{1+6}{2}, \frac{3+1}{2}\right) = \left(3\frac{1}{2}, 2\right)$$

$$\text{Gradient of } AC = \frac{3-1}{1-6} = -\frac{2}{5}$$

Equation of BD:

$$y - 2 = \frac{5}{2}\left(x - 3\frac{1}{2}\right)$$

$\mathbf{y = \frac{5}{2}x - \frac{27}{4}}$

(ii) x-coordinate of B ($y = 0$),

$0 = \frac{5}{2}x - \frac{27}{4}$

$x = \frac{27}{10}$

$\therefore \mathbf{B\left(2\frac{7}{10}, 0\right)}$

(iii) Let the coordinates of D be (x, y).

$$\text{Midpoint of } BD = \left(\frac{2\frac{7}{10}+x}{2}, \frac{0+y}{2}\right)$$

$$\left(3\frac{1}{2}, 2\right) = \left(\frac{2\frac{7}{10}+x}{2}, \frac{0+y}{2}\right)$$

$$\frac{7}{2} = \frac{\frac{27}{10}+x}{2}$$

$x = \frac{43}{10}$

$2 = \frac{y}{2}$

$y = 4$

$\therefore \mathbf{D\left(4\frac{3}{10}, 4\right)}$

10. (i) $$\text{Gradient of } AC = \frac{8-0}{-2-4} = -\frac{4}{3}$$

Equation of AC:

$y - 0 = -\frac{4}{3}(x - 4)$

$\mathbf{y = -\frac{4}{3}x + \frac{16}{3}}$

(ii) $AB = \sqrt{(-3-4)^2 + (1-0)^2} = \sqrt{50}$

$BC = \sqrt{(-2+3)^2 + (8-1)^2} = \sqrt{50}$

Thus $AB = BC$. (shown)

$$\text{Gradient of } AB = \frac{1-0}{-3-4} = -\frac{1}{7}$$

$$\text{Gradient of } BC = \frac{8-1}{-2+3} = 7$$

Since the product of the gradients $= 7\left(-\frac{1}{7}\right) = -1$, AB is perpendicular to BC.

(iii) Let the coordinates of D be (x, y).

$$\text{Midpoint of } AC = \left(\frac{4-2}{2}, \frac{8+0}{2}\right) = (1, 4)$$

$$(1, 4) = \left(\frac{-3+x}{2}, \frac{1+y}{2}\right)$$

$1 = \frac{-3+x}{2}$ and $4 = \frac{1+y}{2}$

$x = 5$ $y = 7$

$\therefore \mathbf{D(5, 7)}$

(iv) Area of square $ABCD = \left(\sqrt{50}\right)^2$

$= \mathbf{50\ units^2}$

11. (i) Gradient of $BC = \frac{6-4}{8-4}$

$= \frac{1}{2}$

Equation of AP:

$y - 5 = -2(x - 1)$

$\mathbf{y = -2x + 7}$

(ii) Equation of BC:

$y - 6 = \frac{1}{2}(x - 8)$

$y = \frac{1}{2}x + 2$ — (1)

$-2x + 7 = \frac{1}{2}x + 2$

$x = 2$

Substitute $x = 2$ into (1):

$y = 3$

$\therefore \mathbf{P(2, 3)}$

(iii) $AP = \sqrt{(1-2)^2 + (5-3)^2}$

$= \mathbf{2.24\ units}$ (to 3 s.f.)

$BC = \sqrt{(8-4)^2 + (6-4)^2}$

$= \mathbf{4.47\ units}$

$AC = \sqrt{(8-1)^2 + (6-5)^2}$

$= \mathbf{7.07\ units}$

(iv) Area of $\triangle ABC = \frac{1}{2}(\sqrt{5})(\sqrt{20})$

$= \mathbf{5\ units^2}$

(v) Length of perpendicular from B to AC

$= \frac{2(5)}{\sqrt{50}}$

$= \mathbf{1.41\ units}$

12. (i) Gradient of BC, $m_{BC} = \frac{8-3}{1-6}$

$= \mathbf{-1}$

Gradient of AB, $m_{AB} = \frac{3-1}{6+3}$

$= \mathbf{\frac{2}{9}}$

(ii) Let coordinates of H be $(0, y)$.

Gradient of $HC = \frac{8-y}{1-0}$

$-1 = 8 - y$

$y = 9$

$\therefore \mathbf{H(0, 9)}$

(iii) Let the coordinates of D be (x, y).

Midpoint of BD = Midpoint of AC

$\left(\frac{6+x}{2}, \frac{3+y}{2}\right) = \left(\frac{-3+1}{2}, \frac{1+8}{2}\right)$

$\frac{6+x}{2} = -1$ and $\frac{3+y}{2} = \frac{9}{2}$

$x = -8$ $\quad$ $y = 6$

$\therefore \mathbf{D(-8, 6)}$

(iv) Midpoint of $BC = \left(\frac{6+1}{2}, \frac{8+3}{2}\right)$

$= \left(3\frac{1}{2}, 5\frac{1}{2}\right)$

Equation of perpendicular bisector of BC:

$y - 5\frac{1}{2} = 1\left(x - 3\frac{1}{2}\right)$

$\mathbf{y = x + 2}$

13. (a) Gradient of $AB = \frac{7-2}{6-1}$

$= 1$

Midpoint of $AB = \left(\frac{1+6}{2}, \frac{2+7}{2}\right)$

$= \left(3\frac{1}{2}, 4\frac{1}{2}\right)$

Equation of perpendicular bisector of AB:

$y - 4\frac{1}{2} = -1\left(x - 3\frac{1}{2}\right)$

$\mathbf{y = -x + 8}$ — (1)

(b) Gradient of $BC = \frac{7-2}{6-7}$

$= -5$

Midpoint of $BC = \left(\frac{6+7}{2}, \frac{7+2}{2}\right)$

$= \left(6\frac{1}{2}, 4\frac{1}{2}\right)$

Equation of perpendicular bisector of BC:

$y - 4\frac{1}{2} = \frac{1}{5}\left(x - 6\frac{1}{2}\right)$

$\mathbf{y = \frac{1}{5}x + \frac{16}{5}}$ — (2)

Substitute (1) into (2):

$-x + 8 = \frac{1}{5}x + \frac{16}{5}$

$x = 4$

Substitute $x = 4$ into (1):

$y = 4$

$\therefore$ the equidistant point from A, B and C is $\mathbf{(4, 4)}$.

14. (i) Gradient of $PQ = \frac{10-4}{-1-2}$

$= -2$

Midpoint of $PQ = \left(\frac{-1+2}{2}, \frac{10+4}{2}\right)$

$= \left(\frac{1}{2}, 7\right)$

Equation of perpendicular bisector of PQ:

$y - 7 = \frac{1}{2}\left(x - \frac{1}{2}\right)$

$\mathbf{y = \frac{1}{2}x + \frac{27}{4}}$

(ii) Let the coordinates of H be $(x, 0)$.
Let the coordinates of K be $(0, y)$.

$$\frac{1}{2}x + \frac{27}{4} = 0$$

$$x = -\frac{27}{2}$$

$$y = \frac{27}{4}$$

$$\therefore H\left(-13\frac{1}{2}, 0\right) \text{ and } K\left(0, 6\frac{3}{4}\right)$$

$$HK = \sqrt{\left(-13\frac{1}{2} - 0\right)^2 + \left(0 - 6\frac{3}{4}\right)^2}$$

= **15.1 units** (to 1 d.p.)

(iii) Let the coordinates of R be (x, y).

$$\text{Midpoint of } HR = \left(\frac{-13\frac{1}{2} + x}{2}, \frac{0 + y}{2}\right)$$

$$\left(\frac{1}{2}, 7\right) = \left(\frac{-13\frac{1}{2} + x}{2}, \frac{0 + y}{2}\right)$$

$$\frac{1}{2} = \frac{-13\frac{1}{2} + x}{2} \quad \text{and} \quad 7 = \frac{y}{2}$$

$$x = \frac{29}{2} \qquad\qquad y = 14$$

$$\therefore \boldsymbol{R\left(14\frac{1}{2}, 14\right)}$$

(iv) Gradient of $HR = \dfrac{14 - 0}{14\frac{1}{2} + 13\frac{1}{2}}$

$$= \frac{1}{2}$$

Gradient of $PQ = -2$

Since the product of the gradients $= \frac{1}{2}(-2) = -1$, HR is perpendicular to PQ.

$$\text{Midpoint of } HR = \left(\frac{-13\frac{1}{2} + 14\frac{1}{2}}{2}, \frac{14 + 0}{2}\right)$$

$$= \left(\frac{1}{2}, 7\right)$$

$$\text{Midpoint of } PQ = \left(\frac{1}{2}, 7\right)$$

$\therefore$ HR and PQ are perpendicular bisectors of each other and $PRQH$ is a rhombus. (shown)

Chapter 6 Graphs of Functions and Graphical Solution

TEACHING NOTES

Suggested Approach

This chapter serves as an introduction to the important concepts of relations and functions. Before plotting graphs of functions, revise the choice of scales and labelling of scales on both axes. Students are often weak in some of these areas. Teachers should encourage the students to draw the curves free hand, as well as to use curved rules to assist them.

It will be worthwhile to ask students to remember the general shapes of quadratic graphs, the "U" shape and the inverted-"U" shape. Also, students should remember the general shapes of cubic, reciprocal and exponential graphs. This will help them to identify and rectify errors when they sketch or plot a few points wrongly and if the shapes of their graphs look odd.

Section 6.1: Graphs of cubic functions

Teachers should help students to see how the coefficient of x^3 affects the shape of the graph. Teachers can ask students who are confident and have grasped the concept behind graphs of cubic functions to explain to the class (see Investigation: Graphs of cubic functions). Teachers will facilitate the class through the activity as the students lead and learn together.

Section 6.2: Graphs of reciprocal functions

When introducing graphs of reciprocal functions, teachers should bring the students' attention to the marginal notes on the four quadrants on the Cartesian plane and the order of rotational symmetry about a particular point (see Information and Recall on page 166). Students need to know why the graphs of $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ do not intersect with both axes and the values obtained when any real number is divided by zero (see Investigation: Graphs of $y = \frac{a}{x}$ and Investigation: Graphs of $y = \frac{a}{x^2}$).

When students are able to understand the shapes of the graphs of reciprocal functions, they can go on to further describe the graphs in the cases where $a < 0$ and where $a > 0$ (see pages 166 and 167 of the textbook).

Section 6.3: Graphs of functions involving $\sqrt{x}$

As an introduction to this section, teachers may first do a quick recap with students about the square root function, which they have learnt in Book 1. Some students might not understand which negative values of x are absent in the graphs. Teachers may highlight to students that negative values within the square root function give an output that is not real.

Teachers should help students to see how the value of a affects the shape of the graphs of $y = a\sqrt{x}$ and $y = \frac{a}{\sqrt{x}}$ (see Investigation: Graphs of $y = a\sqrt{x}$ and Investigation: Graphs of $y = \frac{a}{\sqrt{x}}$). At the same time, teachers may also highlight an interesting fact that when $x < 1$, $\sqrt{x}$ gives a value that is greater than x itself (e.g. $\sqrt{0.09} = 0.3 > 0.09$ but $\sqrt{9} = 3 < 9$). However, there is no break in any of the graphs at $x = 1$.

Section 6.4: Graphs of exponential functions

Teachers should help students to see how the values of a and k affect the shape of the graphs of $y = a^x$ and $y = ka^x$. Teachers can ask students who are confident and have grasped the concept behind the graphs of exponential functions to explain to the class (see Investigation: Graphs of $y = a^x$ and $y = ka^x$) while the teachers facilitate.

After students have learnt the graphs of power and exponential functions, teachers can get them to match such graphs with their corresponding functions and see whether they are able to differentiate between the different types of graphs (see Class Discussion: Matching graphs of power functions with the corresponding functions).

Section 6.5: Graphs of rational functions

This section is an extension of what students have learnt in Section 6.2, which also requires a revisit of the solving of quadratic equations which were covered in Chapters 2 and 3. In this section, students will learn to identify the vertical and horizontal asymptotes, if any, of the graph of a rational function.

It is also important for students to be able to identify the degree of a polynomial. As such, teachers should guide students to see if they are able to correctly identify the degree of a polynomial (see Investigation: Graph of a rational function involving polynomials and its asymptotes).

Section 6.6: Gradient of a curve

Teachers can first revise what was covered in Chapter 5, where the gradient of a straight line is the ratio of the vertical change to the horizontal change. Students may have difficulties in drawing the tangent to the curve. They may either draw the line such that it does not pass through the point or it has a gradient which is very different from the actual gradient. Teachers can guide them along and suggest to them to draw the tangent by first picking two points on the curve, e.g. *A* and *B*, such that each is at an equal distance from and is near the point of contact, before drawing a line to pass through these two points.

Section 6.7: Applications of graphs in real-world contexts

Students tend to make mistakes with the quantities "distance", "time" and "speed". As such, teachers should revise the formula for speed as the distance per unit time. Teachers should highlight to students the difference between instantaneous speed and average speed. The total distance travelled by an object has to include the distance travelled in the return journey while the total time taken has to include the time taken even when the object is at rest or stationary.

Teachers can introduce some applications of graphs in real-world contexts through the worked examples and activities (see Class Discussion: Linear distance-time graphs).

Some common mistakes that students make are that they may write 1 hour 15 minutes as 1.15 h instead of 1.25 h. Students are also prone to mistakes when different units are given in a problem. For example, when a problem gives the speed as v km/h and the time as t minutes. As such, teachers can highlight to the students to convert the units where needed. Students are encouraged to search on the internet for more real-life examples related to graphs so that they can familiarise themselves with the units used in the real world.

Worked Solutions

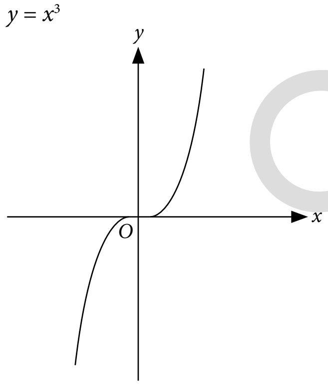
Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 1).

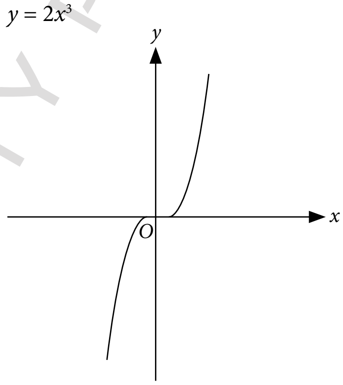
6.1 Graphs of cubic functions

Investigation (Graphs of cubic functions)

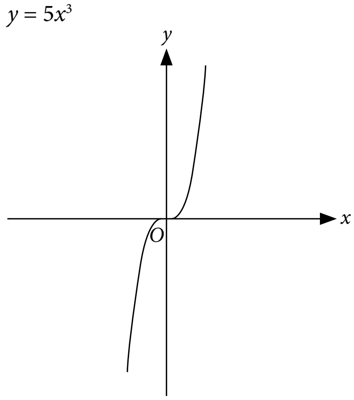
1. **(a)** $y = x^3$



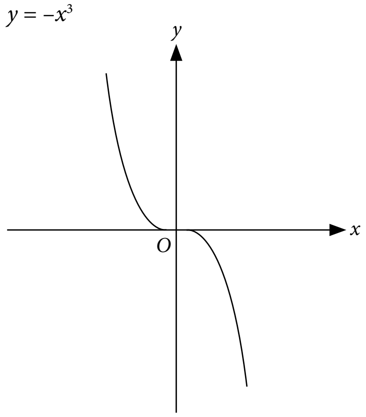
(b) $y = 2x^3$



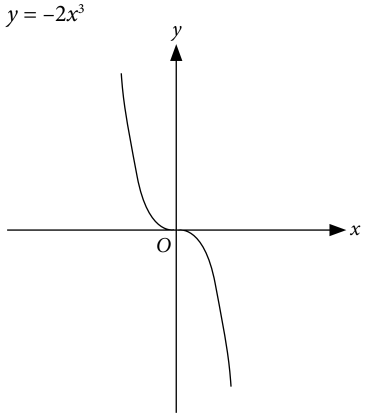
(c) $y = 5x^3$



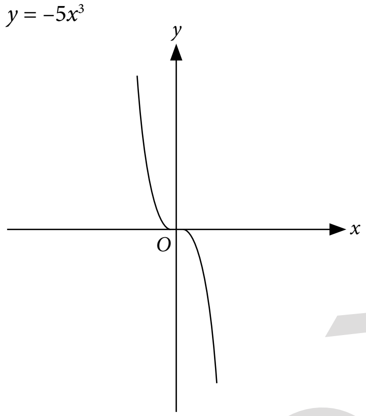
(d) $y = -x^3$



(e) $y = -2x^3$



(f) $y = -5x^3$

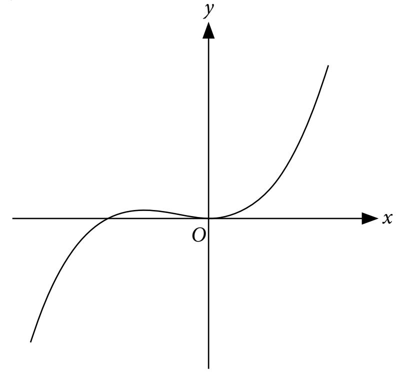


2. **(a)** The graphs of $y = ax^3$ are curves that pass through the origin, and they have rotational symmetry of order 2.
When $a > 0$, y is negative when x is negative, and y increases at a decreasing rate when x increases from $-\infty$ to 0; y is positive when x is positive, and y increases at an increasing rate when x increases from 0 to ∞.
When $a < 0$, y is positive when x is negative, and y decreases at a decreasing rate when x increases from $-\infty$ to 0; y is negative when x is positive, and y decreases at an increasing rate when x increases from 0 to ∞.

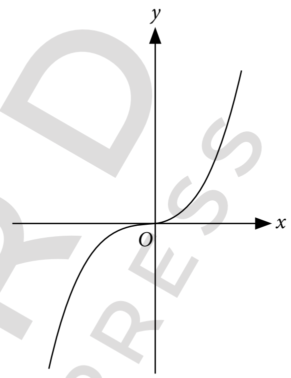
(b) When the absolute value of a increases, the graph becomes narrower/steeper.

(c) The graph of $y = 0.5x^3$ will have shape as $y = x^3$ but it will be broader.

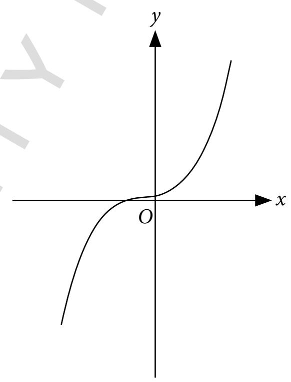
3. **(a)** $y = x^3 + x^2$



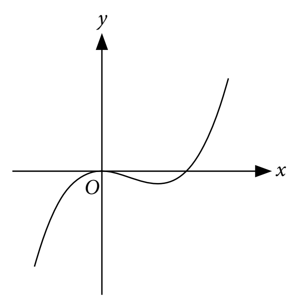
(b) $y = x^3 + x^2 + x$



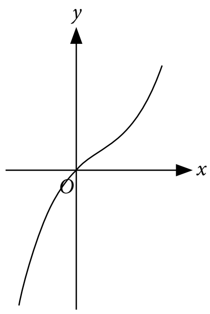
(c) $y = x^3 + x^2 + x + 1$



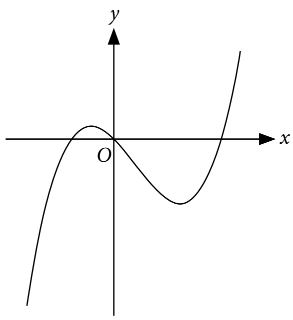
(d) $y = x^3 - x^2$



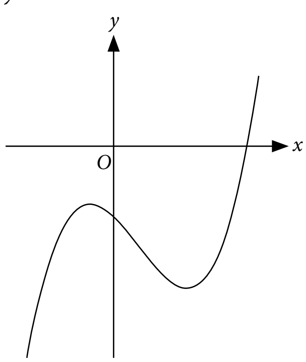
(e) $y = x^3 - x^2 + x$



(f) $y = x^3 - x^2 - x$

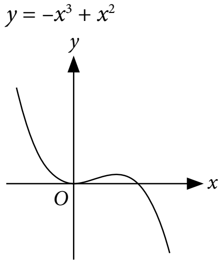


(g) $y = x^3 - x^2 - x - 1$

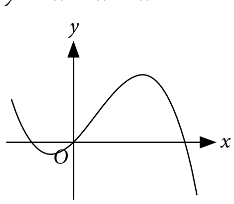


In general, the graph of a cubic function consists of one or two turning points and may or may not pass through the origin.

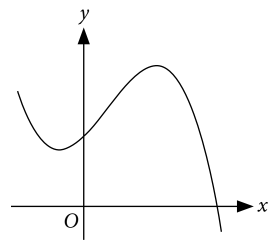
4. $y = -x^3 + x^2$



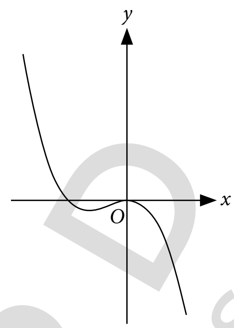
$y = -x^3 + x^2 + x$



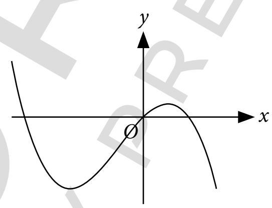
$y = -x^3 + x^2 + x + 1$



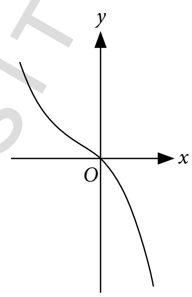
$y = -x^3 - x^2$



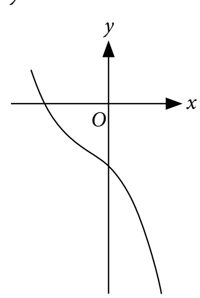
$y = -x^3 - x^2 + x$



$y = -x^3 - x^2 - x$

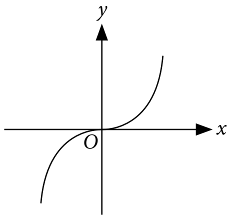


$y = -x^3 - x^2 - x - 1$

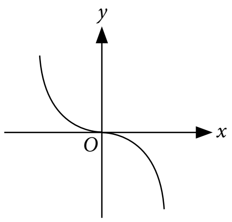


In general, when the coefficient of x^3 changes from 1 to –1, the shape of the resultant curve is similar to that obtained by reflecting the original curve about the y-axis.

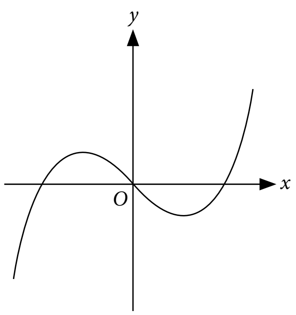
5. In general, the graphs of cubic functions consist of one or two turning points.
For graphs of cubic functions of the form $y = ax^3$, they pass through the origin.
When $a > 0$, it takes the shape of

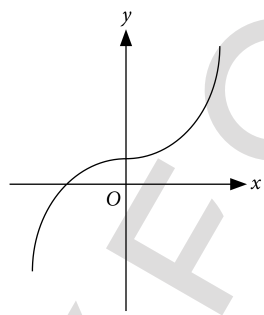


When $a < 0$, it takes the shape of

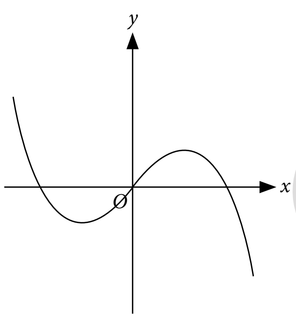


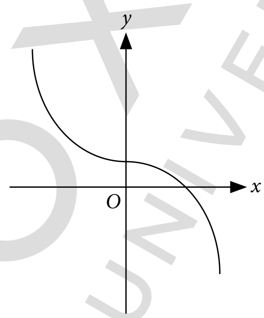
For graphs of cubic functions of the form $y = ax^3 + bx^2 + cx + d$, they may not pass through the origin if $d \neq 0$.
When $a > 0$, it takes the shapes of





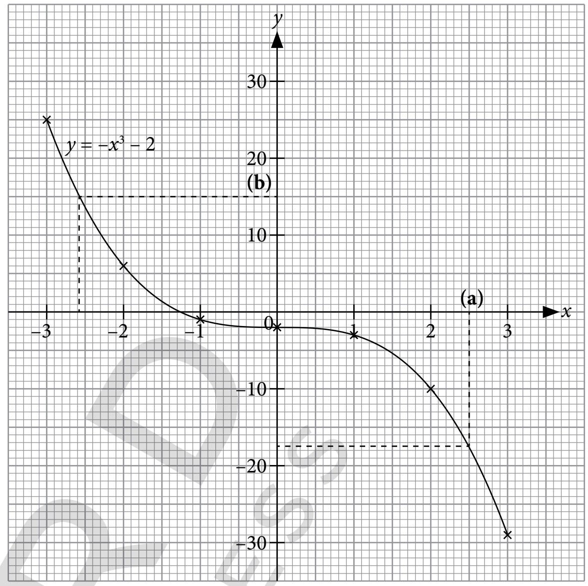
When $a < 0$, it takes the shapes of





Practise Now 1

x	−3	−2	−1	0	1	2	3
$y = -x^3 - 2$	25	6	−1	−2	−3	−10	−29



(a) From the graph, when $x = 2.5$, $y = \mathbf{-17.5}$.
(b) From the graph, when $y = 15$, $x = \mathbf{-2.55}$.

Introductory Problem Revisited

Volume of box, $V = (60 - 2x)(80 - 2x)(x)$
$= x(60 - 2x)(80 - 2x)$
$= x(4800 - 120x - 160x + 4x^2)$
$= x(4x^2 - 280x + 4800)$
$= (4x^3 - 280x^2 + 4800x)$ cm^3

Note that $(60 - 2x)$ cm, $(80 - 2x)$ cm and x cm must be greater than 0 cm.

$60 - 2x > 0$
$60 > 2x$
$30 > x$
$x < 30$

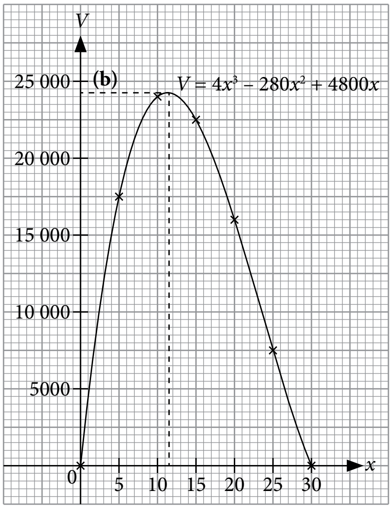
$80 - 2x > 0$
$80 > 2x$
$40 > x$
$x < 40$

$x > 0$
$\therefore\ 0 < x < 30$

Hence we shall look at the graph of $V = 4x^3 - 280x^2 + 4800x$ for $0 \leqslant x \leqslant 30$.

x	0	5	10	15	20	25	30
$V = 4x^3 - 280x^2 + 4800x$	0	17 500	24 000	22 500	16 000	7500	0

Using a scale of 1 cm to represent 5 units on the x-axis and 2 cm to represent 5000 units on the V-axis.

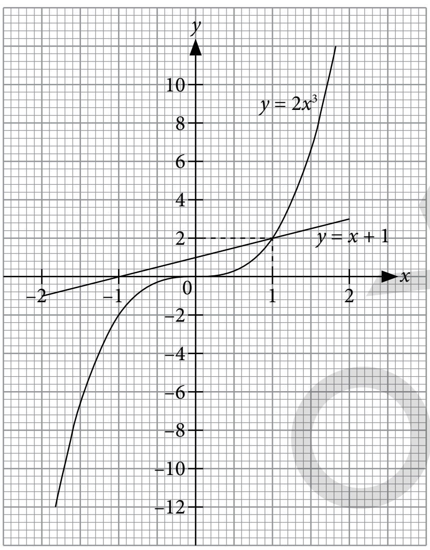


(a) From the graph, the maximum value of V is only around 24 250. Hence, it is **not** possible to build a box with a volume of 50 000 cm^3.

(b) From the graph, when V is maximum, x = **11.5**.

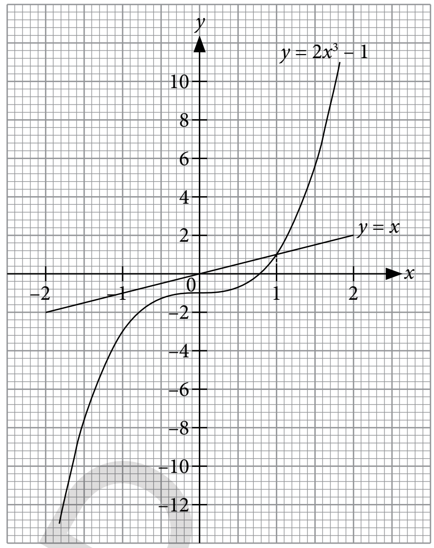
Investigation (Graphical solution to equations)

1.



$\therefore$ the value of x which satisfies the simultaneous equations is **1**.

2.



$\therefore$ the value of x which satisfies the simultaneous equations is **1**.

3. Subtracting 1 from each of the equations in Question 1 yields the equations in Question 2.

4. $2x^3 - x - 1 = 0$

$2x^3 - 1 = x$

The LHS and RHS correspond to the equations in Question 2.

$\therefore x = \mathbf{1}$

5. Simultaneous equations can be solved using the graphical method by drawing the graphs of the simultaneous equations or other equivalent sets of equations that preserve the equivalence of the original equations, and finding the value(s) of x at the intersection point(s) of the graphs.

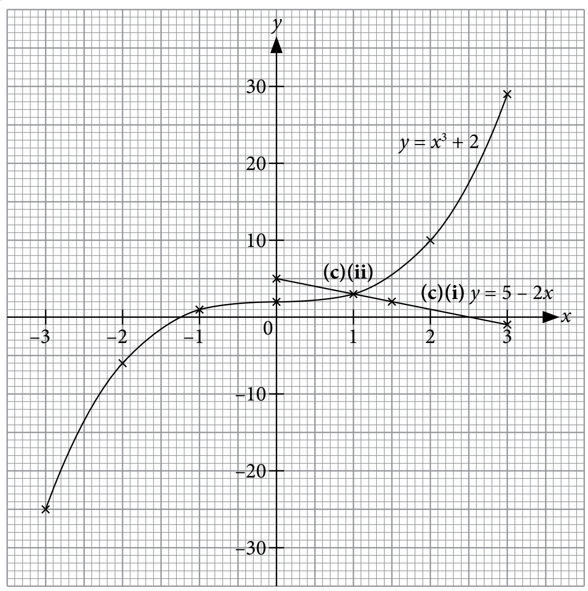
Practise Now 2

(a) When $x = 3$, $y = 3^3 + 2$

$= 29$

$\therefore q = \mathbf{29}$

(b)



(c) **(i)**

x	0	1.5	3
$y = 5 - 2x$	5	2	−1

(ii) From the graph, the line $y = 5 - 2x$ intersects the curve $y = x^3 + 2$ at x = **1**.

(iii) At the point of intersection,

$$x^3 + 2 = 5 - 2x$$
$$x^3 + 2x - 3 = 0$$

Comparing $x^3 + 2x - 3 = 0$ with $x^3 + Ax + B = 0$,

$\boldsymbol{A = 2}$ and $\boldsymbol{B = -3}$

Practise Now 3

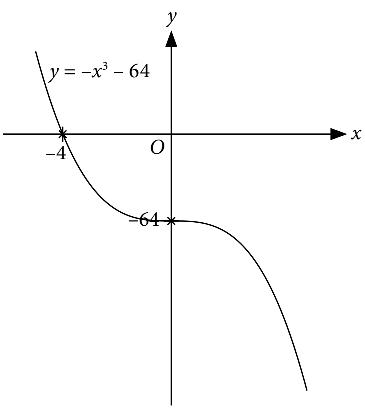
Find y-intercept:

When $x = 0$, $y = -x^3 - 64$
$= -0^3 - 64$
$= -64$

Find x-intercept(s):

When $y = 0$, $-x^3 - 64 = 0$
$x^3 = -64$
$x = -4$

Sketch:



Practise Now 4

Find y-intercept:

When $x = 0$, $y = -x^3 - 6x^2 + 9x$
$= -0^3 - 6(0)^2 + 9(0)$
$= 0$

Find x-intercept(s):

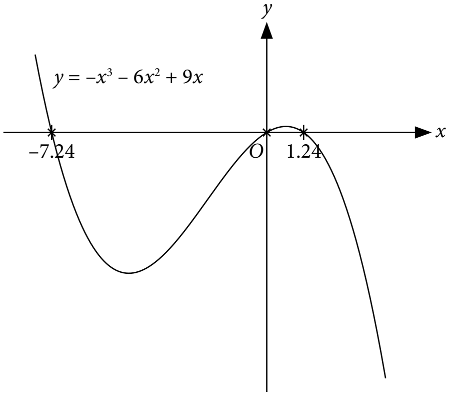
When $y = 0$, $-x^3 - 6x^2 + 9x = 0$
$x^3 + 6x^2 - 9x = 0$
$x(x^2 + 6x - 9) = 0$
$x = 0$

or

$x^2 + 6x - 9 = 0$

$$x = \frac{-(-6) \pm \sqrt{6^2 - 4(1)(-9)}}{2(1)}$$
$= 1.24$ or -7.24 (to 3 s.f.)

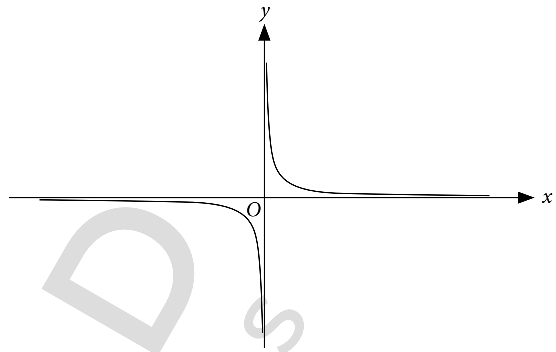
Sketch:



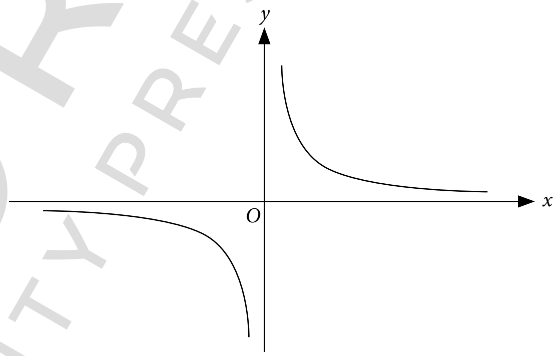
6.2 Graphs of reciprocal functions

Investigation (Graphs of $y = \frac{a}{x}$)

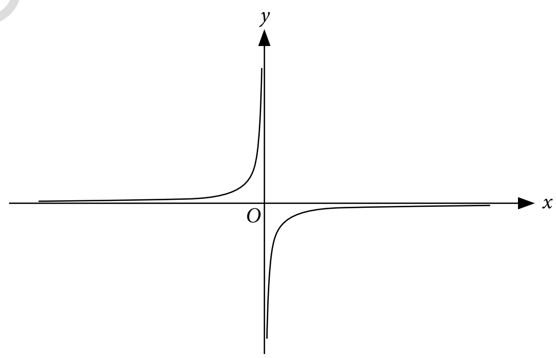
(a) $y = \frac{1}{x}$



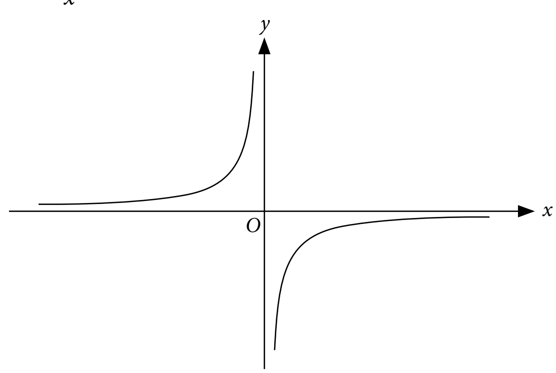
(b) $y = \frac{5}{x}$



(c) $y = -\frac{1}{x}$



(d) $y = -\frac{5}{x}$



1. **(i)** For $a > 0$, the graph consists of two parts that lie in the **1st** and **3rd** quadrants respectively.
   (ii) For $a < 0$, the graph consists of two parts that lie in the **2nd** and **4th** quadrants respectively.
2. There is rotational symmetry of order 2 about the origin, i.e. it maps onto itself twice by rotation in 360°.
3. **No**. The curves get very close to the x-axis and y-axis but never touch them.

Thinking Time (Page 167)

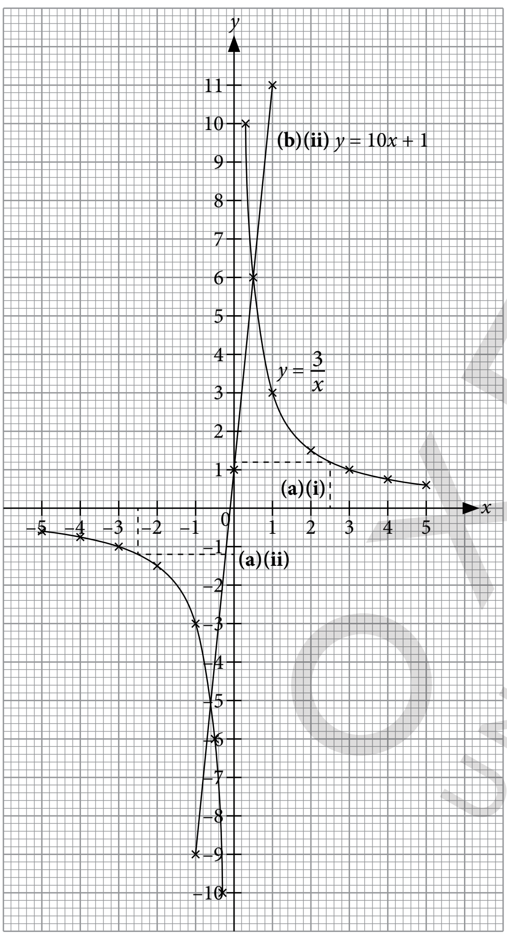
(a) When $a > 0$, the equations of the lines of symmetry are $\boldsymbol{y = x}$ and $\boldsymbol{y = -x}$.

(b) When $a < 0$, the equations of the lines of symmetry are $\boldsymbol{y = x}$ and $\boldsymbol{y = -x}$.

Practise Now 5

(a)

x	−5	−4	−3	−2	−1	−0.5	−0.3	0.3	0.5	1	2	3	4	5
y	−0.6	−0.75	−1	−1.5	−3	−6	−10	10	6	3	1.5	1	0.75	0.6



(i) From the graph, when $x = 2.5$, $y = \mathbf{1.2}$.
(ii) From the graph, when $y = -1.2$, $x = \mathbf{-2.5}$.

(b) **(i)** $\dfrac{3}{x} - 10x - 1 = 0$

$$\frac{3}{x} = 10x + 1$$

$\therefore$ the straight line to be drawn is $\boldsymbol{y = 10x + 1}$.

(ii)

x	−1	0	1
$y = 10x + 1$	−9	1	11

From the graph, the line $y = 10x + 1$ and the curve $y = \dfrac{3}{x}$ intersect at (0.5, 6) and (−0.6, −5).
$\therefore$ $x = \mathbf{0.5}$ or $x = \mathbf{-0.6}$

Practise Now 6

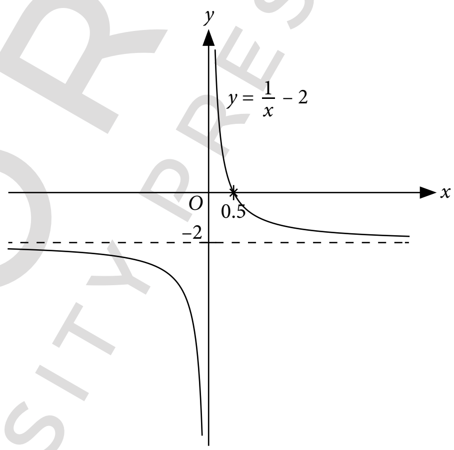
Find asymptote(s):
When $x = 0$, y is undefined.
$\therefore$ $x = 0$ is an asymptote of the graph.
When $y = -2$, x is undefined.
$\therefore$ $y = -2$ is an asymptote of the graph.
Find x-intercept(s):

When $y = 0$, $\dfrac{1}{x} - 2 = 0$

$$\frac{1}{x} = 2$$

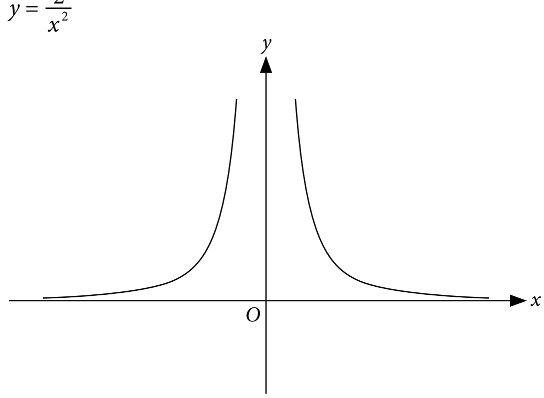
$$x = 0.5$$

Sketch:

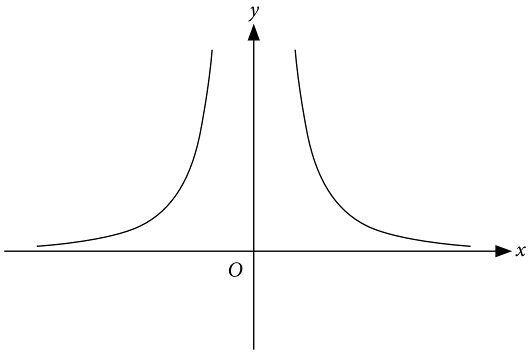


Investigation (Graphs of $y = \dfrac{a}{x^2}$)

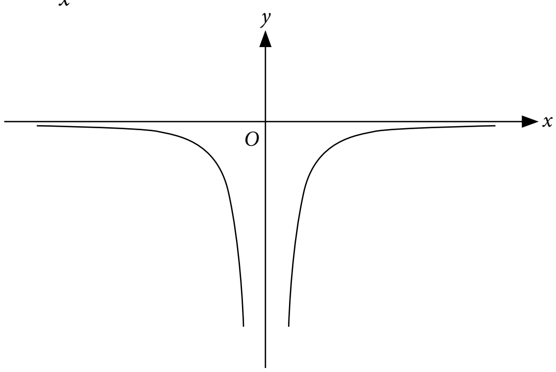
(a) $y = \dfrac{2}{x^2}$



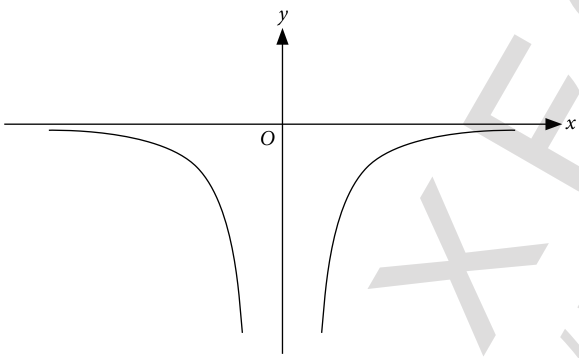
(b) $y = \frac{4}{x^2}$



(c) $y = -\frac{1}{x^2}$



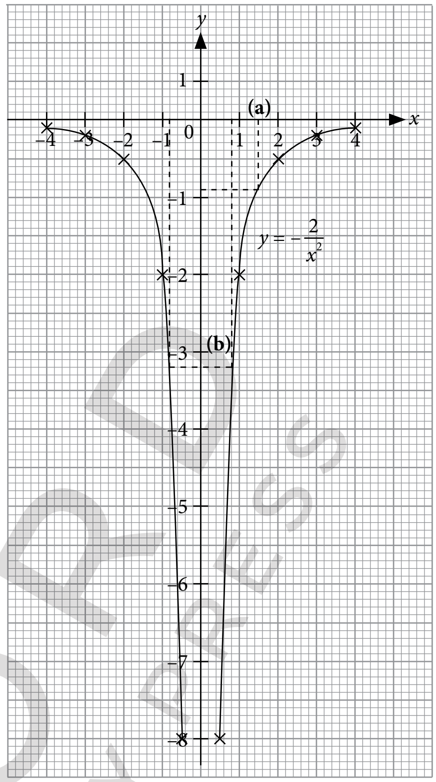
(d) $y = -\frac{3}{x^2}$



1. **(i)** For $a > 0$, the graph consists of two parts that lie in the **1st** and **2nd** quadrants respectively.
   (ii) For $a < 0$, the graph consists of two parts that lie in the **3rd** and **4th** quadrants respectively.
2. The y-axis is the line of symmetry, i.e. the graph is symmetrical about the y-axis.
3. **No**. The curves get very close to the x-axis and y-axis but never touch them.

Practise Now 7

x	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
y	−0.13	−0.22	−0.5	−2	−8	−8	−2	−0.5	−0.22	−0.13

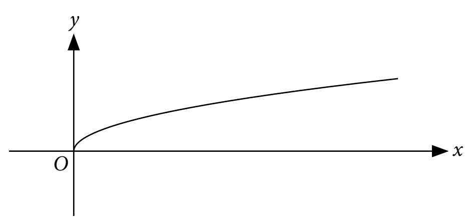


(a) From the graph, when $x = 1.5$, $y =$ **−0.9**.
(b) From the graph, when $y = -3.2$, $x =$ **−0.8** or $x =$ **0.8**.

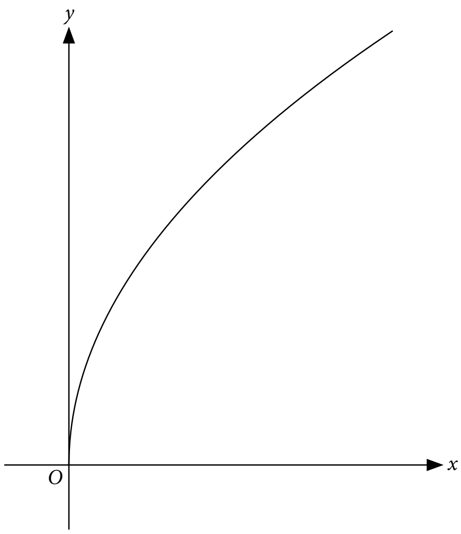
6.3 Graphs of functions involving $\sqrt{x}$

Investigation (Graphs of $y = a\sqrt{x}$)

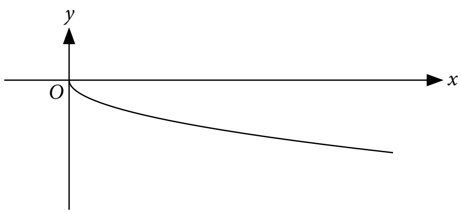
(a) $y = \sqrt{x}$



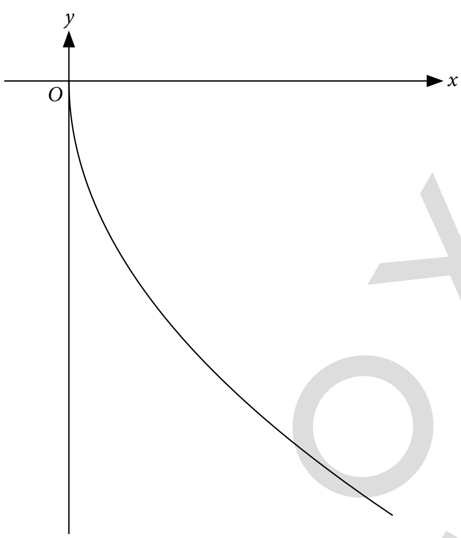
(b) $y = 6\sqrt{x}$



(c) $y = -\sqrt{x}$



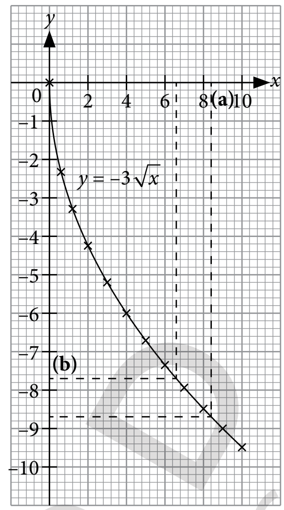
(d) $y = -6\sqrt{x}$



1. **(i)** For $a > 0$, the graph lies in the **1st** quadrant.
   (ii) For $a < 0$, the graph lies in the **4th** quadrant.
2. **(a)** When $a > 0$, y increases at a decreasing rate when x increases from 0 to ∞.
   When $a < 0$, y decreases at a decreasing rate when x increases from 0 to ∞.
   (b) When the absolute value of a increases, the graph becomes steeper.
3. Since $\sqrt{0} = 0$, the graphs of $y = a\sqrt{x}$ for various values of a always intersect the origin.

Practise Now 8

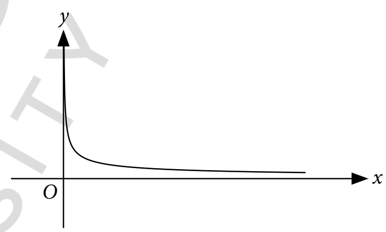
x	0	0.6	1.2	2	3	4	5	6	7	8	9	10
y	0	–2.3	–3.3	–4.2	–5.2	–6	–6.7	–7.3	–7.9	–8.5	–9	–9.5



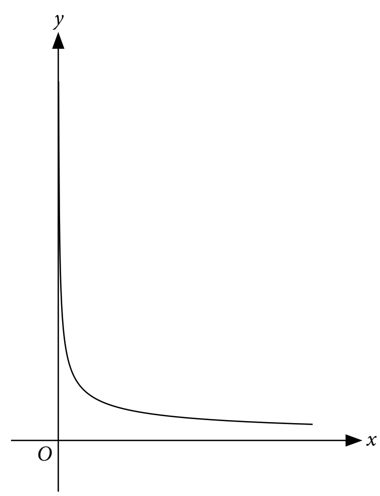
(a) From the graph, when $x = 8.4$, $y =$ **–8.7**.
(b) From the graph, when $y = -7.7$, $x =$ **6.6**.

Investigation (Graphs of $y = \frac{a}{\sqrt{x}}$)

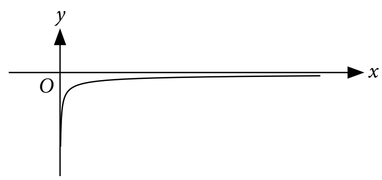
(a) $y = \frac{2}{\sqrt{x}}$



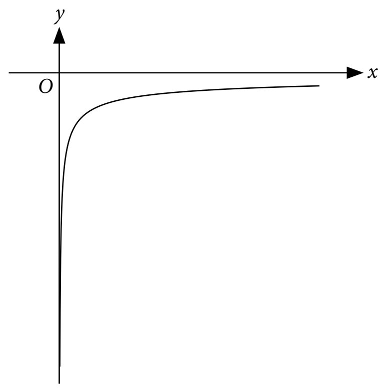
(b) $y = \frac{5}{\sqrt{x}}$



(c) $y = -\dfrac{1}{\sqrt{x}}$



(d) $y = -\dfrac{4}{\sqrt{x}}$



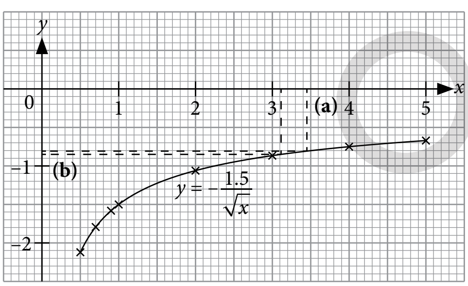
1. **(i)** For $a > 0$, the graph lies in the **1st** quadrant.
(ii) For $a < 0$, the graph lies in the **4th** quadrant.

2. **(a)** When $a > 0$, y decreases at a decreasing rate when x increases from 0 to ∞.
When $a < 0$, y increases at a decreasing rate when x increases from 0 to ∞.
(b) When the absolute value of a increases, the graph becomes steeper.

3. **No**. The curves get very close to the x-axis and y-axis but never touch them.

Practise Now 9

x	0.5	0.7	0.9	1	2	3	4	5
y	–2.12	–1.79	–1.58	–1.5	–1.06	–0.87	–0.75	–0.67



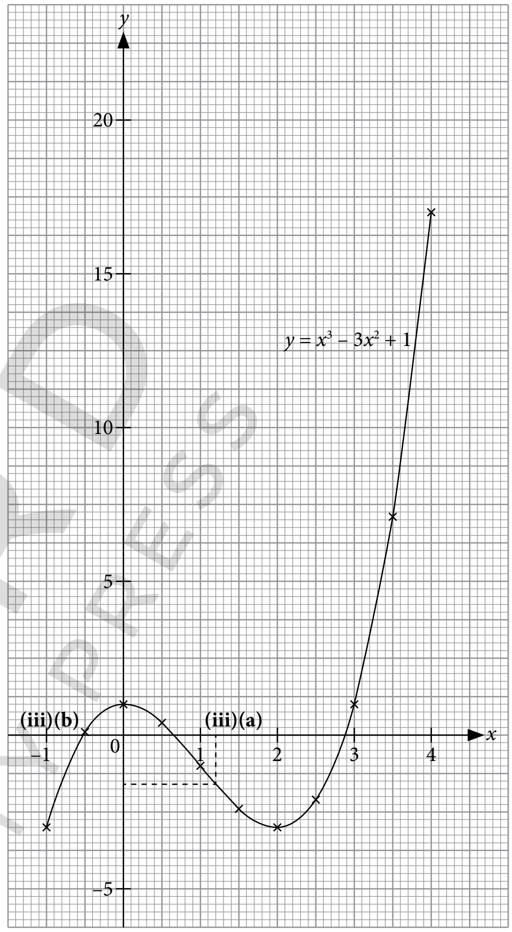
(a) From the graph, when $x = 3.45$, $y = \mathbf{-0.8}$.
(b) From the graph, when $y = -0.85$, $x = \mathbf{3.1}$.

Exercise 6A

1. **(i)** $y = x^3 - 3x^2 + 1$

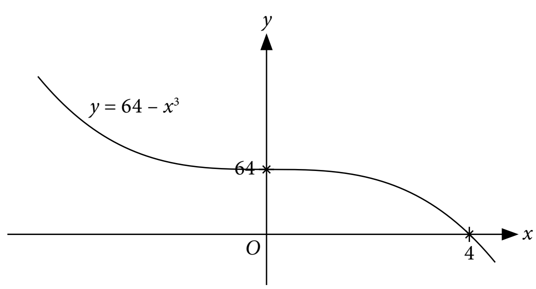
x	–1	–0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
y	**–3**	**0.1**	1	**0.4**	–1	**–2.4**	–3	**–2.1**	**1**	**7.1**	17

(ii)



(iii) **(a)** From the graph, when $x = 1.2$, $y = \mathbf{-1.625}$.
(b) From the graph, when $y = 0$, $x = \mathbf{-0.55}$, $x = \mathbf{0.65}$ or $x = \mathbf{2.9}$.

2. **(a)** Find y-intercept:
When $x = 0$, $y = 64 - x^3$
$= 64 - 0^3$
$= 64$
Find x-intercept(s):
When $y = 0$, $64 - x^3 = 0$
$x^3 = 64$
$x = 4$
Sketch:



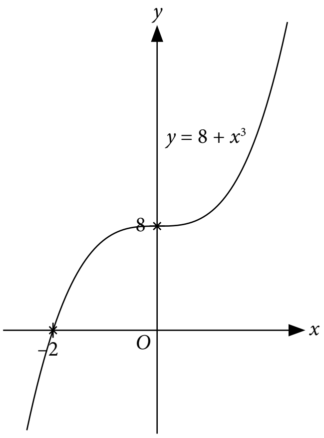
(b) Find y-intercept:

When $x = 0$, $y = 8 + x^3$
$= 8 + 0^3$
$= 8$

Find x-intercept(s):

When $y = 0$, $8 + x^3 = 0$
$x^3 = -8$
$x = -2$

Sketch:



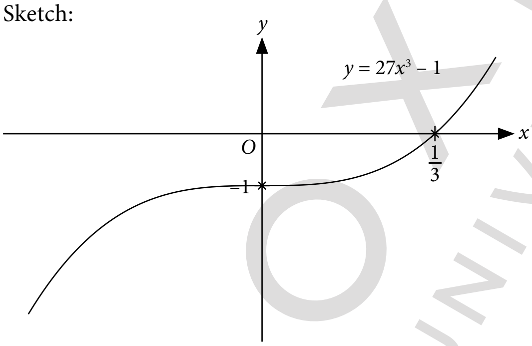
(c) Find y-intercept:

When $x = 0$, $y = 27x^3 - 1$
$= 27(0^3) - 1$
$= -1$

Find x-intercept(s):

When $y = 0$, $27x^3 - 1 = 0$
$27x^3 = 1$
$x^3 = \frac{1}{27}$
$x = \frac{1}{3}$

Sketch:



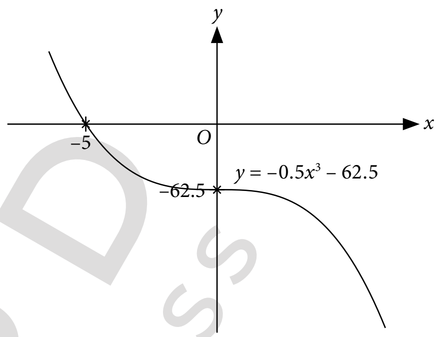
(d) Find y-intercept:

When $x = 0$, $y = -0.5x^3 - 62.5$
$= -0.5(0^3) - 62.5$
$= -62.5$

Find x-intercept(s):

When $y = 0$, $-0.5x^3 - 62.5 = 0$
$0.5x^3 = -62.5$
$x^3 = -125$
$x = -5$

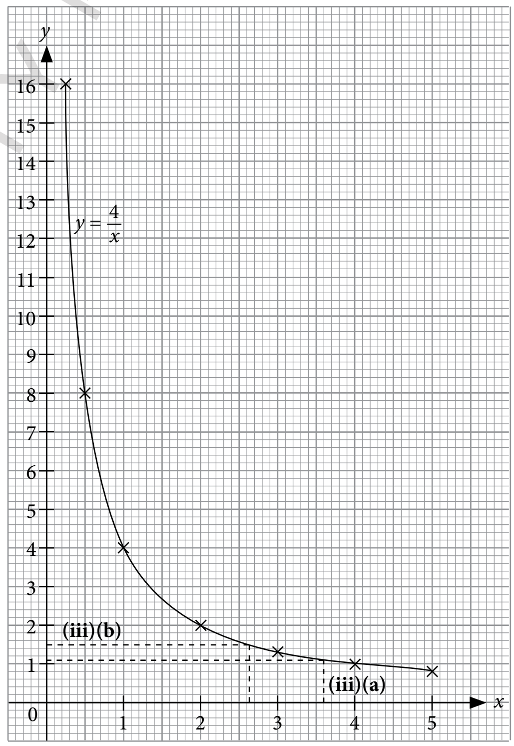
Sketch:



3. **(i)** $y = \frac{4}{x}$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5
y	16	**8**	**4**	2	**1.3**	1	**0.8**

(ii)



(iii) **(a)** From the graph, when $x = 3.6$, $y = $ **1.1**.

(b) From the graph, when $y = 1.5$, $x = $ **2.65**.

4. **(a)** Find asymptote(s):

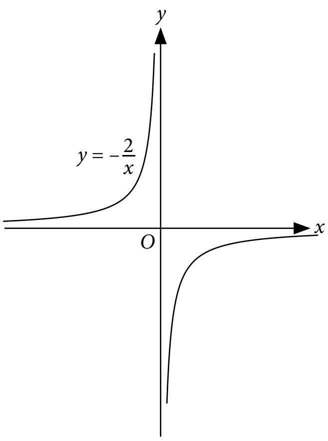
When $x = 0$, y is undefined.

$\therefore\ x = 0$ is an asymptote of the graph.

When $y = 0$, x is undefined.

$\therefore\ y = 0$ is an asymptote of the graph.

Sketch:



(b) Find asymptote(s):

When $x = 0$, y is undefined.

$\therefore\ x = 0$ is an asymptote of the graph.

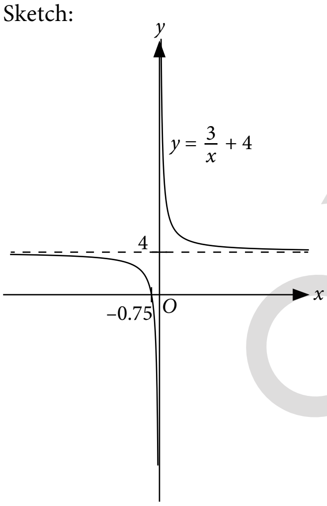
When $y = 4$, x is undefined.

$\therefore\ y = 4$ is an asymptote of the graph.

Find x-intercept(s):

$$\begin{aligned}\text{When } y = 0,\ \frac{3}{x} + 4 &= 0\\ \frac{3}{x} &= -4\\ x &= -0.75\end{aligned}$$

Sketch:



(c) Find asymptote(s):

When $x = 0$, y is undefined.

$\therefore\ x = 0$ is an asymptote of the graph.

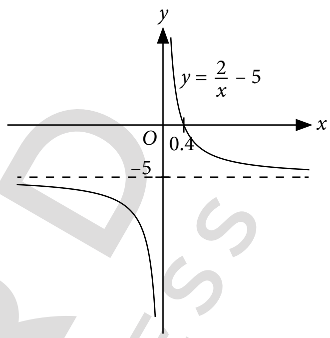
When $y = -5$, x is undefined.

$\therefore\ y = -5$ is an asymptote of the graph.

Find x-intercept(s):

$$\begin{aligned}\text{When } y = 0,\ \frac{2}{x} - 5 &= 0\\ \frac{2}{x} &= 5\\ x &= 0.4\end{aligned}$$

Sketch:



(d) Find asymptote(s):

When $x = 0$, y is undefined.

$\therefore\ x = 0$ is an asymptote of the graph.

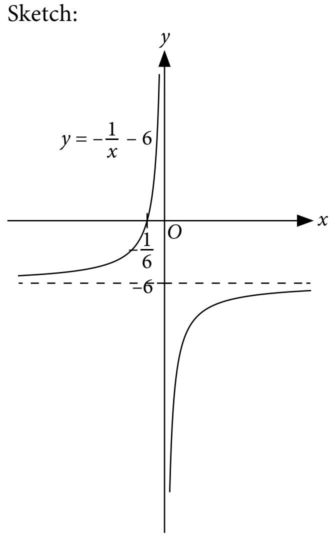
When $y = -6$, x is undefined.

$\therefore\ y = -6$ is an asymptote of the graph.

Find x-intercept(s):

$$\begin{aligned}\text{When } y = 0,\ -\frac{1}{x} - 6 &= 0\\ \frac{1}{x} &= -6\\ x &= -\frac{1}{6}\end{aligned}$$

Sketch:



5. **(i)** When $x = 3$, $y = \frac{10}{3^2}$

$= 1.1$ (to 1 d.p.)

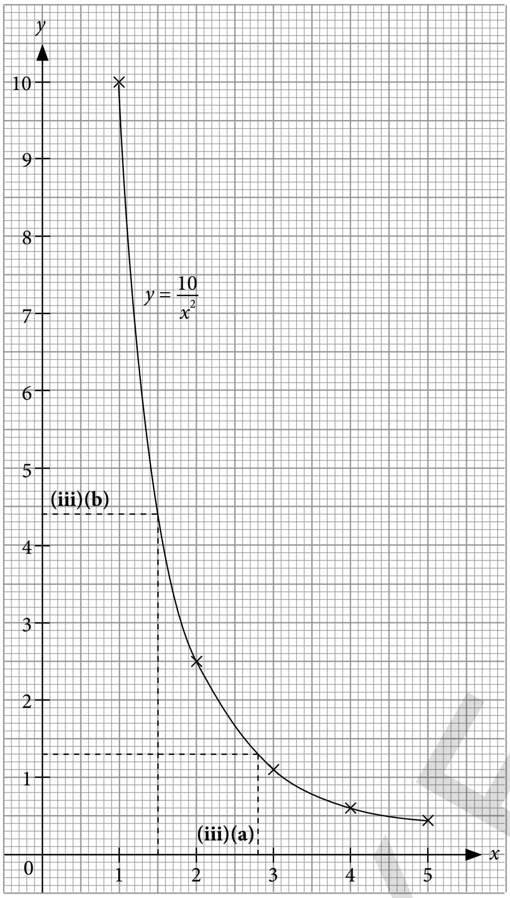
$\therefore$ $\boldsymbol{a = 1.1}$

When $x = 5$, $y = \frac{10}{5^2}$

$= 0.4$

$\therefore$ $\boldsymbol{b = 0.4}$

(ii)

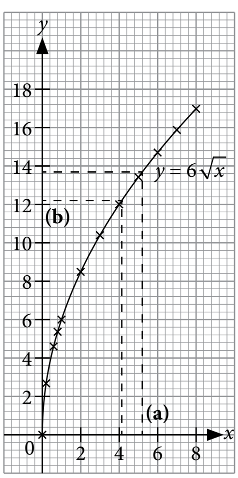


(iii) **(a)** From the graph, when $x = 2.8$, $y = \mathbf{1.3}$.

(b) From the graph, when $y = 4.4$, $x = \mathbf{1.5}$.

6.

x	0	0.2	0.6	0.8	1	2	3	4	5	6	7	8
y	0	2.7	4.6	5.4	6	8.5	10.4	12	13.4	14.7	15.9	17

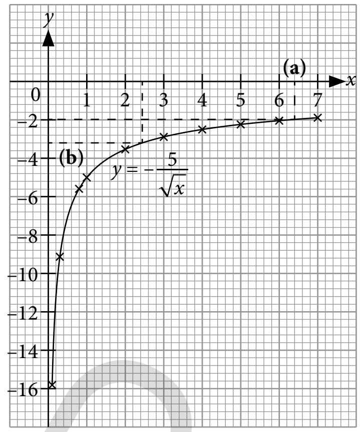


(a) From the graph, when $x = 5.2$, $y = \mathbf{13.6}$.

(b) From the graph, when $y = 12.2$, $x = \mathbf{4.2}$.

7.

x	0.1	0.3	0.8	1	2	3	4	5	6	7
y	–15.8	–9.1	–5.6	–5	–3.5	–2.9	–2.5	–2.2	–2	–1.9

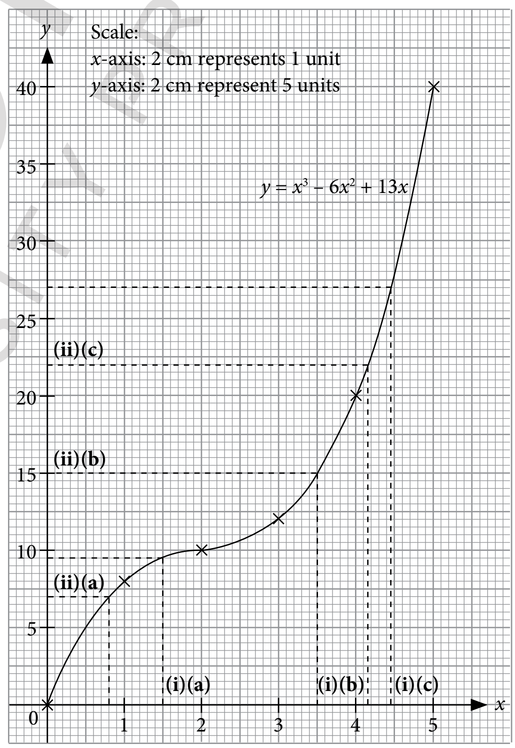


(a) From the graph, when $x = 6.4$, $y = \mathbf{-2}$.

(b) From the graph, when $y = -3.2$, $x = \mathbf{2.4}$.

8. $y = x^3 - 6x^2 + 13x$

x	0	1	2	3	4	5
y	0	8	10	12	20	40



(i) **(a)** From the graph, when $x = 1.5$, $y = \mathbf{9.5}$.

(b) From the graph, when $x = 3.5$, $y = \mathbf{15}$.

(c) From the graph, when $x = 4.45$, $y = \mathbf{27}$.

(ii) **(a)** From the graph, when $y = 7$, $x = \mathbf{0.8}$.

(b) From the graph, when $y = 15$, $x = \mathbf{3.5}$.

(c) From the graph, when $y = 22$, $x = \mathbf{4.15}$.

9. **(a)** Find y-intercept:

When $x = 0$, $y = x^3 - x^2 - 2x$
$= 0^3 - 0^2 - 2(0)$
$= 0$

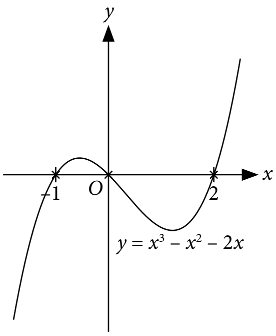
Find x-intercept(s):

When $y = 0$, $x^3 - x^2 - 2x = 0$
$x(x^2 - x - 2) = 0$
$x = 0$

or

$x^2 - x - 2 = 0$
$(x - 2)(x + 1) = 0$
$x - 2 = 0$ or $x + 1 = 0$
$x = 2$ $\quad$ $x = -1$

Sketch:



(b) Find y-intercept:

When $x = 0$, $y = -x^3 - 2x^2 + 15x$
$= -0^3 - 2(0^2) + 15(0)$
$= 0$

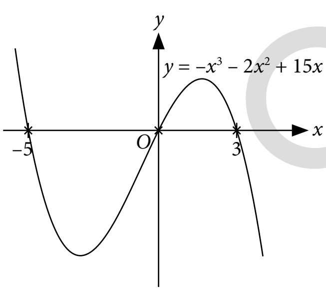
Find x-intercept(s):

When $y = 0$, $-x^3 - 2x^2 + 15x = 0$
$-x(x^2 + 2x - 15) = 0$
$x = 0$

or

$x^2 + 2x - 15 = 0$
$(x + 5)(x - 3) = 0$
$x + 5 = 0$ or $x - 3 = 0$
$x = -5$ $\quad$ $x = 3$

Sketch:



(c) Find y-intercept:

When $x = 0$, $y = -\frac{1}{3}x^3 + \frac{4}{3}x^2 - x$
$= -\frac{1}{3}(0^3) + \frac{4}{3}(0^2) - 0$
$= 0$

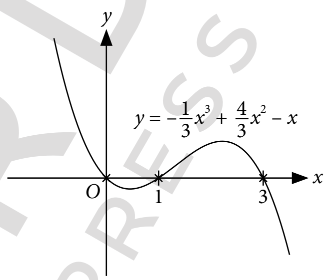
Find x-intercept(s):

When $y = 0$, $-\frac{1}{3}x^3 + \frac{4}{3}x^2 - x = 0$
$x^3 - 4x^2 + 3x = 0$
$x(x^2 - 4x + 3) = 0$
$x = 0$

or

$x^2 - 4x + 3 = 0$
$(x - 3)(x - 1) = 0$
$x - 3 = 0$ or $x - 1 = 0$
$x = 3$ $\quad$ $x = 1$

Sketch:



(d) Find y-intercept:

When $x = 0$, $y = 0.5x^3 - x^2 - 4x$
$= 0.5(0^3) - 0^2 - 4(0)$
$= 0$

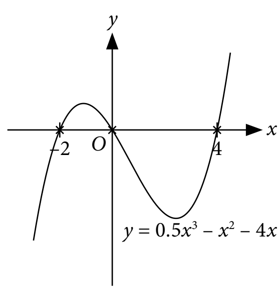
Find x-intercept(s):

When $y = 0$, $0.5x^3 - x^2 - 4x = 0$
$x^3 - 2x^2 - 8x = 0$
$x(x^2 - 2x - 8) = 0$
$x = 0$

or

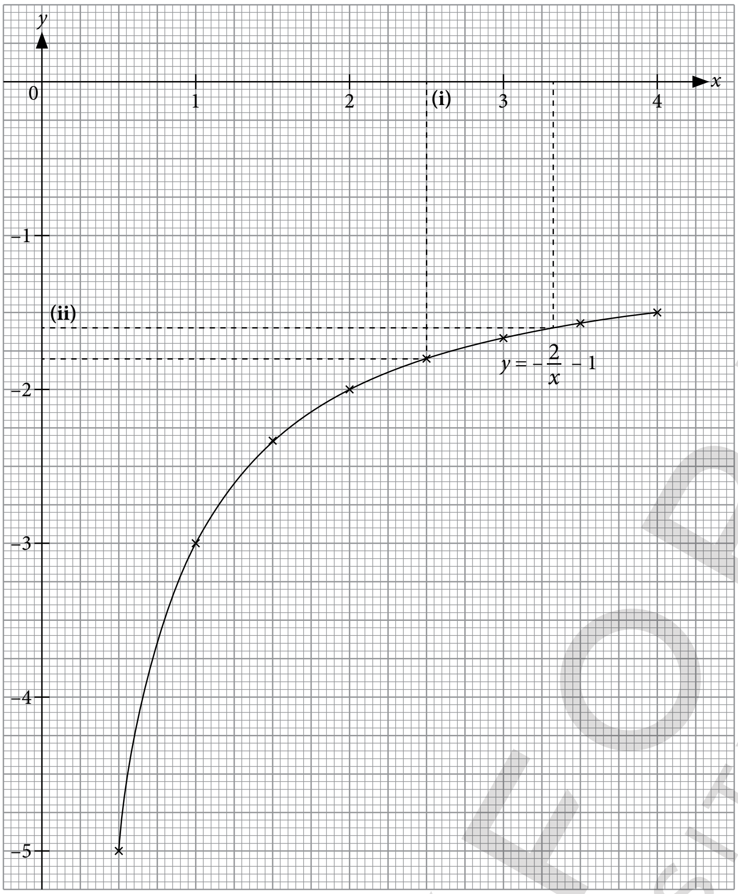
$x^2 - 2x - 8 = 0$
$(x - 4)(x + 2) = 0$
$x - 4 = 0$ or $x + 2 = 0$
$x = 4$ $\quad$ $x = -2$

Sketch:



10. $y = -\frac{2}{x} - 1$

x	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
y	–5	–3	–2.333	–2	–1.8	–1.667	–1.571	–1.5



(i) From the graph, when $x = 2.5$, $y =$ **–1.8**.

(ii) From the graph, when $y = -1.6$, $x =$ **3.325**.

11. (i) When $x = 3$, $y = 3 - \frac{3}{3}$

$= 2$

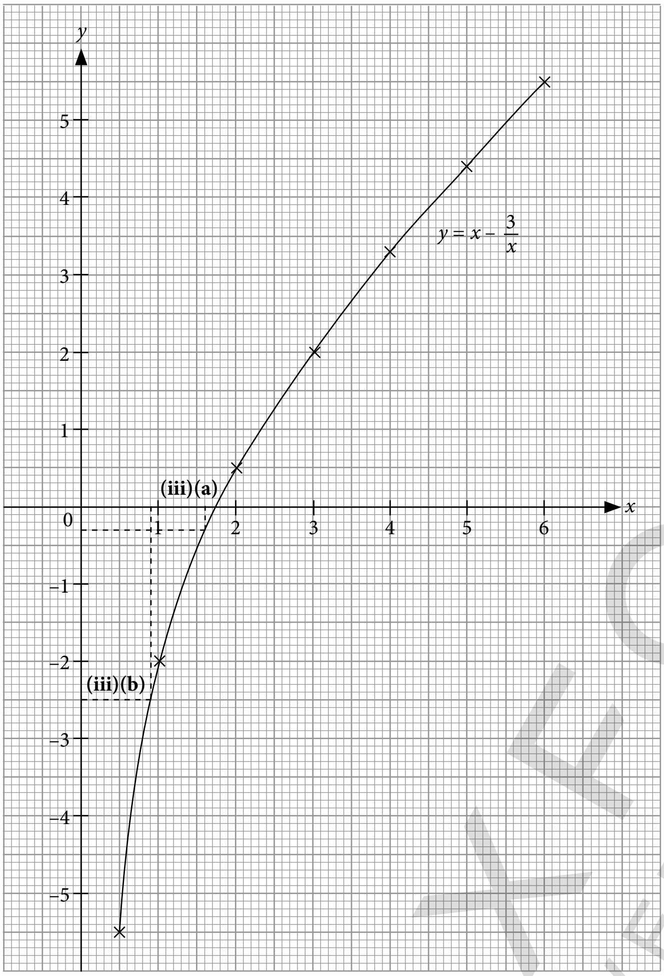
$\therefore$ $\boldsymbol{h = 2}$

When $x = 6$, $y = 6 - \frac{3}{6}$

$= 5.5$

$\therefore$ $\boldsymbol{k = 5.5}$

(ii)

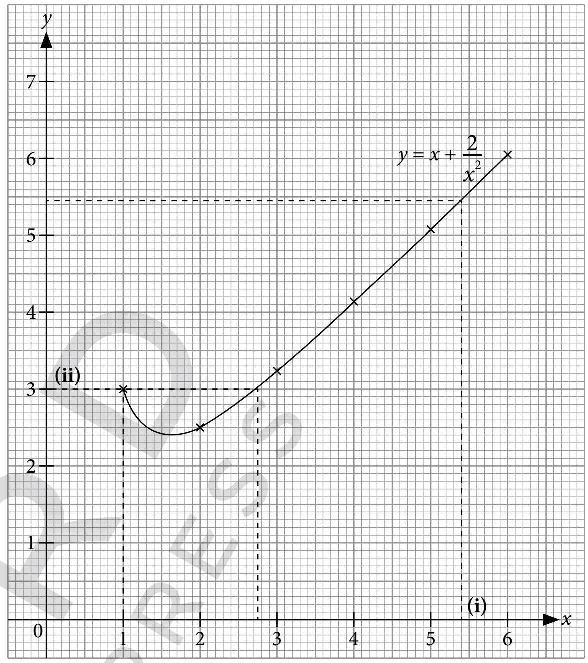


(iii) (a) From the graph, when $x = 1.6$, $y = \mathbf{-0.3}$.

(b) From the graph, when $y = -2.5$, $x = \mathbf{0.9}$.

12. $y = x + \frac{2}{x^2}$

x	1	2	3	4	5	6
y	3	2.5	3.22	4.13	5.08	6.06

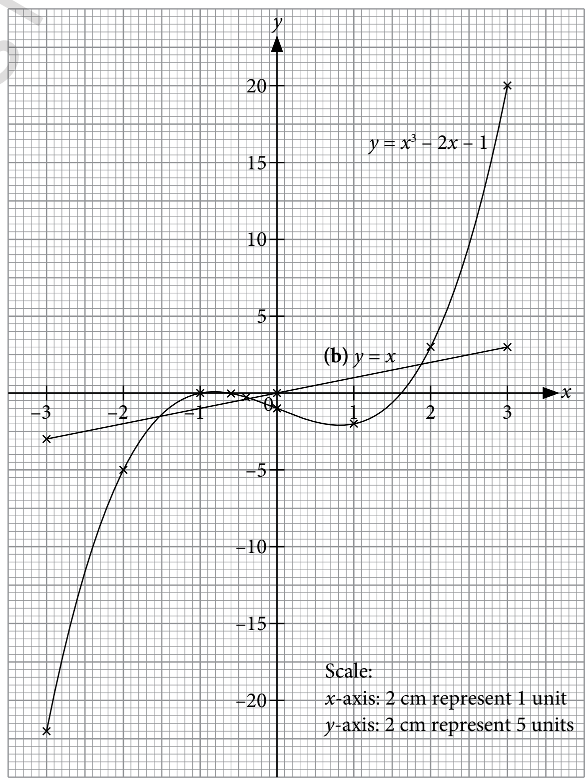


(i) From the graph, when $x = 5.4$, $y = \mathbf{5.45}$.

(ii) From the graph, when $y = 3$, $x = \mathbf{1}$ or $x = \mathbf{2.75}$.

13. $y = x^3 - 2x - 1$

x	−3	−2	−1	−0.6	−0.4	0	1	2	3
y	−22	−5	0	−0.016	−0.264	−1	−2	3	20



(a) From the graph, the x-coordinates of the points of intersection of the curve with the x-axis are **–1**, **–0.6** and **1.6**.

(b) **(i)**

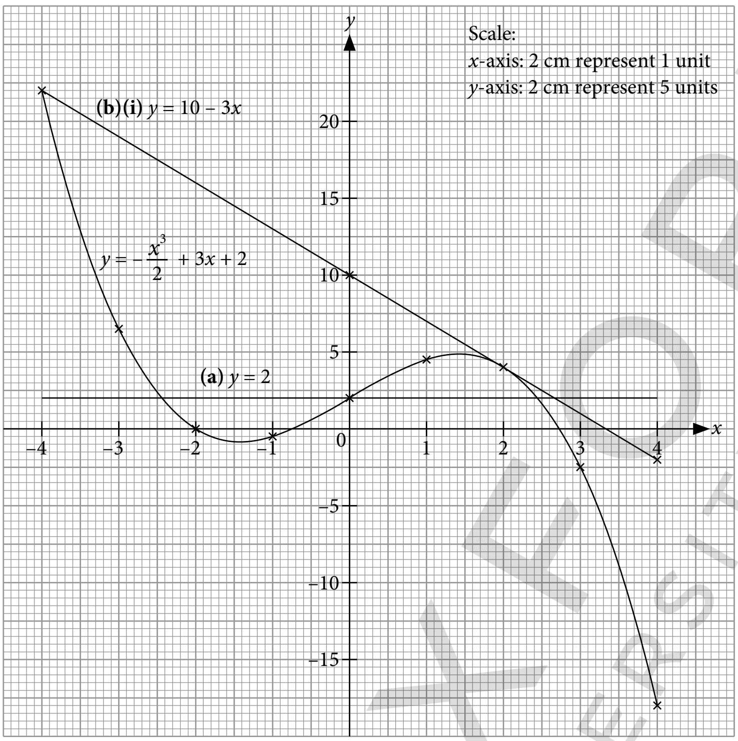
x	–3	0	3
$y = x$	–3	0	3

From the graph, the x-coordinates of the points at which the line meets the curve are **–1.55**, **–0.35** and **1.9**.

(ii) The solutions of the equation $x^3 - 2x - 1 = x$ are the x-coordinates of the points at which the curve $y = x^3 - 2x - 1$ meets the line $y = x$. Hence the solutions of the equation are **–1.55**, **–0.35** and **1.9**.

14. $y = -\frac{x^3}{2} + 3x + 2$

x	–4	–3	–2	–1	0	1	2	3	4
y	22	6.5	0	–0.5	2	4.5	4	–2.5	–18



(a) $-\frac{x^3}{2} + 3x = 0$

$-\frac{x^3}{2} + 3x + 2 = 2$

Since there are three points of intersection between the line $y = 2$ and the curve $y = -\frac{x^3}{2} + 3x + 2$, hence the equation has three solutions.

(b) **(i)** $y = 10 - 3x$

x	–4	0	4
y	22	10	–2

(ii) From the graph, the x-coordinates of the points at which the line intersects the curve are **–4** and **2**.

(iii) Since the values of x in **(b)(ii)** are the solutions of the equation,

$-\frac{x^3}{2} + 3x + 2 = 10 - 3x$

$-\frac{x^3}{2} + 6x - 8 = 0$

$x^3 - 12x + 16 = 0$

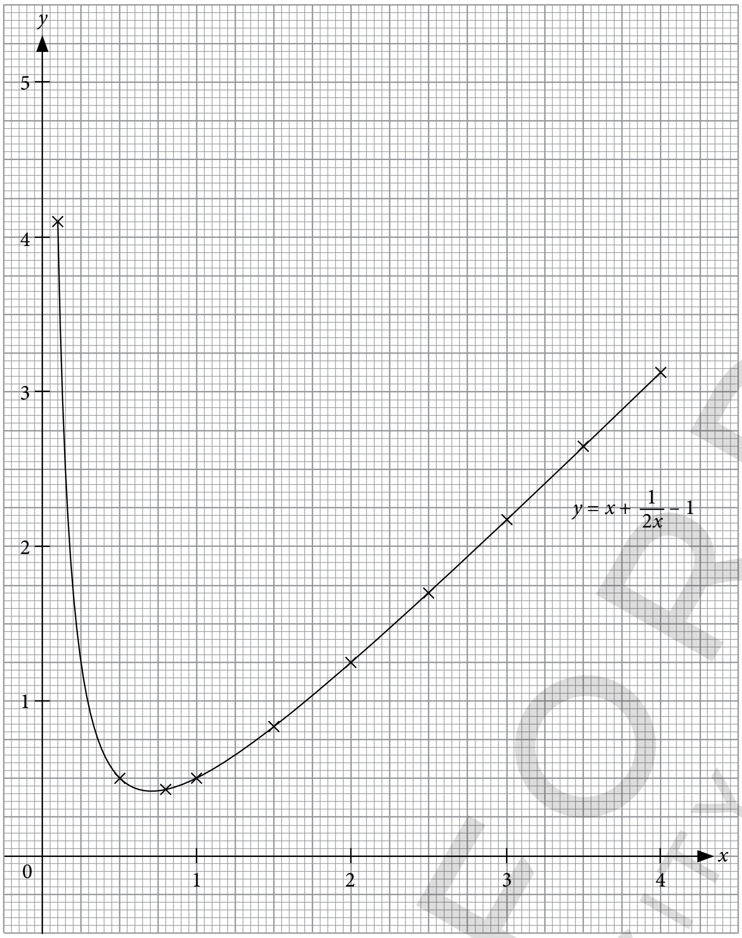
Comparing $x^3 - 12x + 16 = 0$ with $x^3 + Ax + B = 0$, $\boldsymbol{A = -12}$ and $\boldsymbol{B = 16}$

15. **(i)** When $x = 2.5$, $y = 2.5 + \frac{1}{2(2.5)} - 1$

$= 1.7$

$\therefore p = \mathbf{1.7}$

(ii)



(iii) $x + \frac{1}{2x} = 1$

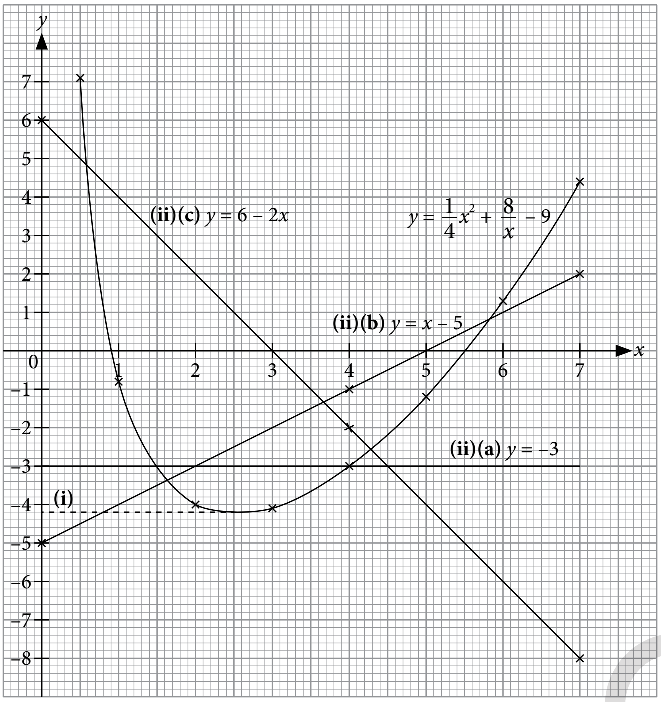
$x + \frac{1}{2x} - 1 = 0$

From the graph, the line $y = 0$, i.e. the x-axis, and the curve $y = x + \frac{1}{2x} - 1$ do not intersect.

$\therefore$ there is **no solution**.

16. $y = \frac{1}{4}x^2 + \frac{8}{x} - 9$

x	0.5	1	2	3	4	5	6	7
y	7.1	−0.8	−4	−4.1	−3	−1.2	1.3	4.4



(i) From the graph, the minimum value of y in the given range $0.5 \leqslant x \leqslant 7$ is **−4.2**.

(ii) (a) $\frac{1}{4}x^2 + \frac{8}{x} = 6$

$\frac{1}{4}x^2 + \frac{8}{x} - 9 = 6 - 9$

$\frac{1}{4}x^2 + \frac{8}{x} - 9 = -3$

$\therefore$ the straight line to be drawn is $y = -3$.

From the graph, the line $y = -3$ and the curve $y = \frac{1}{4}x^2 + \frac{8}{x} - 9$ intersect at (1.45, −3) and (4, −3).

$\therefore x = \mathbf{1.45}$ or $x = \mathbf{4}$

(b) $\frac{1}{4}x^2 + \frac{8}{x} = x + 4$

$\frac{1}{4}x^2 + \frac{8}{x} - 9 = x + 4 - 9$

$\frac{1}{4}x^2 + \frac{8}{x} - 9 = x - 5$

$\therefore$ the straight line to be drawn is $y = x - 5$.

x	0	4	7
y = x − 5	−5	−1	2

From the graph, the line $y = x - 5$ and the curve $y = \frac{1}{4}x^2 + \frac{8}{x} - 9$ intersect at (1.6, −3.4) and (5.8, 0.8).

$\therefore x = \mathbf{1.6}$ or $x = \mathbf{5.8}$

(c) $\frac{1}{4}x^2 + 2x = 15 - \frac{8}{x}$

$\frac{1}{4}x^2 + \frac{8}{x} + 2x = 15$

$\frac{1}{4}x^2 + \frac{8}{x} = 15 - 2x$

$\frac{1}{4}x^2 + \frac{8}{x} - 9 = 15 - 2x - 9$

$\frac{1}{4}x^2 + \frac{8}{x} - 9 = 6 - 2x$

$\therefore$ the straight line to be drawn is $y = 6 - 2x$.

x	0	4	7
y = 6 − 2x	6	−2	−8

From the graph, the line $y = 6 - 2x$ and the curve $y = \frac{1}{4}x^2 + \frac{8}{x} - 9$ intersect at (0.6, 4.8) and (4.3, −2.6).

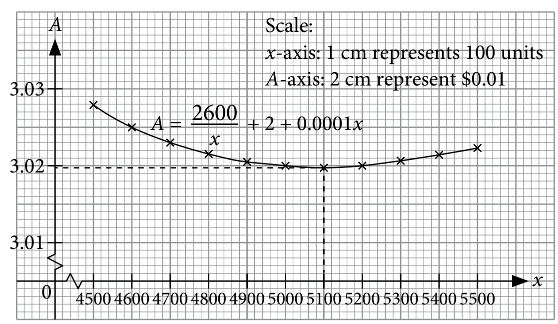
$\therefore x = \mathbf{0.6}$ or $x = \mathbf{4.3}$

17. (i) $A = \frac{2600 + 2x + 0.0001x^2}{x}$

$= \mathbf{\frac{2600}{x} + 2 + 0.0001x}$

(ii) $A = \frac{2600}{x} + 2 + 0.0001x$

x	4500	4600	4700	4800	4900	5000	5100	5200	5300	5400	5500
A	3.0278	3.0252	3.0232	3.0217	3.0206	3.02	3.0198	3.02	3.0206	3.0215	3.0227

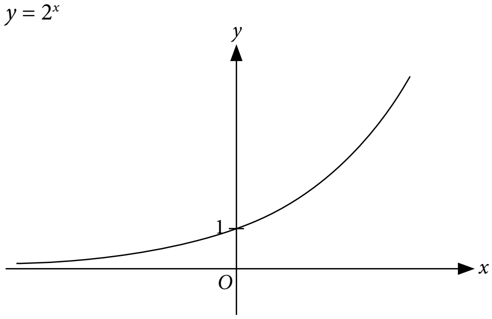


From the graph, the production level at which the average cost is the lowest is x = **5100**. The minimum average cost is **\$3.02**.

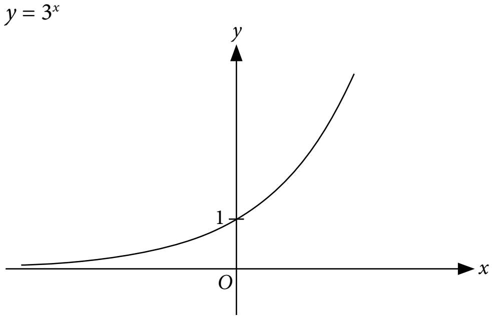
6.4 Graphs of exponential functions

Investigation (Graphs of $y = a^x$ and $y = ka^x$)

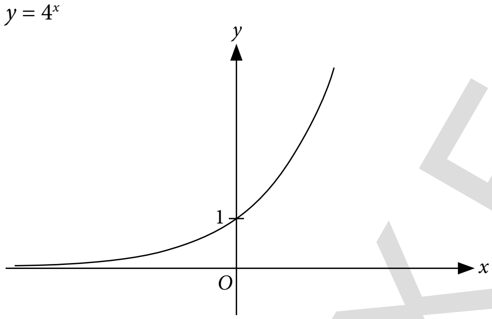
1. **(a)** $y = 2^x$



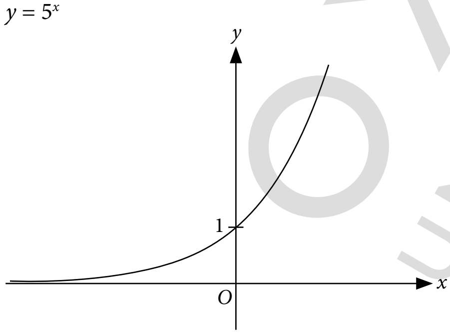
(b) $y = 3^x$



(c) $y = 4^x$



(d) $y = 5^x$



2. **(a)** **(0, 1)**

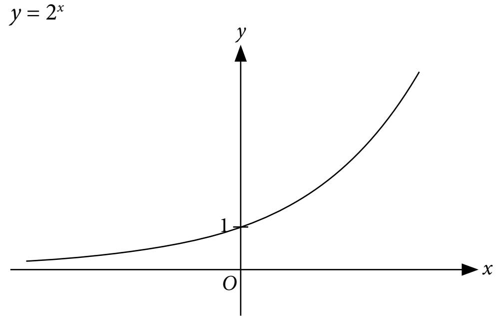
(b) As x increases, y increases at an increasing rate.

(c) **No**, it does not.

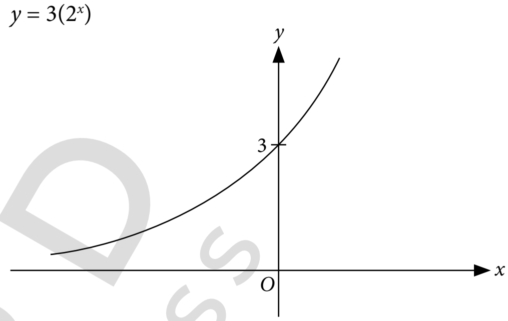
3. **(a)** As a increases, the graph becomes steeper.

(b) Since $1^x = 1$ for all real values of x, the graph of $y = 1^x$ is a horizontal line with y-intercept of 1.

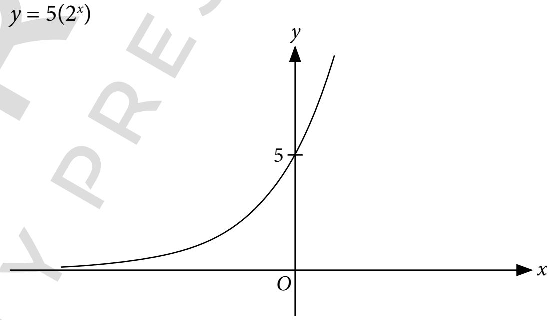
4. **(a)** $y = 2^x$



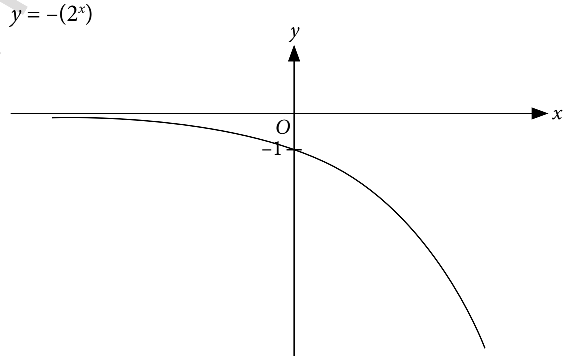
(b) $y = 3(2^x)$



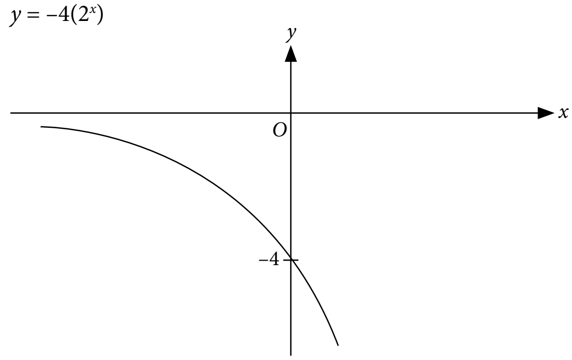
(c) $y = 5(2^x)$



(d) $y = -(2^x)$



(e) $y = -4(2^x)$



5. **(a)** The coordinates of the point where the graph intersects the y-axis in Question **4(a)** are **(0, 1)**.
The coordinates of the point where the graph intersects the y-axis in Question **4(b)** are **(0, 3)**.
The coordinates of the point where the graph intersects the y-axis in Question **4(c)** are **(0, 5)**.
The coordinates of the point where the graph intersects the y-axis in Question **4(d)** are **(0, –1)**.
The coordinates of the point where the graph intersects the y-axis in Question **4(e)** are **(0, –4)**.
(b) In the graphs of Questions **4(a)**, **4(b)** and **4(c)**, as x increases, y increases at an increasing rate while in the graphs of Questions **4(d)** and **4(e)**, as x increases, y decreases at an increasing rate.
(c) **No**, the graphs do not intersect the x-axis.

6. If $k > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis.
If $k < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis.

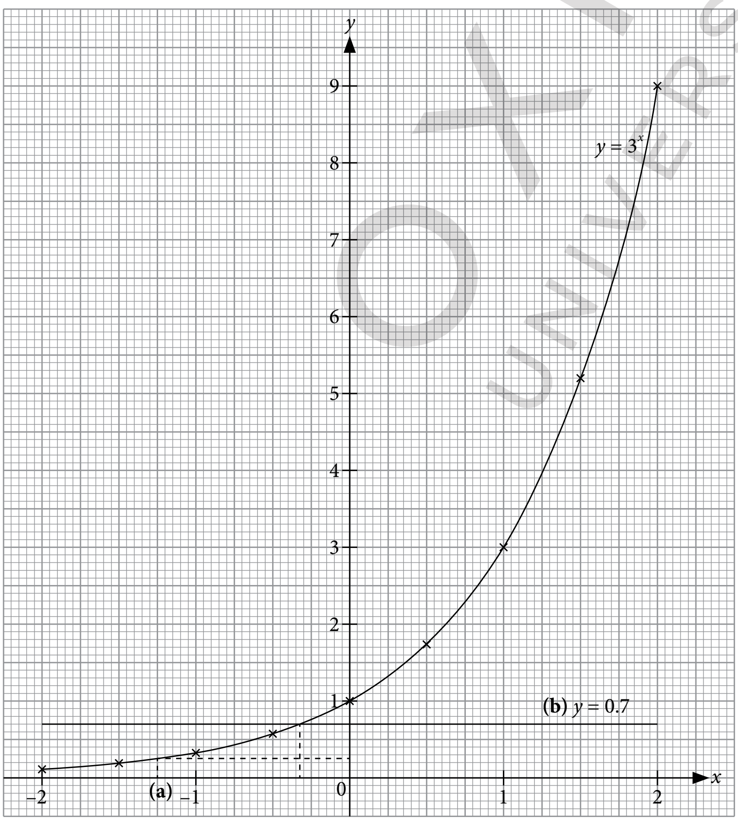
Journal Writing (Page 179)

(i) As x increases, the number of members y increases at an increasing rate.
(ii) More real-life applications of exponential graphs include bacterial decay, population growth, money investment, half-lives of unstable, radioactive atoms.
Teachers should take note that the list is not exhaustive.

Practise Now 10

$y = 3^x$

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2
y	0.11	0.19	0.33	0.58	1	1.73	3	5.2	9



(a) From the graph, when $x = -1.25$, $y = \mathbf{0.25}$.
(b) $3^x + 1 = 1.7$
$3^x = 0.7$
From the graph, the line $y = 0.7$ and the curve $y = 3^x$ intersect at (–0.325, 0.7).
$\therefore$ the solution is $x = \mathbf{-0.325}$.

Practise Now 11

(a) Find asymptote:

$y = 8$ is an asymptote of the graph of $y = 8 - 2^x$.

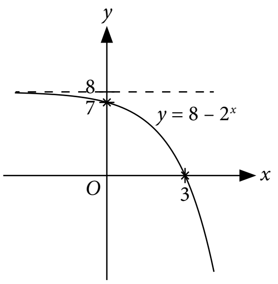
Find y-intercept:

When $x = 0$, $y = 8 - 2^x$
$= 8 - 2^0$
$= 7$

Find x-intercept:

When $y = 0$, $8 - 2^x = 0$
$2^x = 8$
$2^x = 2^3$
$x = 3$

Sketch:



(b) Find asymptote:

$y = 2$ is an asymptote of the graph of $y = 4.5^x + 2$.

Find y-intercept:

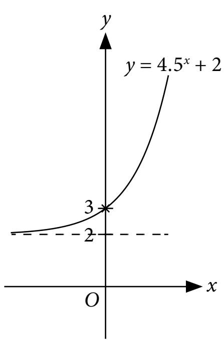
When $x = 0$, $y = 4.5^x + 2$
$= 4.5^0 + 2$
$= 3$

Find x-intercept:

When $y = 0$, $4.5^x + 2 = 0$
$4.5^x = -2$

There are no real values of x for which $4.5^x = -2$.

Sketch:



(c) Find asymptote:

$y = -3$ is an asymptote of the graph of $y = -4^x - 3$.

Find y-intercept:

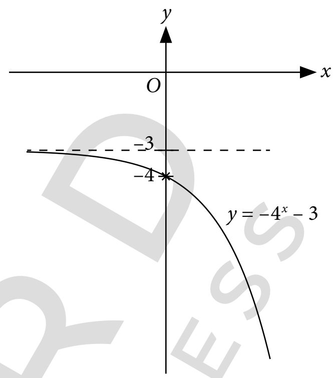
When $x = 0$, $y = -4^x - 3$
$= -4^0 - 3$
$= -4$

Find x-intercept:

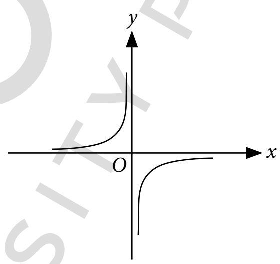
When $y = 0$, $-4^x - 3 = 0$
$4^x = -3$

There are no real values of x for which $4^x = -3$.

Sketch:



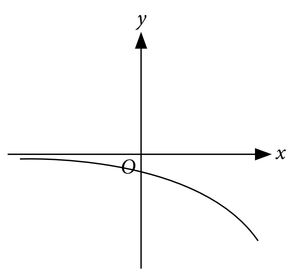
Class Discussion (Matching graphs of power functions with the corresponding functions)



Graph 1

Graph 1 is a graph of $y = \frac{a}{x}$, where $a < 0$.

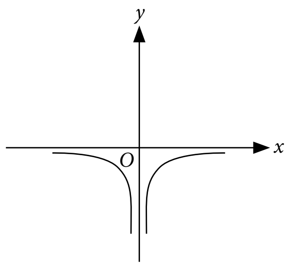
$\therefore$ the function of Graph 1 is **B**: $y = -\frac{6}{x}$.



Graph 2

Graph 2 is a graph of $y = ka^x$, where $k < 0$.

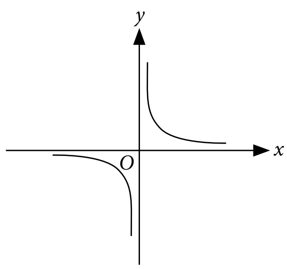
$\therefore$ the function of Graph 2 is **F**: $y = -2(6^x)$.



Graph 3

Graph 3 is a graph of $y = \frac{a}{x^2}$, where $a < 0$.

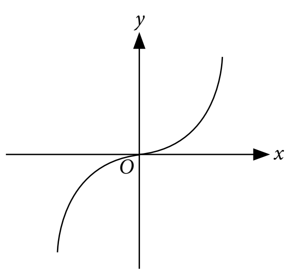
$\therefore$ the function of Graph 3 is **E**: $y = -\frac{3}{x^2}$.



Graph 4

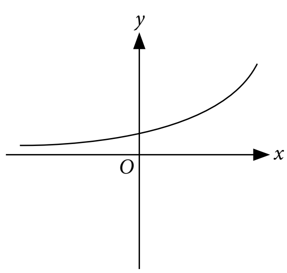
Graph 4 is a graph of $y = \frac{a}{x}$, where $a > 0$.

$\therefore$ the function of Graph 4 is **G**: $y = \frac{1}{2x}$.



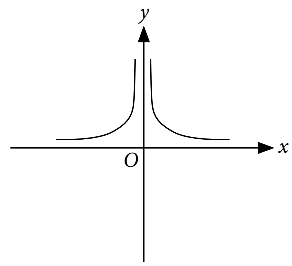
Graph 5

Graph 5 is a graph of $y = ax^3$, where $a > 0$.
$\therefore$ the function of Graph 5 is **A**: $y = 2x^3$.



Graph 6

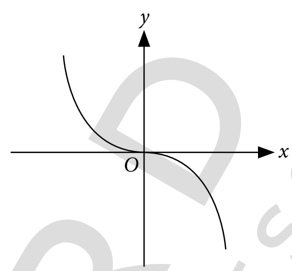
Graph 6 is a graph of $y = ka^x$, where $k > 0$.
$\therefore$ the function of Graph 6 is **D**: $y = 5^x$.



Graph 7

Graph 7 is a graph of $y = \frac{a}{x^2}$, where $a > 0$.

$\therefore$ the function of Graph 7 is **C**: $y = \frac{5}{2x^2}$.



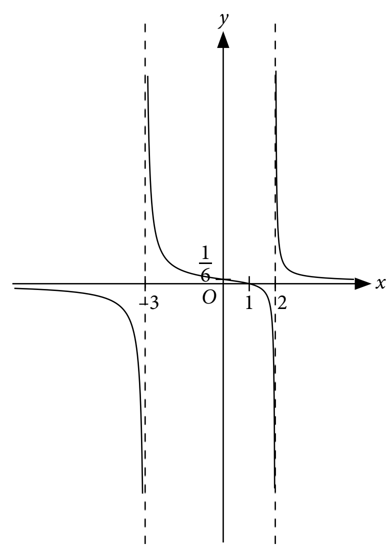
Graph 8

Graph 8 is a graph of $y = ax^3$, where $a < 0$.
$\therefore$ the function of Graph 8 is **H**: $y = -3x^3$.

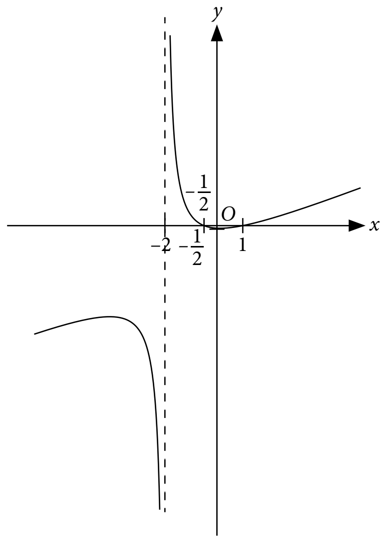
6.5 Graphs of rational functions

Investigation (Graph of a rational function involving polynomials and its asymptotes)

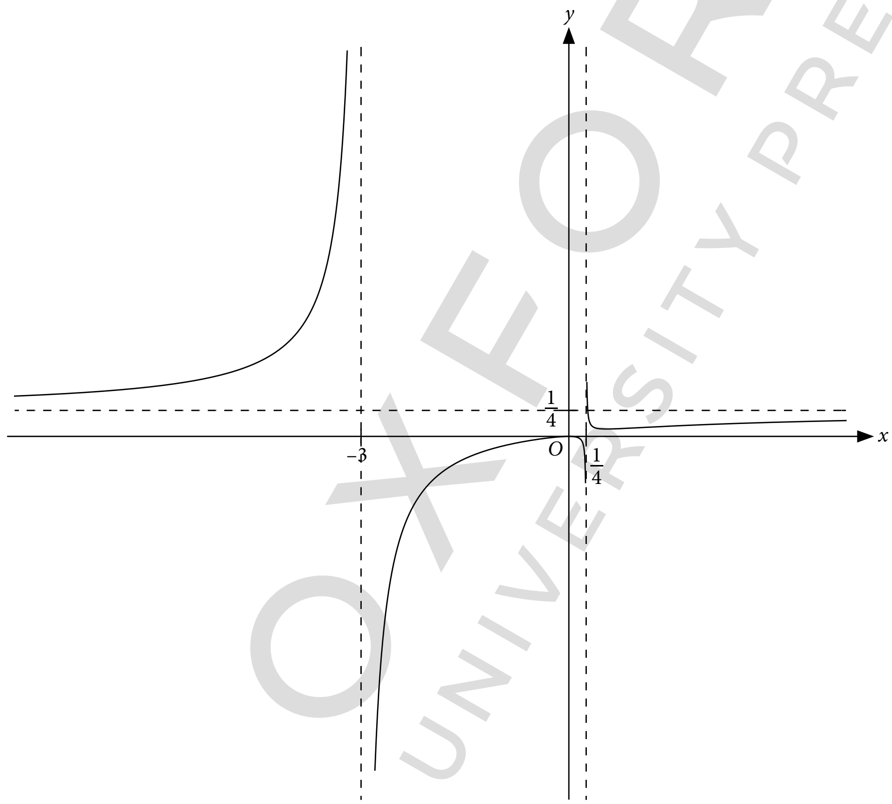
1. **(a)** $y = \frac{x-1}{(x-2)(x+3)}$



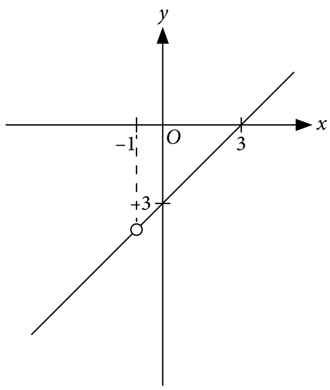
(b) $y = \dfrac{(2x+1)(x-1)}{x+2}$



(c) $y = \dfrac{x^2}{(4x-1)(x+3)}$



(d) $y = \dfrac{(x+1)(x-3)}{x+1}$



2. **(a)** For Question 1(**a**):

$(x-2)(x+3) = 0$

$x - 2 = 0$ or $x + 3 = 0$

$x = \mathbf{2}$ $\qquad x = \mathbf{-3}$

For Question 1(**b**):

$x + 2 = 0$

$x = \mathbf{-2}$

For Question 1(**c**):

$(4x-1)(x+3) = 0$

$4x - 1 = 0$ or $x + 3 = 0$

$4x = 1$ $\qquad x = \mathbf{-3}$

$x = \mathbf{\dfrac{1}{4}}$

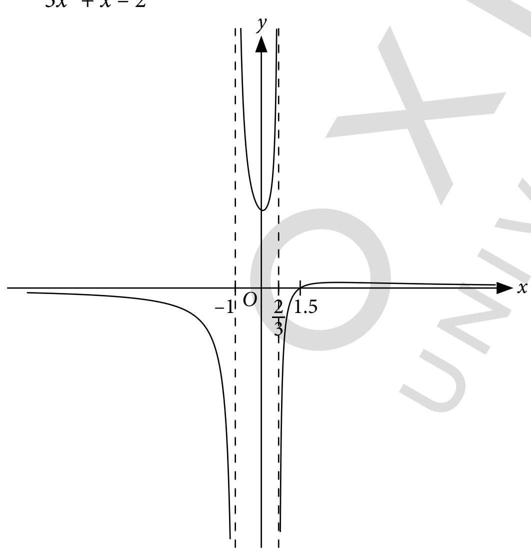
For Question 1(**d**):

$x + 1 = 0$

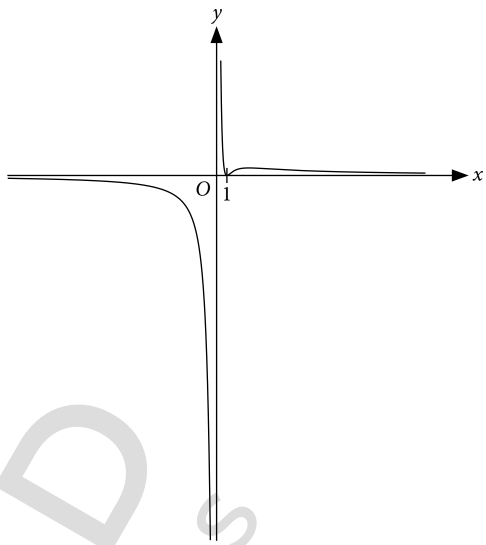
$x = \mathbf{-1}$

(b) **No**

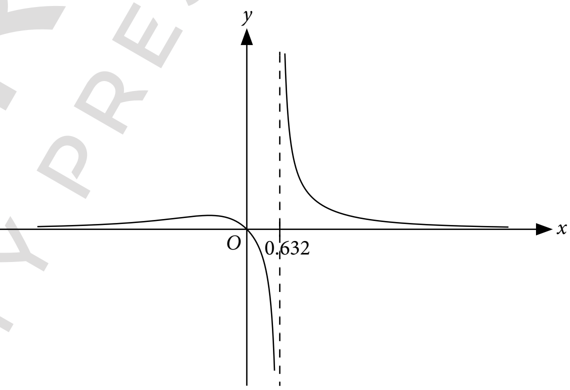
3. **(a)** $y = \dfrac{2x-3}{3x^2+x-2}$



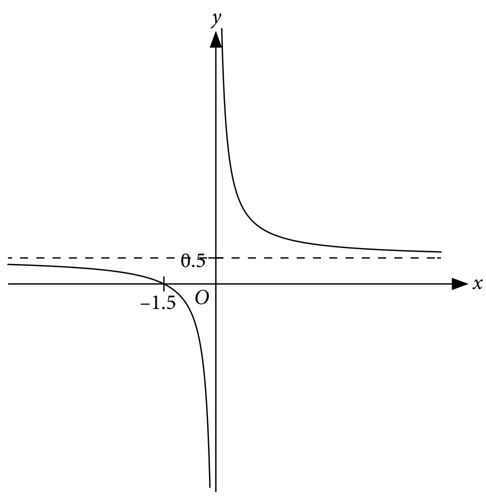
(b) $y = \dfrac{x^2-2x+1}{4x^3+x}$



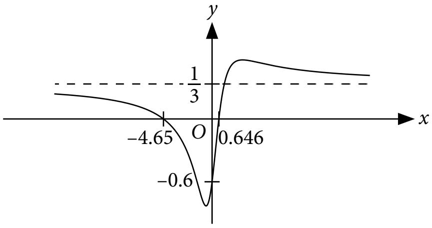
(c) $y = \dfrac{2x}{2x^3-x^2+3x-2}$



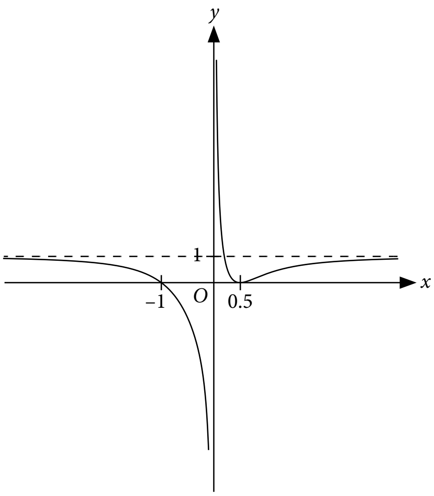
(d) $y = \dfrac{2x+3}{4x}$



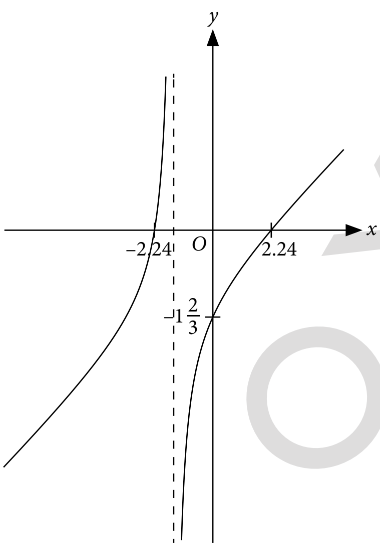
(e) $y = \dfrac{x^2 + 4x - 3}{3x^2 + 5}$



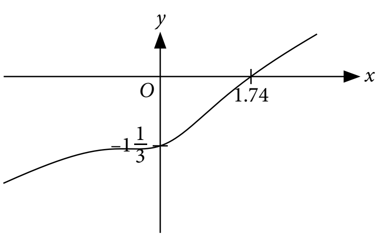
(f) $y = \dfrac{4x^3 - 3x + 1}{4x^3 + 2x}$



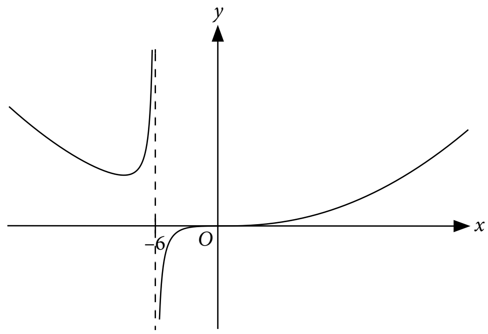
(g) $y = \dfrac{x^2 - 5}{2x + 3}$



(h) $y = \dfrac{x^3 - x^2 + x - 4}{2x^2 + 3}$



(i) $y = \dfrac{2x^3 + x}{x + 6}$



4. **(a)** For Question 3(**a**): $\boldsymbol{m < n}$
For Question 3(**b**): $\boldsymbol{m < n}$
For Question 3(**c**): $\boldsymbol{m < n}$
For Question 3(**d**): $\boldsymbol{m = n}$
For Question 3(**e**): $\boldsymbol{m = n}$
For Question 3(**f**): $\boldsymbol{m = n}$
For Question 3(**g**): $\boldsymbol{m > n}$
For Question 3(**h**): $\boldsymbol{m > n}$
For Question 3(**i**): $\boldsymbol{m > n}$

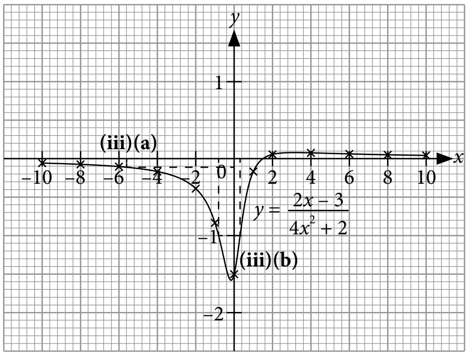
(b) For Question 3(**a**): $\boldsymbol{y = 0}$
For Question 3(**b**): $\boldsymbol{y = 0}$
For Question 3(**c**): $\boldsymbol{y = 0}$
For Question 3(**d**): $\boldsymbol{y = \frac{1}{2}}$
For Question 3(**e**): $\boldsymbol{y = \frac{1}{3}}$
For Question 3(**f**): $\boldsymbol{y = 1}$
For Question 3(**g**): **No horizontal asymptote**
For Question 3(**h**): **No horizontal asymptote**
For Question 3(**i**): **No horizontal asymptote**

Practise Now 12

1. **(i)** Find vertical asymptote(s):
$4x^2 + 2 = 0$
$4x^2 = -2$
Since $4x^2 \geqslant 0$, there are no real solutions.
$\therefore$ the function does not have a vertical asymptote.
Find horizontal asymptote:
The degree of $2x - 3$ is 1.
The degree of $4x^2 + 2$ is 2.
$\therefore$ the horizontal asymptote of the function is $\boldsymbol{y = 0}$.

(ii)

x	−10	−8	−6	−4	−2	−1	0	1	2	4	6	8	10
y	−0.06	−0.07	−0.1	−0.17	−0.39	−0.83	−1.5	−0.17	0.06	0.08	0.06	0.05	0.04



(iii) **(a)** From the graph, when $x = -5.6$, $y = \mathbf{-0.1}$.

(b) From the graph, when $y = -1$, $x = \mathbf{-0.8}$ or $x = \mathbf{0.4}$.

2. **(i)**

x	−8	−6	−4	−2	0	2	4	4.4
y	−5.1	**−3.3**	**−1.6**	0	**1.2**	**1.3**	−6	**−14.9**

(ii) Find vertical asymptote(s):

$x - 5 = 0$

$x = 5$

$\therefore$ the vertical asymptote of the function is $\boldsymbol{x = 5}$.

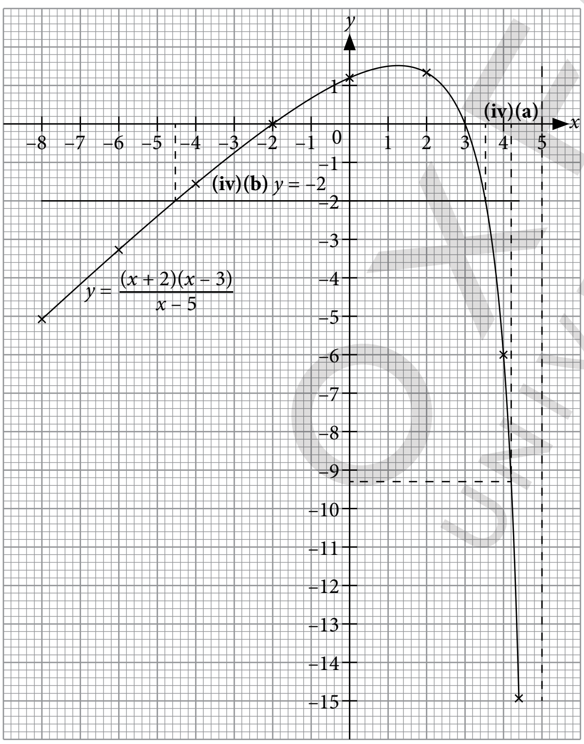
Find horizontal asymptote:

The degree of $(x + 2)(x - 3)$ is 2.

The degree of $x - 5$ is 1.

$\therefore$ the function does not have a horizontal asymptote.

(iii)



(iv) **(a)** From the graph, when $x = 4.2$, $y = \mathbf{-9.3}$.

(b) $(x + 2)(x - 3) = 10 - 2x$

$(x + 2)(x - 3) = -2(-5 + x)$

$\frac{(x+2)(x-3)}{x-5} = -2$

From the graph, the line $y = -2$ and the curve $y = \frac{(x+2)(x-3)}{x-5}$ intersect at $(-4.5, -2)$ and $(3.5, -2)$.

$\therefore x = \mathbf{-4.5}$ or $x = \mathbf{3.5}$

3. **(i)**

x	−7	−5	−3	−2	−1.5	−0.5	0	0.5	1	2.5	3.5	4	6	8
y	**0.8**	**0.63**	**0**	**−1.4**	−4.33	**7.86**	**5**	**4.2**	**4**	7.86	**−4.33**	**−1.4**	0.43	**0.73**

(ii) Find vertical asymptote(s):

$(x + 1)(x - 3) = 0$

$x + 1 = 0$ or $x - 3 = 0$

$x = -1$ $\quad$ $x = 3$

$\therefore$ the vertical asymptotes of the function are $\boldsymbol{x = -1}$ and $\boldsymbol{x = 3}$.

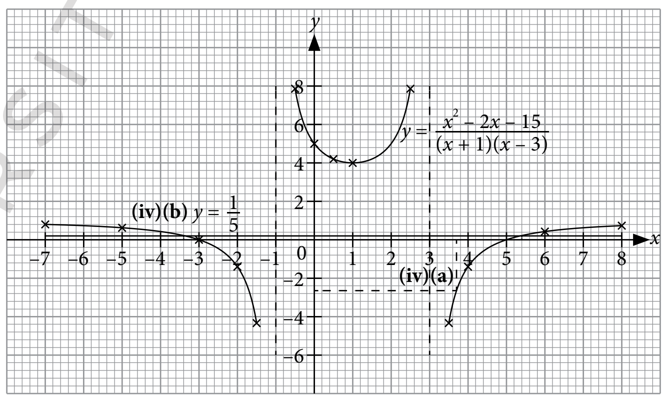
Find horizontal asymptote:

The degree of $x^2 - 2x - 15$ is 2.

The degree of $(x + 1)(x - 3)$ is 2.

$\therefore$ the horizontal asymptote of the function is $\boldsymbol{y = 1}$.

(iii)



(iv) **(a)** From the graph, when $x = 3.7$, $y = \mathbf{-2.6}$.

(b) $5(x^2 - 2x - 15) = (x + 1)(x - 3)$

$\frac{x^2 - 2x - 15}{(x+1)(x-3)} = \frac{1}{5}$

From the graph, the line $y = \frac{1}{5}$ and the curve

$y = \frac{x^2 - 2x - 15}{(x+1)(x-3)}$ intersect at $\left(-3.4, \frac{1}{5}\right)$ and $\left(5.4, \frac{1}{5}\right)$.

$\therefore x = \mathbf{-3.4}$ or $x = \mathbf{5.4}$

6.6 Gradient of a curve

Practise Now 13

(a) When $x = -1$,

$y = (-1)^2 - 4(-1)$

$= 5$

$\therefore$ $\boldsymbol{a = 5}$

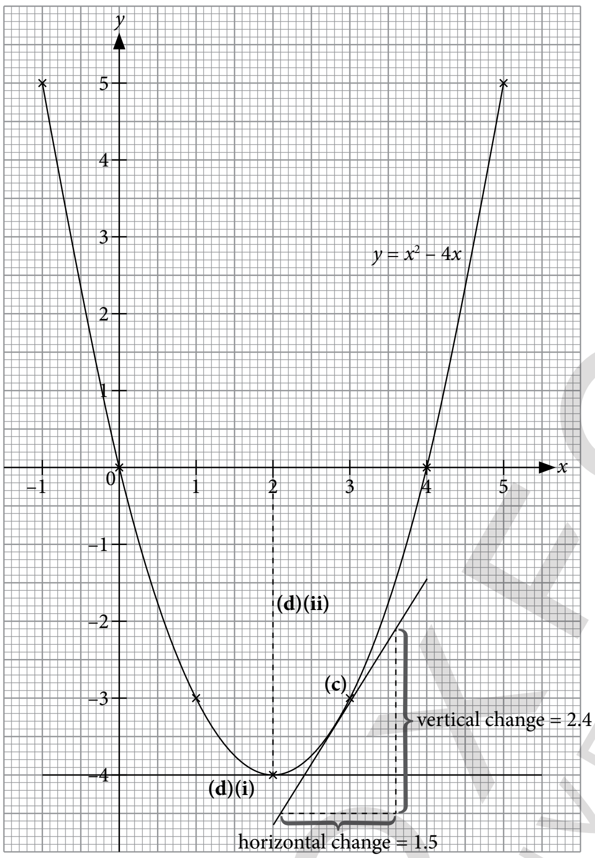
When $x = 3$,

$y = 3^2 - 4(3)$

$= -3$

$\therefore$ $\boldsymbol{b = -3}$

(b)



(c) A tangent is drawn to the curve at the point (2.8, –3.36).
From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{2.4}{1.5} = \mathbf{1.6}$$

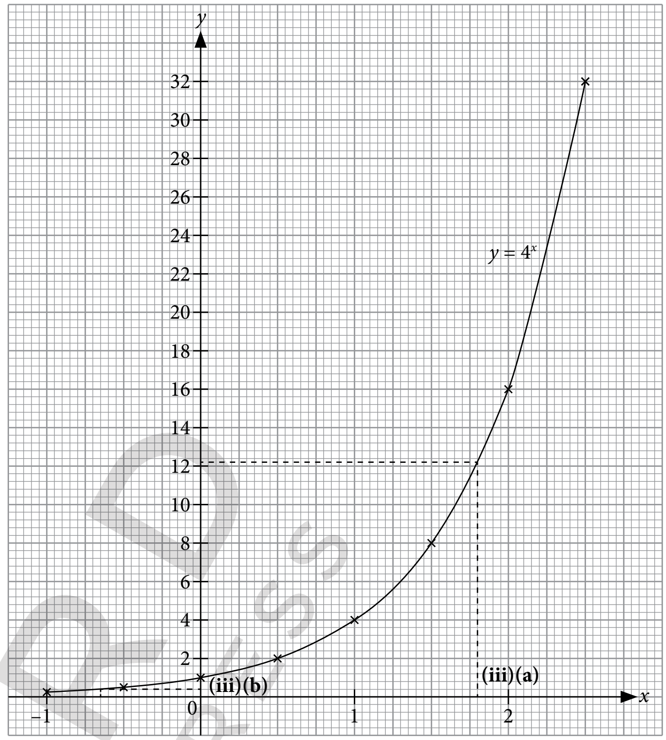
(d) **(ii)** From the graph and table, $\boldsymbol{h = 2}$, $\boldsymbol{k = -4}$.

Exercise 6B

1. **(i)** $y = 4^x$

x	–1	–0.5	0	0.5	1	1.5	2	2.5
y	0.25	**0.5**	1	2	4	**8**	**16**	**32**

(ii)



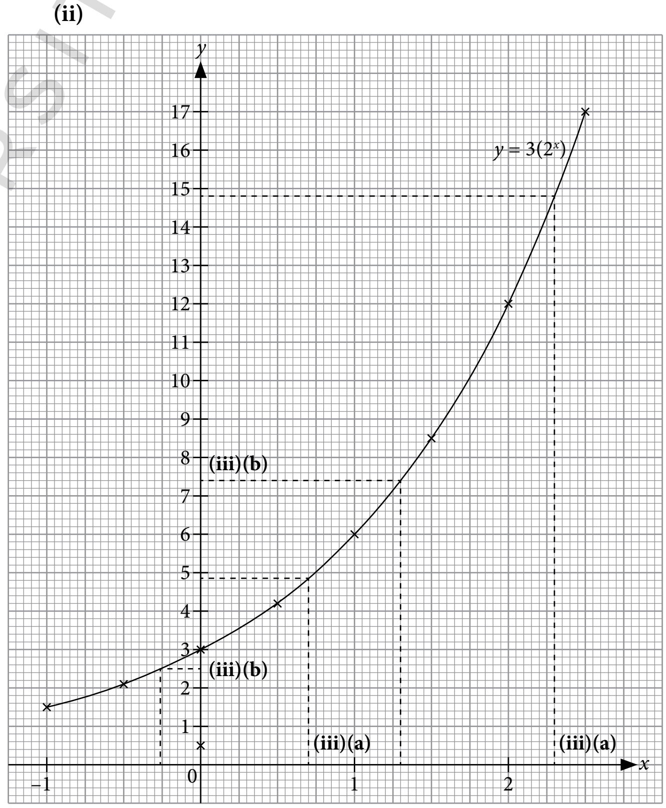
(iii) **(a)** From the graph, when $x = 1.8$, $y =$ **12.2**.

(b) From the graph, when $y = 0.4$, $x =$ **–0.65**.

2. **(i)** $y = 3(2^x)$

x	–1	–0.5	0	0.5	1	1.5	2	2.5
y	1.5	2.1	**3**	**4.2**	6	**8.5**	12	17.0

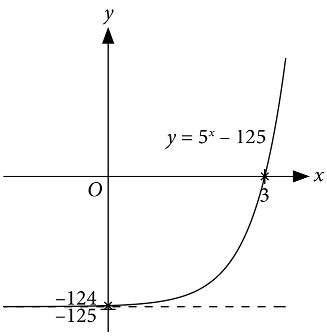
(ii)



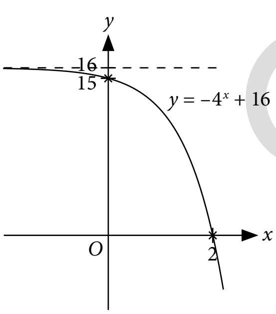
(iii) (a) From the graph, when $x = 0.7$, $y =$ **4.9**.
From the graph, when $x = 2.3$, $y =$ **14.8**.

(b) From the graph, when $y = 2.5$, $x =$ **−0.275**.
From the graph, when $y = 7.4$, $x =$ **1.3**.

3. (a) Find asymptote:
$y = -125$ is an asymptote of the graph of $y = 5^x - 125$.
Find y-intercept:
When $x = 0$, $y = 5^x - 125$
$= 5^0 - 125$
$= -124$
Find x-intercept:
When $y = 0$, $5^x - 125 = 0$
$5^x = 125$
$5^x = 5^3$
$x = 3$
Sketch:

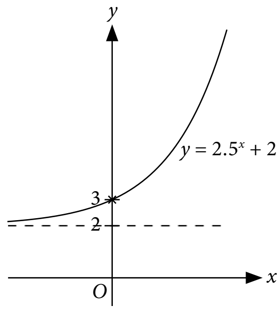


(b) Find asymptote:
$y = 16$ is an asymptote of the graph of $y = -4^x + 16$.
Find y-intercept:
When $x = 0$, $y = -4^x + 16$
$= -4^0 + 16$
$= 15$
Find x-intercept:
When $y = 0$, $-4^x + 16 = 0$
$4^x = 16$
$4^x = 4^2$
$x = 2$
Sketch:

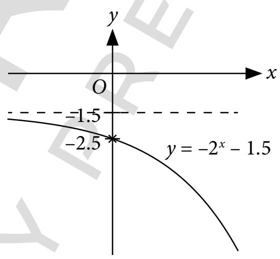


(c) Find asymptote:
$y = 2$ is an asymptote of the graph of $y = 2.5^x + 2$.
Find y-intercept:
When $x = 0$, $y = 2.5^x + 2$
$= 2.5^0 + 2$
$= 3$
Find x-intercept:
When $y = 0$, $2.5^x + 2 = 0$
$2.5^x = -2$
There are no real values of x for which $2.5^x = -2$.

Sketch:



(d) Find asymptote:
$y = -1.5$ is an asymptote of the graph of $y = -2^x - 1.5$.
Find y-intercept:
When $x = 0$, $y = -2^x - 1.5$
$= -2^0 - 1.5$
$= -2.5$
Find x-intercept:
When $y = 0$, $-2^x - 1.5 = 0$
$2^x = -1.5$
There are no real values of x for which $2^x = -1.5$.
Sketch:



4. (i) Find vertical asymptote(s):
$2x^2 + 3x + 2 = 0$

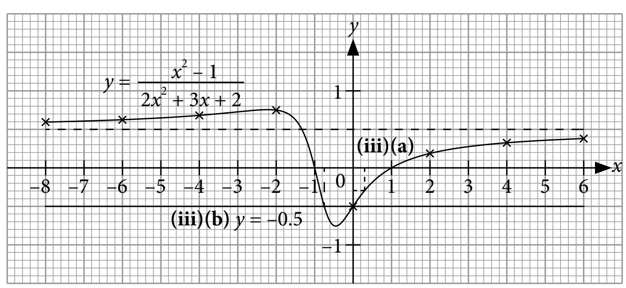
$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(2)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{-7}}{2(2)}$$

$\therefore$ the function does not have a vertical asymptote.
Find horizontal asymptote:
The degree of $x^2 - 1$ is 2.
The degree of $2x^2 + 3x + 2$ is 2.
$\therefore$ the horizontal asymptote of the function is $\boldsymbol{y = \frac{1}{2}}$.

(ii)

x	−8	−6	−4	−2	0	2	4	6
y	0.59	0.63	0.68	0.75	−0.5	0.19	0.33	0.38



(iii) (a) From the graph, when $x = 0.3$, $y =$ **−0.3**.

(b) From the graph, the line $y = -0.5$ and the curve
$y = \dfrac{x^2 - 1}{2x^2 + 3x + 2}$ intersect at $(-0.8, -0.5)$ and $(0, -0.5)$.
$\therefore x =$ **−0.8** or $x =$ **0**

5. **(i)**

x	–5	–3	–1	0	1	1.5	2
y	**1.47**	**1.45**	1.43	**1.4**	**1.33**	1.25	**1**

x	2.2	2.8	3	3.5	4	6	8
y	0.67	**2.33**	**2**	1.75	**1.67**	**1.57**	**1.55**

(ii) Find vertical asymptote(s):

$2x - 5 = 0$

$2x = 5$

$x = \frac{5}{2}$

$\therefore$ the vertical asymptote of the function is $\boldsymbol{x = \frac{5}{2}}$.

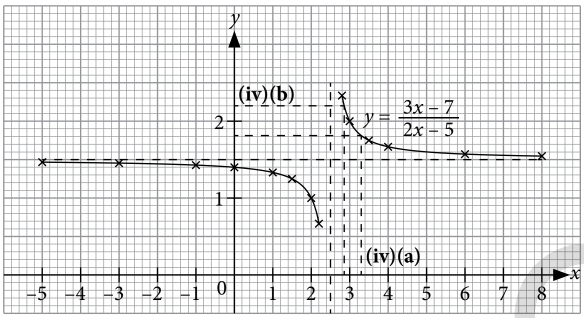
Find horizontal asymptote:

The degree of $3x - 7$ is 1.

The degree of $2x - 5$ is 1.

$\therefore$ the horizontal asymptote of the function is $\boldsymbol{y = \frac{3}{2}}$.

(iii)



(iv) **(a)** From the graph, when $x = 3.3$, $y =$ **1.8**.

(b) From the graph, when $y = 2.2$, $x =$ **2.9**.

6. **(i)**

x	–4	–2	0	2	4	4.5	5	5.5	6	6.8
y	–0.18	**–0.22**	**–0.29**	**–0.4**	**–0.67**	**–0.8**	**–1**	–1.33	**–2**	**–10**

x	7.2	8	8.5	9	9.5	10	12	14	16	18
y	**10**	**2**	1.33	**1**	**0.8**	0.67	**0.4**	**0.29**	0.22	**0.18**

(ii) Find vertical asymptote(s):

$x - 7 = 0$

$x = 7$

$\therefore$ the vertical asymptote of the function is $\boldsymbol{x = 7}$.

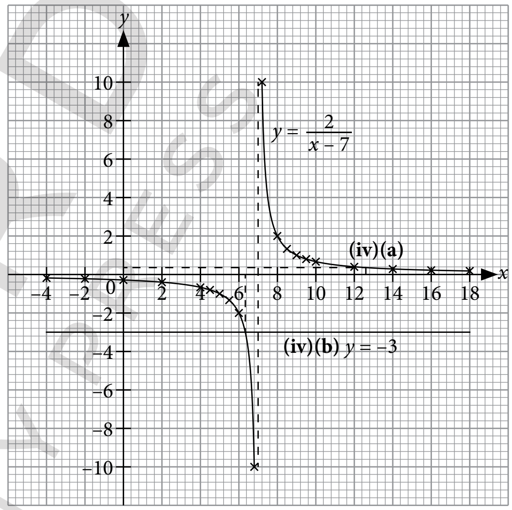
Find horizontal asymptote:

The degree of 2 is 0.

The degree of $x - 7$ is 1.

$\therefore$ the horizontal asymptote of the function is $\boldsymbol{y = 0}$.

(iii)



(iv) **(a)** From the graph, when $x = 12.6$, $y =$ **0.4**.

(b) From the graph, the line $y = –3$ and the curve $y = \frac{2}{x-7}$ intersect at (6.4, –3).

$\therefore x =$ **6.4**

7. (i) When $x = -1$,

$y = 2 + 2^{-1}$

$= 2.5$

$\therefore$ ***a* = 2.5**

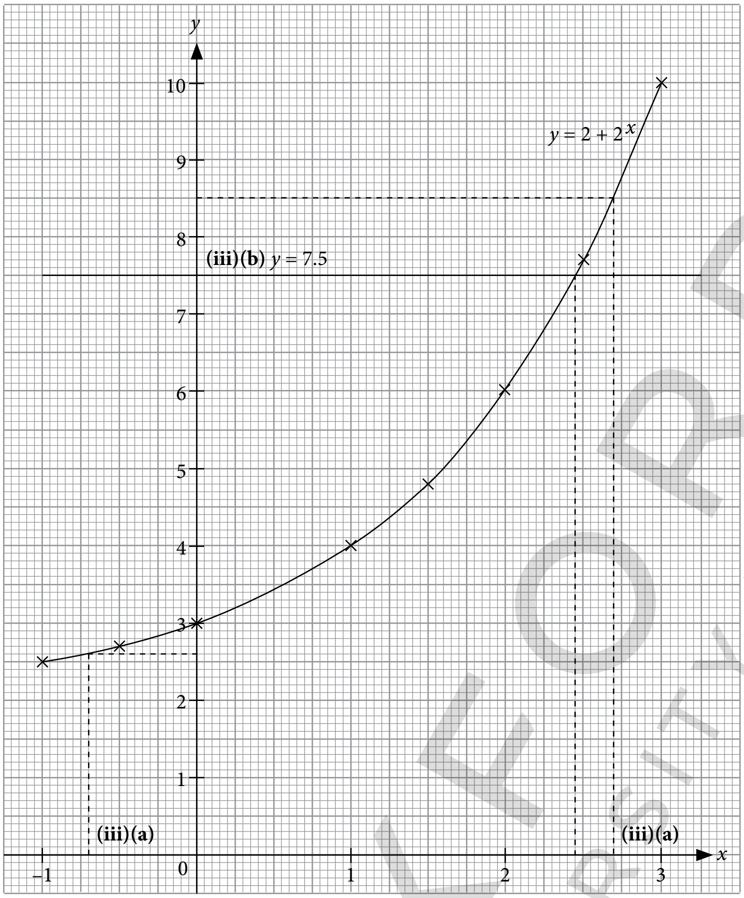
When $x = 2.5$,

$y = 2 + 2^{2.5}$

$= 7.7$ (to 1 d.p.)

$\therefore$ ***b* = 7.7**

(ii)



(iii) (a) From the graph, when $x = -0.7$, $y =$ **2.6**.

From the graph, when $x = 2.7$, $y =$ **8.5**.

(b) $2^x = 5.5$

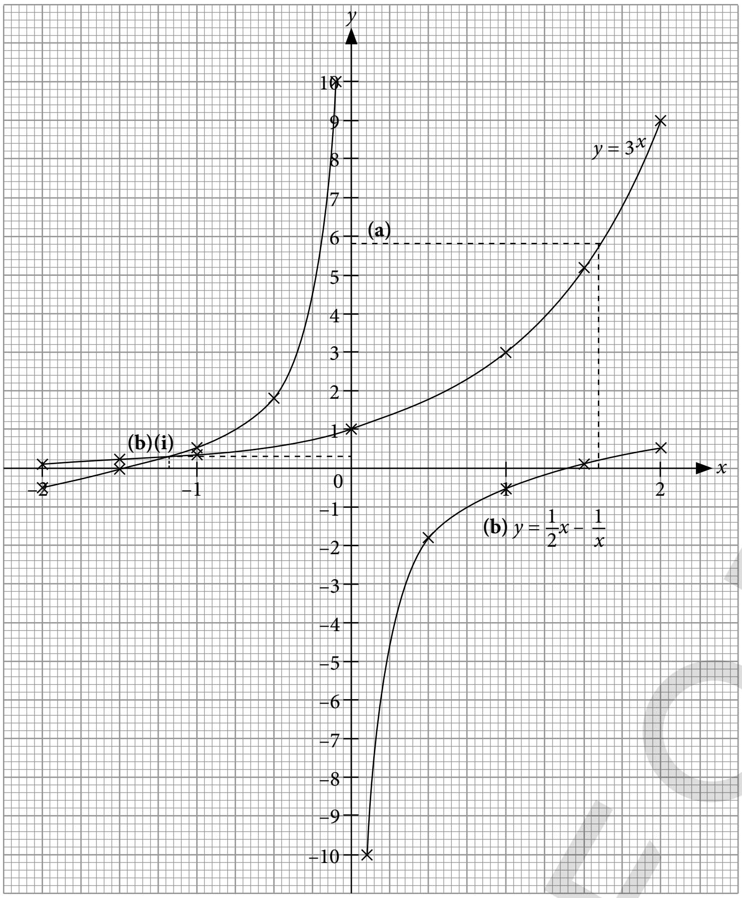
$2 + 2^x = 7.5$

From the graph, the line $y = 7.5$ and the curve $y = 2 + 2^x$ intersect at $(2.45, 7.5)$.

$\therefore$ the solution is $x =$ **2.45**.

8. $y = 3^x$

x	−2	−1.5	−1	0	1	1.5	2
y	0.1	0.2	0.3	1	3	5.2	9



(a) From the graph, when $y = 5.8$, $x =$ **1.6**.

(b) $y = \frac{1}{2}x - \frac{1}{x}$

x	−2	−1.5	−1	−0.5	−0.1	0.1	0.5	1	1.5	2
y	−0.5	−0.1	0.5	1.8	10	−10	−1.8	−0.5	0.1	0.5

(i) From the graph, the coordinates of the point are **(−1.175, 0.3)**.

(ii) $3^x + \frac{1}{x} - \frac{1}{2}x = 0$

$$3^x = \frac{1}{2}x - \frac{1}{x}$$

Hence the solution of the equation is **−1.175**.

9. $y = 3^x$

Find asymptote:

$y = 0$ is an asymptote of the graph of $y = 3^x$.

Find y-intercept:

When $x = 0$, $y = 3^x$
$= 3^0$
$= 1$

Find x-intercept:

When $y = 0$, $3^x = 0$

There are no real values of x for which $3^x = 0$.

$y = 6^x + 1$

Find asymptote:

$y = 1$ is an asymptote of the graph of $y = 6^x + 1$.

Find y-intercept:

When $x = 0$, $y = 6^x + 1$
$= 6^0 + 1$
$= 2$

Find x-intercept:

When $y = 0$, $6^x + 1 = 0$
$6^x = -1$

There are no real values of x for which $6^x = -1$.

$y = -2^x + 4$

Find asymptote:

$y = 4$ is an asymptote of the graph of $y = -2^x + 4$.

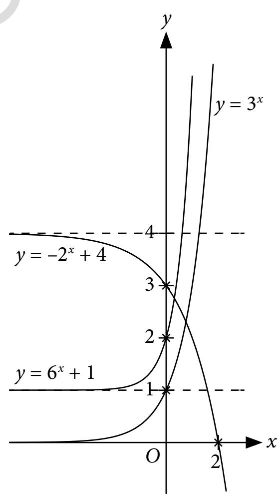
Find y-intercept:

When $x = 0$, $y = -2^x + 4$
$= -2^0 + 4$
$= 3$

Find x-intercept:

When $y = 0$, $-2^x + 4 = 0$
$2^x = 4$
$2^x = 2^2$
$x = 2$

Sketch:



10. $y = ka^x$

Substitute $x = 0$ and $y = 3$ into the equation:

$3 = ka^0$

$\therefore k = \mathbf{3}$

11. **(i)**

x	-3	-2	-1	0	1	1.9	2	2.1
y	-0.31	$\mathbf{-0.36}$	$\mathbf{-0.44}$	$\mathbf{-0.56}$	$\mathbf{-0.75}$	-0.99	$\mathbf{-1}$	$\mathbf{-0.99}$

x	2.6	2.8	3.3	4	4.5	5	6	7	8
y	$\mathbf{1.25}$	$\mathbf{15}$	17.78	$\mathbf{3}$	$\mathbf{1.78}$	$\mathbf{1.25}$	0.78	$\mathbf{0.56}$	$\mathbf{0.44}$

(ii) Find vertical asymptote(s):

$x^2 - 6x + 9 = 0$

$(x - 3)^2 = 0$

$x = 3$

$\therefore$ the vertical asymptote of the function is $\boldsymbol{x = 3}$.

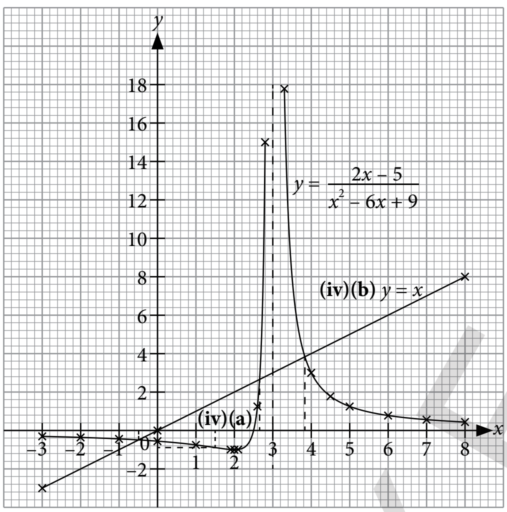
Find horizontal asymptote:

The degree of $2x - 5$ is 1.

The degree of $x^2 - 6x + 9$ is 2.

$\therefore$ the horizontal asymptote of the function is $\boldsymbol{y = 0}$.

(iii)



(iv) **(a)** From the graph, when $x = 1.5$, $y = \mathbf{-0.8}$.

(b)

x	-3	0	8
$y = x$	-3	0	8

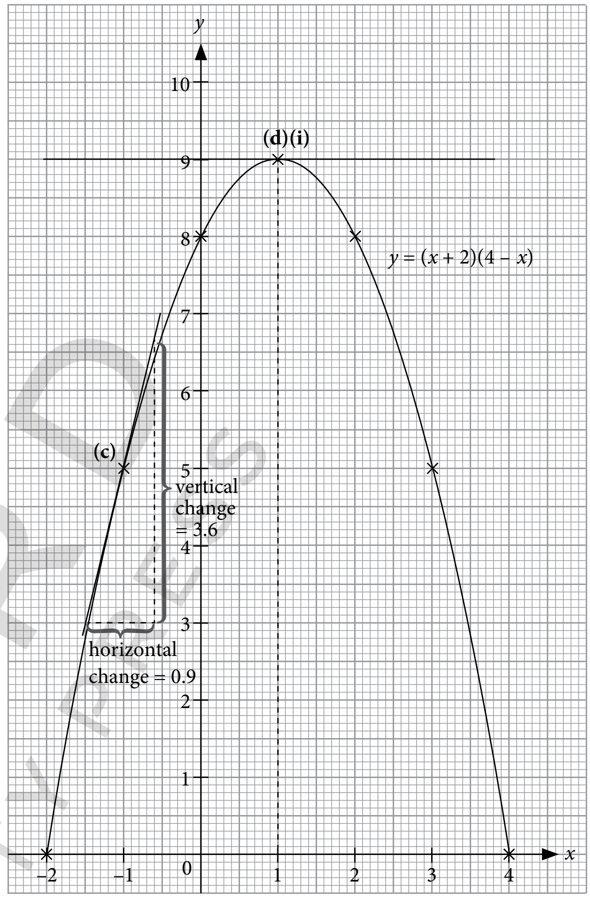
From the graph, the line $y = x$ and the curve $y = \dfrac{x^2 - 1}{2x^2 + 3x + 2}$ intersect at $(-0.5, -0.4)$, $(2.7, 2.6)$ and $(3.8, 3.8)$.

$\therefore x = \mathbf{-0.5}$, $x = \mathbf{2.7}$ or $x = \mathbf{3.8}$

12. **(a)** $y = (x + 2)(4 - x)$

x	-2	-1	0	1	2	3	4
y	0	$\mathbf{5}$	8	9	$\mathbf{8}$	$\mathbf{5}$	0

(b)



(c) A tangent is drawn to the curve at the point $(-1, 5)$.

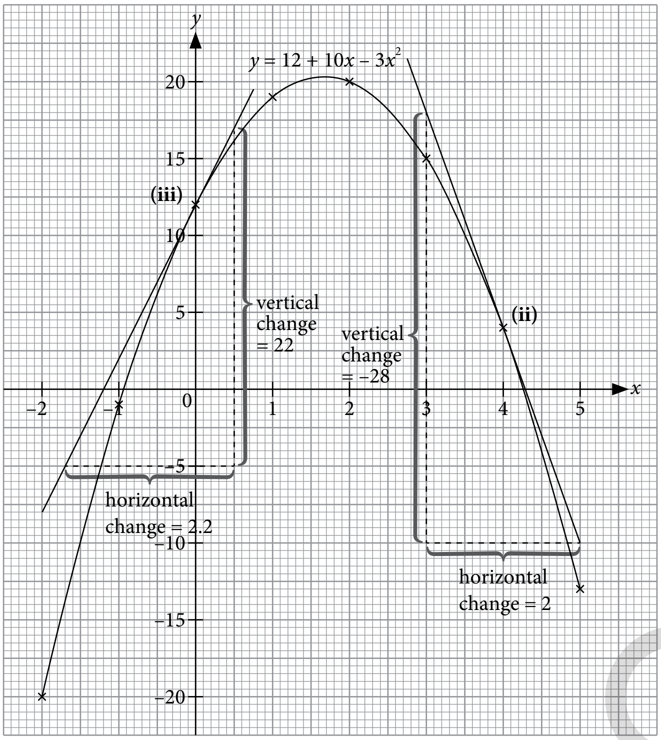
From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{3.6}{0.9} = \mathbf{4}$$

(d) **(ii)** From the graph, $\boldsymbol{h = 1}$, $\boldsymbol{k = 9}$.

13. **(i)** $y = 12 + 10x - 3x^2$

x	−2	−1	0	1	2	3	4	5
y	−20	−1	12	19	20	15	4	−13



(ii) A tangent is drawn to the curve at the point (4, 4).
From the graph,

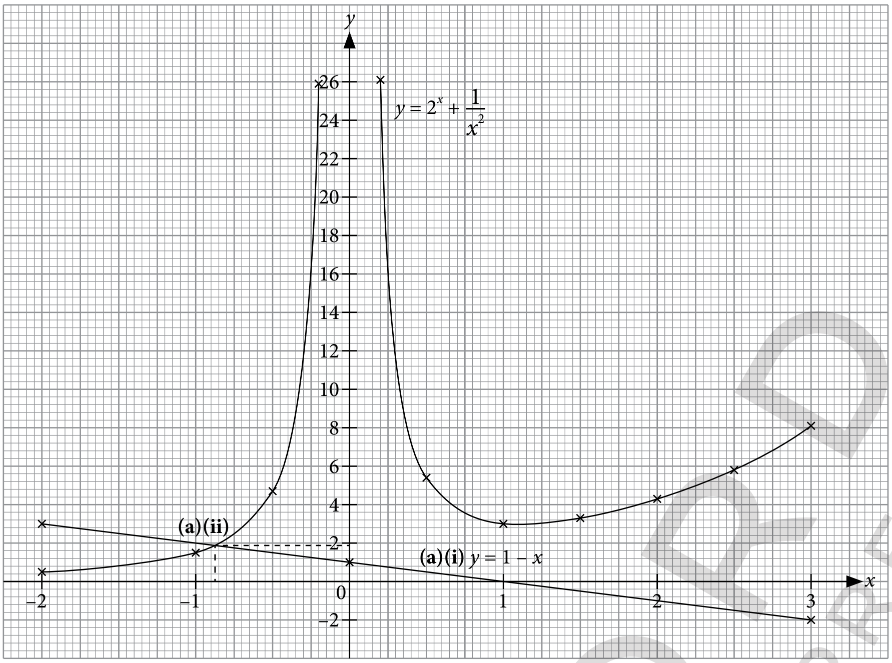
$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{-28}{2}$$
$$= \mathbf{-14}$$

(iii) A tangent is drawn to the curve at the point (0, 12).
From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{22}{2.2}$$
$$= \mathbf{10}$$

14. $y = 2^x + \frac{1}{x^2}$

x	−2	−1	−0.5	−0.2	0.2	0.5	1	1.5	2	2.5	3
y	0.5	1.5	4.7	25.9	26.1	5.4	3	3.3	4.3	5.8	8.1



(a) **(i)**

x	−2	0	3
$y = 1 - x$	3	1	−2

(ii) $2^x + \frac{1}{x^2} - 1 + x = 0$

$$2^x + \frac{1}{x^2} = 1 - x$$

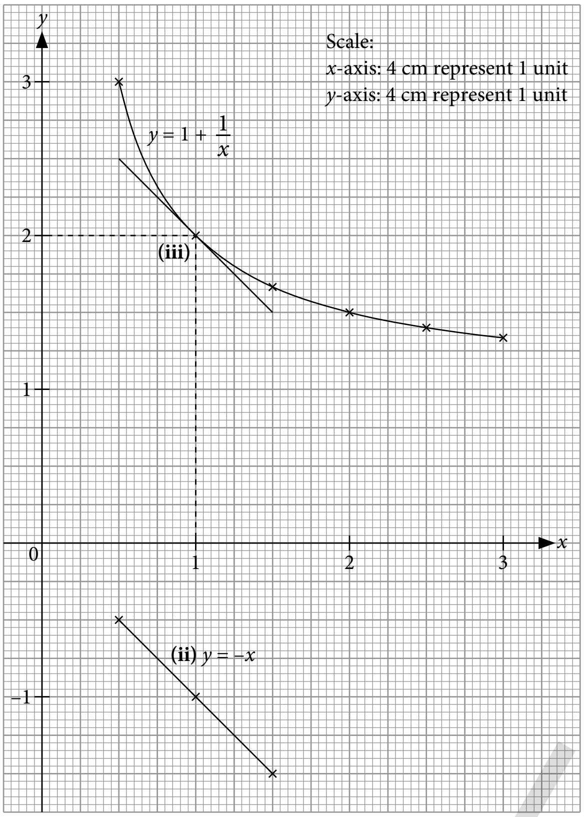
From the graph, the line $y = 1 - x$ and the curve $y = 2^x + \frac{1}{x^2}$ intersect at (−0.875, 1.8).

$\therefore$ the solution is $x = \mathbf{-0.875}$.

(b) For all real values of x, 2^x and $\frac{1}{x^2}$ are always positive, hence $y = 2^x + \frac{1}{x^2}$ is always positive and the graph will not lie below the x-axis.

15. (i) $y = 1 + \frac{1}{x}$

x	0.5	1	1.5	2	2.5	3
y	3	2	1.667	1.5	1.4	1.333



(ii)

x	0.5	1	1.5
$y = -x$	−0.5	−1	−1.5

(iii) A line parallel to the line $y = -x$ at a point of the curve has a gradient equal to −1.
From the graph, the coordinates are **(1, 2)**.

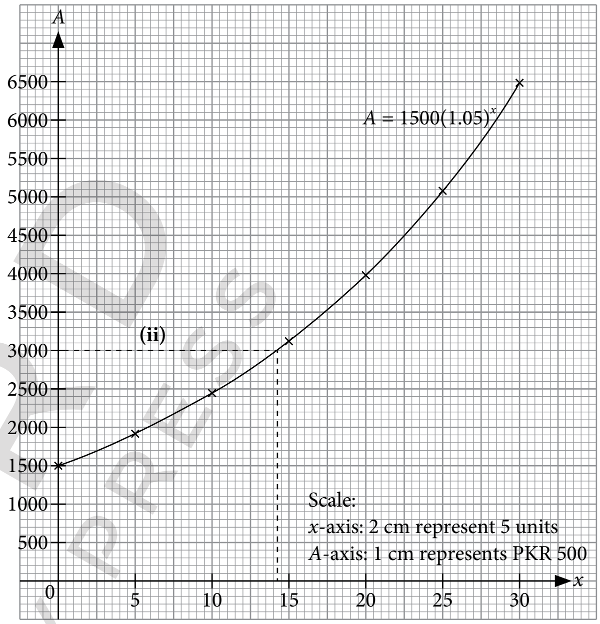
16. (i) $A = P\left(1 + \frac{r}{100}\right)^n$

$= \text{PKR } 1500\left(1 + \frac{5}{100}\right)^x$

$= \text{PKR } 1500(1.05)^x$ (shown)

(ii) $A = 1500(1.05)^x$

x	0	5	10	15	20	25	30
A	1500	1914	2443	3118	3980	5080	6483



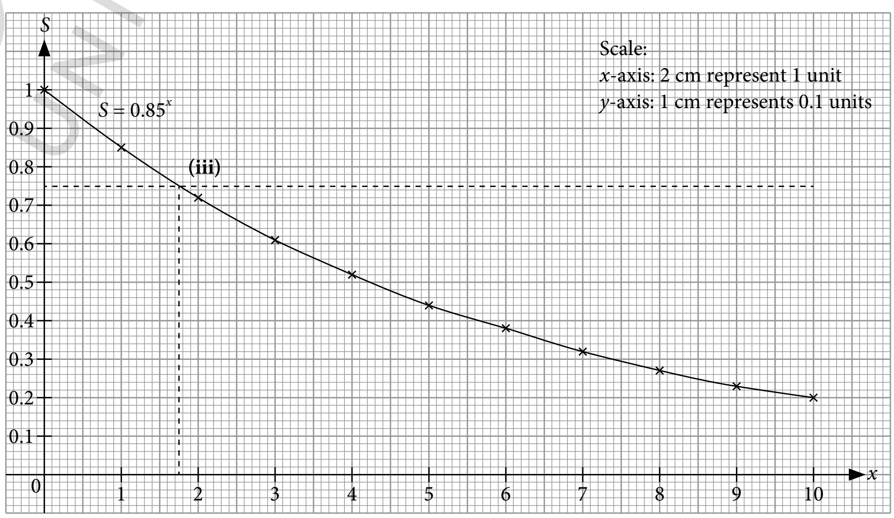
From the graph, when $A = 3000$, $x = 14.25$.
∴ the number of years needed for the investment to double in value is **15**.

17. (i) Initial percentage of S = 100%
= 1

After x hours, percentage of $S = 1(100\% - 15\%)^x$
$= (85\%)^x$
$= \mathbf{0.85^x}$

(ii) $S = 0.85^x$

x	0	1	2	3	4	5	6	7	8	9	10
S	1	0.85	0.72	0.61	0.52	0.44	0.38	0.32	0.27	0.23	0.2



(iii) When at least 25% of the substance has decayed,
$S = 1 - 0.25$
$= 0.75$
From the graph, when $S = 0.75$, $x = 1.75$.
$\therefore$ the amount of time needed for the substance to reach this safe threshold is **1.75 hours**.

18. (i)

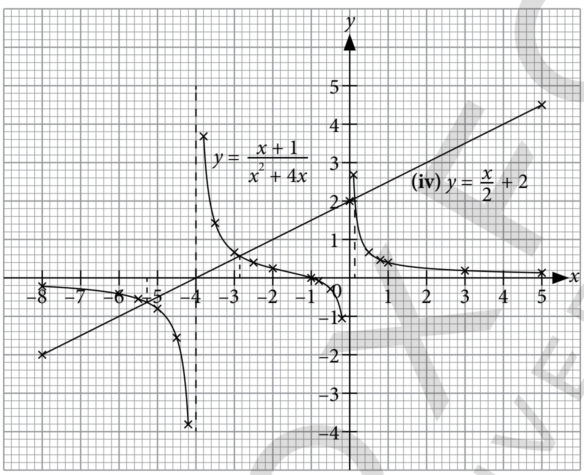
x	−8	−6	−5.5	−5	−4.5	−4.2	−3.8	−3.5
y	−0.2	**−0.4**	−0.5	**−0.8**	−1.6	**−3.8**	3.7	**1.4**

x	−3	−2.5	−2	−1	−0.8	−0.5	−0.2	0.1
y	0.7	**0.4**	0.3	**0**	−0.1	**−0.3**	−1.1	**2.7**

x	0.5	0.8	1	3	5
y	0.7	**0.5**	0.4	**0.2**	0.1

(ii) Find vertical asymptote(s):
$x^2 + 4x = 0$
$x(x + 4) = 0$
$x = 0$ or $x + 4 = 0$
$x = -4$
$\therefore$ the vertical asymptotes of the function are $\boldsymbol{x = 0}$ and $\boldsymbol{x = -4}$.
Find horizontal asymptote:
The degree of $x + 1$ is 1.
The degree of $x^2 + 4x$ is 2.
$\therefore$ the horizontal asymptote of the function is $\boldsymbol{y = 0}$.

(iii)



(iv)
$$\frac{2x+2}{x} = (x+4)^2$$
$$\frac{2(x+1)}{x} = (x+4)^2$$
$$\frac{x+1}{x(x+4)} = \frac{x+4}{2}$$
$$\frac{x+1}{x^2+4x} = \frac{x}{2} + 2$$

x	−8	0	5
$\boldsymbol{y = \frac{x}{2} + 2}$	−2	2	4.5

From the graph, the line $y = \frac{x}{2} + 2$ and the curve $y = \frac{x+1}{x^2+4x}$ intersect at(−5.3, −0.6), (−2.9, 0.6) and (0.1, 2).
$\therefore x = \mathbf{-5.3}$, $x = \mathbf{-2.9}$ or $x = \mathbf{0.1}$

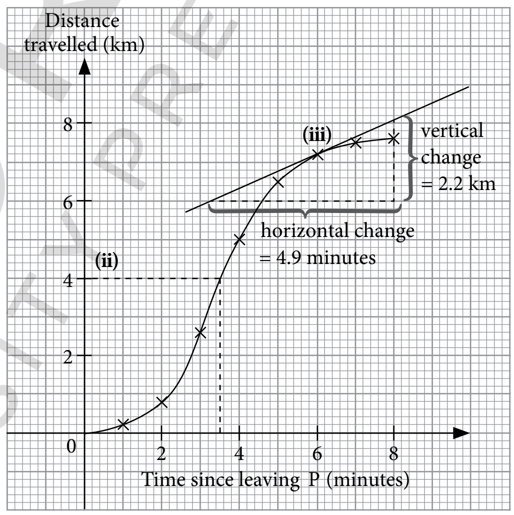
6.7 Applications of graphs in real-world contexts

Class Discussion (Linear distance-time graphs)

1. From 0900 to 0930 hours, since the graph is a horizontal line, gradient = **0**. This gradient represents the speed of the cyclist, which is 0 km/h. The cyclist is resting from 0900 to 0930 hours.
2. From 0930 to 1030 hours, gradient = $\frac{50-20}{1}$ = **30 km/h**. This means that the cyclist travels at a constant speed of 30 km/h in the direction away from home.
3. From 1030 to 1200 hours, gradient = $-\frac{50}{1.5} = \mathbf{-33\frac{1}{3}}$ **km/h**. The negative gradient indicates that the cyclist is travelling in the opposite direction.
   This means that the cyclist travels at a constant speed of $33\frac{1}{3}$ km/h in the direction towards home.

Practise Now 14

(i)



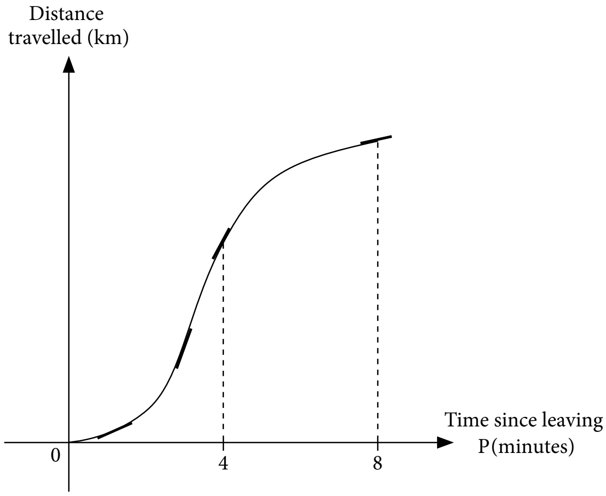
(ii) From the graph, the train takes approximately **3.5 minutes** to travel the first 4 km.

(iii) From the graph,

$$\text{Gradient of tangent at 6 min} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{2.2 \text{ km}}{4.9 \text{ minutes}} = \frac{2.2 \text{ km}}{\frac{4.9}{60} \text{ h}} = 27 \text{ km/h (to the nearest integer)}$$

$\therefore$ the speed of the train 6 minutes after it has left station P is approximately **27 km/h**.

(iv) During the first 4 minutes, the speed of the train increases as the gradient of the curve increases.
During the last 4 minutes, the speed of the train decreases as the gradient of the curve decreases.

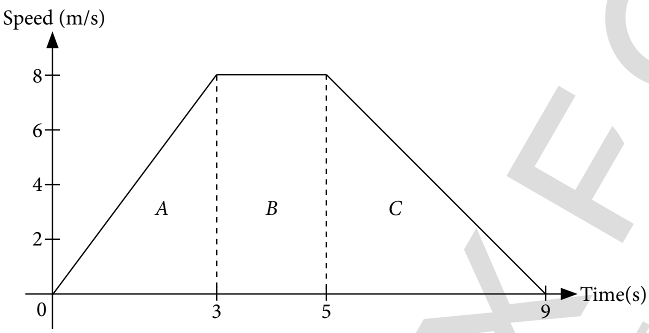


Practise Now 15

(i) Acceleration $= \dfrac{8 \text{ m/s}}{3 \text{ s}}$

$= \mathbf{2\frac{2}{3}\ m/s^2}$ or $\mathbf{2.67\ m/s^2}$ (to 3 s.f.)

(ii)



Total distance = area under graph

$= \text{area of } (A + B + C)$

$= \left(\frac{1}{2} \times 3 \times 8\right) + (2 \times 8) + \left(\frac{1}{2} \times 4 \times 8\right)$

$= 12 + 16 + 16$

$= 44$ m

Alternatively,

Total distance = area under graph

$= \dfrac{(9 + 2)(8)}{2}$

$= 44$ m

Average speed $= \dfrac{\text{total distance}}{\text{total time}}$

$= \dfrac{44}{9}$

$= \mathbf{4.89\ m/s}$ (to 3 s.f.)

(iii) Deceleration in the last 3 seconds

= deceleration in the last 4 seconds

$= \dfrac{8 - 0}{9 - 5}$

$= \mathbf{2\ m/s^2}$

Practise Now 16

(i) When $t = 2$,

$v = 2(2)^2 - 9(2) + 12$

$= 2$

$\therefore \boldsymbol{a = 2}$

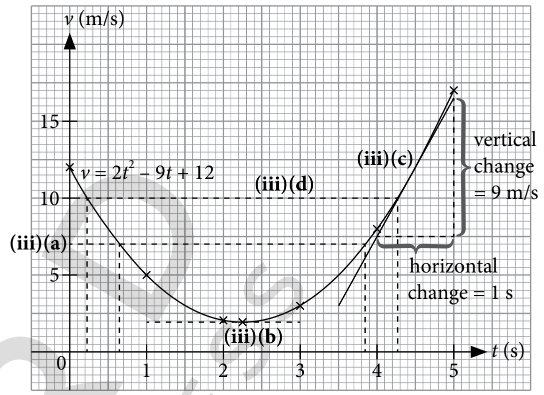
When $t = 5$,

$v = 2(5)^2 - 9(5) + 12$

$= 17$

$\therefore \boldsymbol{b = 17}$

(ii)



(iii) **(a)** From the graph, when $v = 7$, $t = \mathbf{0.65}$ or $t = \mathbf{3.85}$.

(b) The acceleration is zero when the gradient of the curve is zero.
From the graph, the acceleration is zero at $t = 2.25$.
$\therefore$ the acceleration is zero **2.25 seconds** after leaving P.

(c) Gradient of tangent at $t = 4.5$

$= \dfrac{\text{vertical change}}{\text{horizontal change}}$

$= \dfrac{9}{1}$

$= 9$

$\therefore$ the acceleration of the particle at $t = 4.5$ is approximately $\mathbf{9\ m/s^2}$.

(d) From the graph, when $v < 10$, $\mathbf{0.25 < \boldsymbol{t} < 4.25}$.

Practise Now 17

(i) From the graph, the resting heart rate is **60 beats/minute**.

(ii) To find the rate of increase in her heart rate during the first brisk walk, we need to calculate the gradient of the line from the 10th minute to the 20th minute.

Gradient $= \dfrac{120 - 60}{20 - 10}$

$= 6 \text{ beats/minute}^2$

$\therefore$ the rate of increase in her heart rate during the first brisk walk is $\mathbf{6\ beats/minute^2}$.

(iii) To find the rate of decrease in her heart rate as she slows down in the last 20 minutes, we need to calculate the gradient of the line from the 40th minute to the 60th minute.

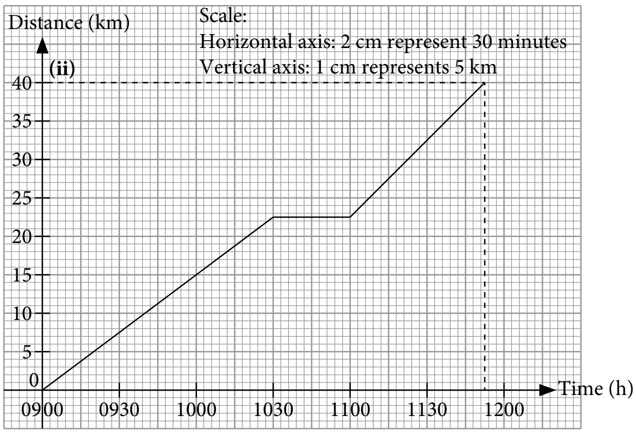
Gradient $= \dfrac{140 - 120}{60 - 40}$

$= 1 \text{ beat/minute}^2$

$\therefore$ the rate of decrease in her heart rate as she slows down in the last 20 minutes is $\mathbf{1\ beat/minute^2}$.

Exercise 6C

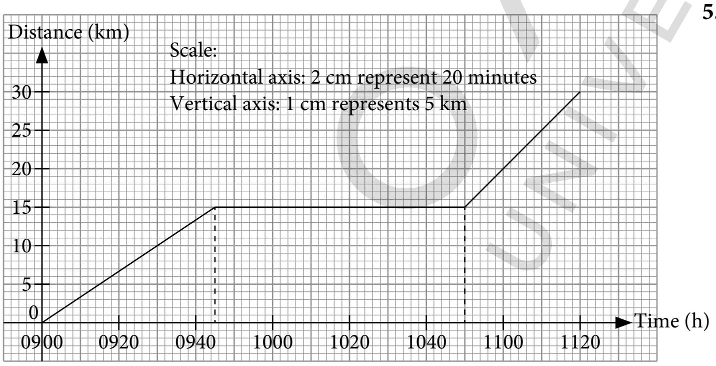
1. **(i)** Distance travelled between 9 a.m. and 10.30 a.m. $= 15 \times 1.5$
$= 22.5$ km



(ii) Distance travelled in the second part of the journey
$= 40 - 22.5$
$= 17.5$ km
Time taken for the second part of the journey
$= \frac{53}{60}$ h
Constant speed of the cyclist in the second part of the journey
$= \dfrac{17.5}{\frac{53}{60}}$
$=$ **19.8 km/h** (to 3 s.f.)

2. **(i)** Distance travelled between 0900 and 0945 $= 20 \times \frac{45}{60}$
$=$ **15 km**

(ii) Distance travelled in the second part of the journey
$= 30 - 15$
$= 15$ km
Time taken for the second part of the journey $= \frac{15}{30}$
$= 0.5$ h
Time at which he begins the second part of the journey
$= 1050$



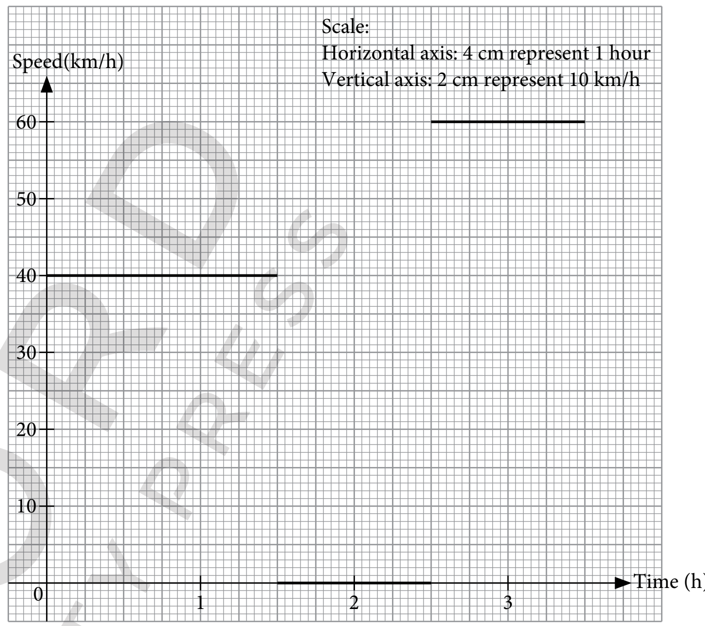
(iii) From the graph, the duration of his rest is **65 minutes**.

3. **(i)** From the graph, the car is not moving between $t = 1.5$ and $t = 2.5$.
$\therefore$ the duration during which the car is not moving is **1 hour**.

(ii) Average speed of car in first 2 hours $= \frac{\text{total distance}}{\text{total time}}$
$= \frac{60}{2}$
$=$ **30 km/h**

(iii) Average speed of car for the whole journey
$= \frac{\text{total distance}}{\text{total time}}$
$= \frac{120}{3.5}$
$=$ **34.3 km/h** (to 3 s.f.)

(iv) Speed of car between $t = 0$ and $t = 1.5 = \frac{60}{1.5}$
$= 40$ km/h
Speed of car between $t = 2.5$ and $t = 3.5 = \frac{120 - 60}{1}$
$= 60$ km/h



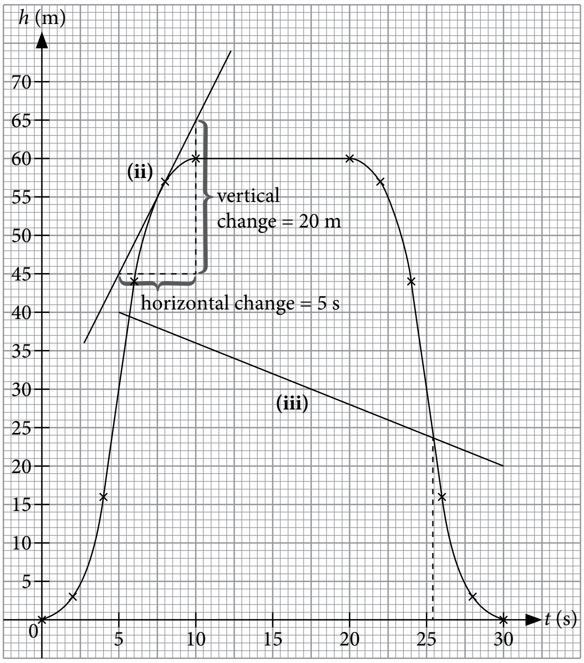
4. **(i)** Acceleration $= \frac{10}{2}$
$=$ **5 m/s²**

(ii) Total distance = area under graph
$= \frac{(6 + 3)(10)}{2}$
$=$ **45 m**

5. **(i)** The gradient of OA represents the acceleration of the car between $t = 0$ and $t = 30$.

(ii) Acceleration between $t = 0$ and $t = 30 = \frac{14 \text{ m/s}}{30 \text{ s}}$
$= \frac{7}{15}$ m/s²
Let the speed of the car when $t = 15$ be x m/s.
Acceleration between $t = 0$ and $t = 15$ = Acceleration between $t = 0$ and $t = 30$
$\frac{x}{15} = \frac{7}{15}$
$x = 7$
$\therefore$ the speed of the car when $t = 15$ is **7 m/s**.

6. **(i)**



(ii) From the graph, when $t = 8$,

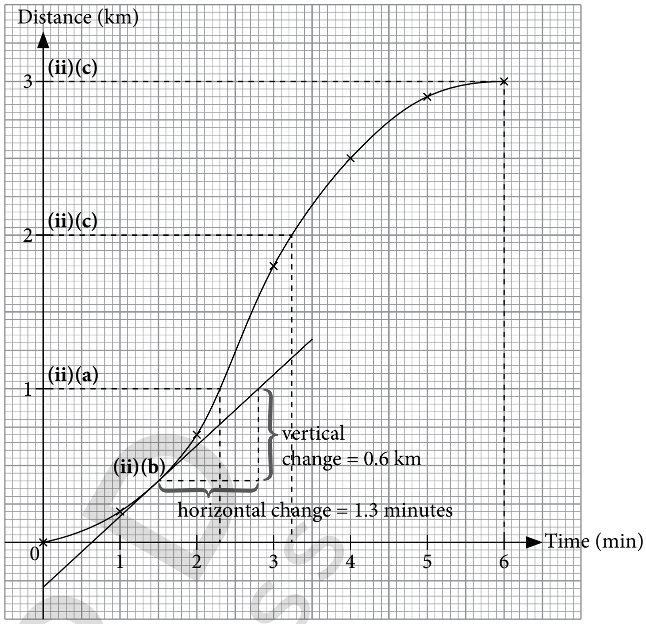
Gradient $= \dfrac{20 \text{ m}}{5 \text{ s}}$

$= \mathbf{4 \text{ m/s}}$

The gradient of the tangent at $t = 8$ represents the instantaneous speed at that particular point.

(iii) After 25 s, the worker will have descended $0.8 \times 25 = 20$ m. Hence at $t = 30$, the worker is at the 20-metre level. The distance-time graph of the worker is drawn. From the graph, the worker and the lift are at the same height at $t =$ **25.5** during the downward journey of the lift.

7. **(i)**



(ii) **(a)** From the graph,

The approximate time taken to travel the first 1 km

= **2.3 minutes**

(b) From the graph, when $t = 1\frac{1}{2}$,

Gradient $= \dfrac{0.6 \text{ km}}{1.3 \text{ min}}$

$= 0.462$ km/min (to 3 s.f.)

The instantaneous speed of the vehicle at $t = 1\frac{1}{2}$ is **0.462 km/min**.

(c) From the graph,

Time taken to travel the last 1 km $= 6 - 3.25$

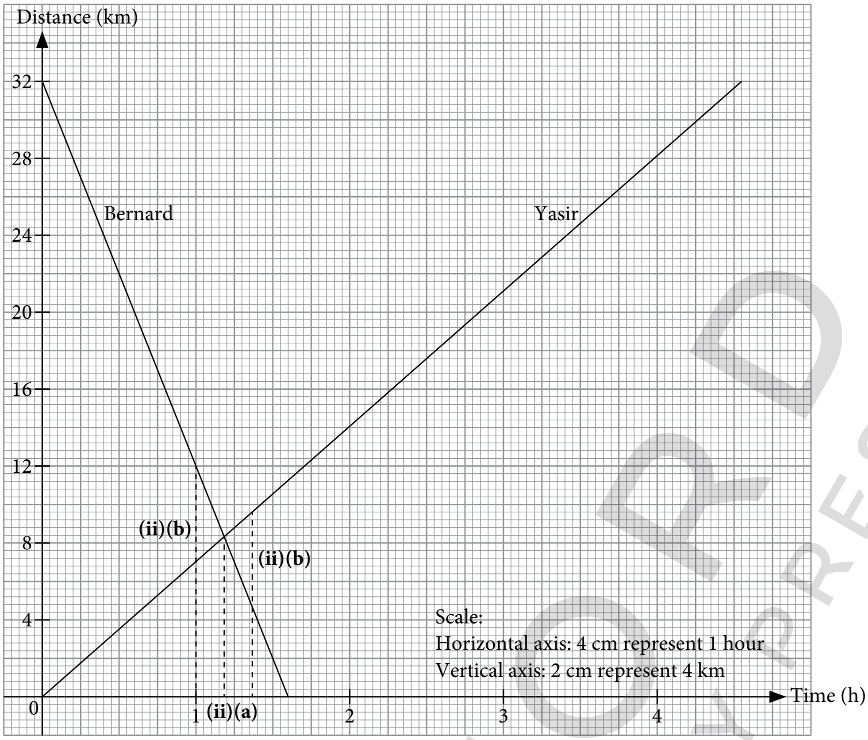
= **2.75 minutes**

8. **(i)** Time taken for Yasir to travel 32 km $= \frac{32}{7}$

$$= 4\frac{4}{7} \text{ h}$$

Time taken for Bernard to travel 32 km $= \frac{32}{20}$

$$= 1.6 \text{ h}$$



(ii) **(a)** From the graph,
Time taken for Bernard and Yasir to pass each other = **1 hour 11 minutes**

(b) From the graph,
Times when Bernard and Yasir are 5 km apart = **1 hour** and **1 hour 22 minutes**

9. **(i)** Distance travelled by Cheryl for the first half hour $= 18 \times \frac{1}{2}$

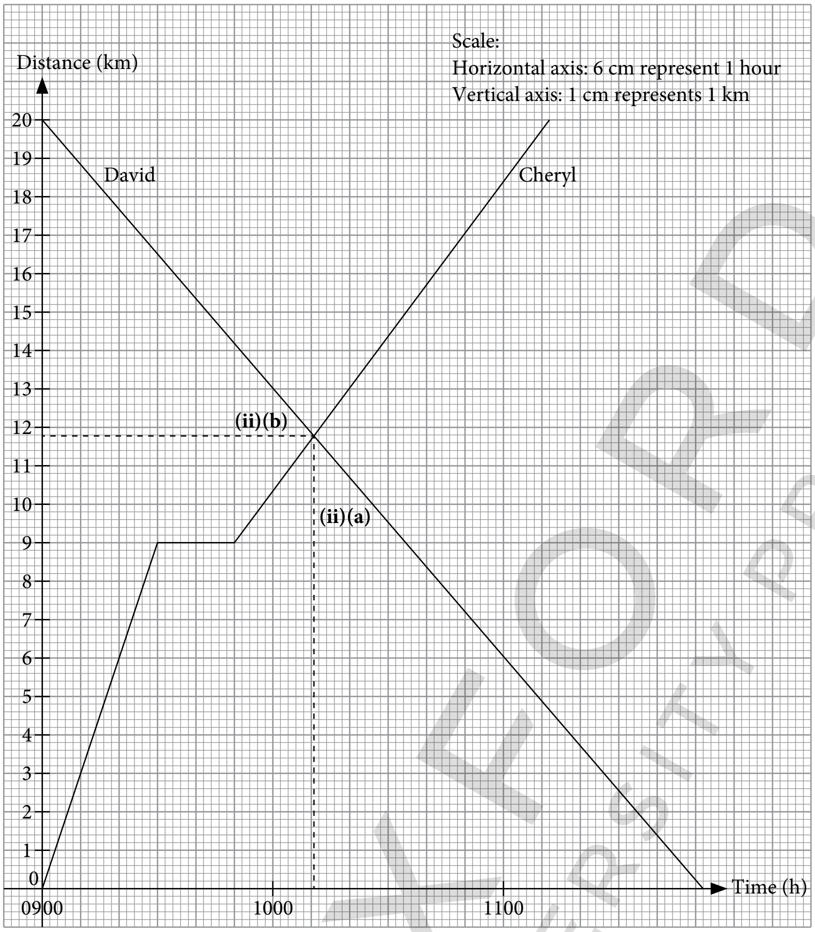
$= 9$ km

Time taken for Cheryl to travel the second part of the journey $= \frac{11}{8}$

$= 1.375$ h

Time taken for David to travel 20 km $= \frac{20}{7}$

$= 2.8571$ h (to 5 s.f.)



(ii) **(a)** From the graph, the time at which Cheryl and David meet is **1011 hours**.

(b) From the graph, the distance from David's home at the meeting point is **8.2 km**.

10. (i) Since the object slows down at a rate of 12 m/s², Deceleration = 12 m/s²

$$\frac{36-0}{t-6} = 12$$
$$36 = 12(t-6)$$
$$= 12t - 72$$
$$12t = 108$$
$$t = \mathbf{9}$$

(ii) Total distance travelled = area under speed-time graph

$$= \frac{1}{2} \times (6+9) \times 36$$
$$= \mathbf{270\ m}$$

11. (i) Acceleration during the first 20 s $= \dfrac{30-0}{20}$

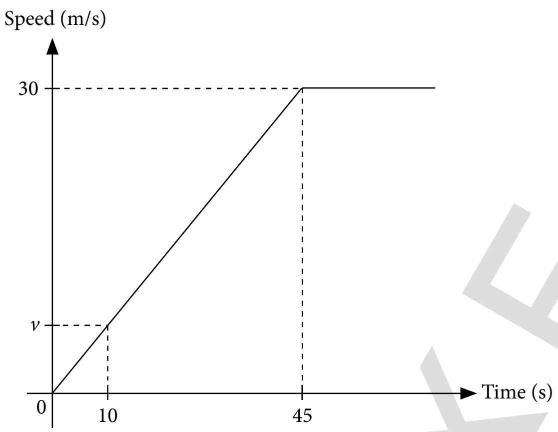
$$= \mathbf{1.5\ m/s^2}$$

(ii) Since the train decelerates at a rate of 0.75 m/s², Deceleration = 0.75 m/s²

$$\frac{30-0}{t-60} = 0.75$$
$$30 = 0.75(t-60)$$
$$= 0.75t - 45$$
$$0.75t = 75$$
$$t = 100$$

∴ the time taken for the whole journey is **100 seconds**.

12.



Since the gradient of a speed-time graph gives the acceleration,

$$\frac{v}{10} = \frac{30}{45}$$
$$v = \frac{2}{3} \times 10$$
$$= 6\frac{2}{3} \text{ or } 6.67 \text{ (to 3 s.f.)}$$

Hence the speed after 10 seconds is $\mathbf{6\frac{2}{3}}$ **m/s or 6.67 m/s**.

13. (i) When $t = 3$,

$v = 3^2 - 7(3) + 16$
$= 4$

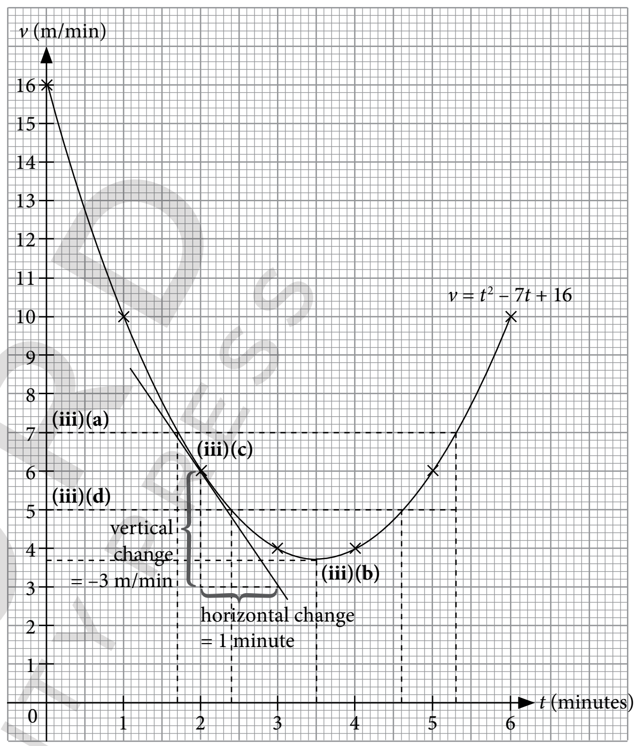
∴ $\boldsymbol{a = 4}$

When $t = 6$,

$v = 6^2 - 7(6) + 16$
$= 10$

∴ $\boldsymbol{b = 10}$

(ii)



(iii) (a) From the graph,
When $v = 7$, $t =$ **1.7** or $t =$ **5.3**.

(b) From the graph,
The time at which the speed is a minimum is **3.5 minutes**.

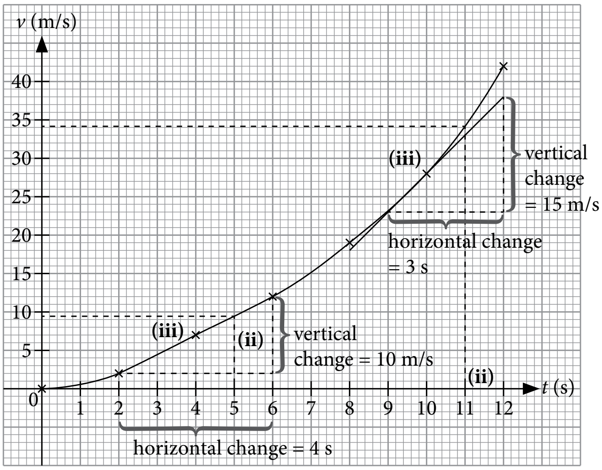
(c) A tangent is drawn to the curve at $t = 2$.
From the graph,

$$\text{Gradient} = \frac{-3 \text{ m/min}}{1 \text{ min}}$$
$$= \mathbf{-3\ m/min^2}$$

The value represents the deceleration of the particle at $t = 2$.

(d) From the graph, the time interval when the speed is not more than 5 m/min is $\mathbf{2.4 \leqslant t \leqslant 4.6}$.

14. **(i)**



(ii) From the graph, when $t = 5$, speed = **9.5 m/s**.
From the graph, when $t = 11$, speed = **34 m/s**.

(iii) A tangent is drawn to the curve at $t = 4$.
From the graph,

$$\text{Gradient} = \frac{10\text{ m/s}}{4\text{ s}} = 2.5\text{ m/s}^2$$

$\therefore$ acceleration at $t = 4$ = **2.5 m/s²**

A tangent is drawn to the curve at $t = 10$.
From the graph,

$$\text{Gradient} = \frac{15\text{ m/s}}{3\text{ s}} = 5\text{ m/s}^2$$

$\therefore$ acceleration at $t = 10$ = **5 m/s²**

15. **(i)** When $t = 2$,

$v = 3(2)^2 - 17(2) + 30$
$= 8$

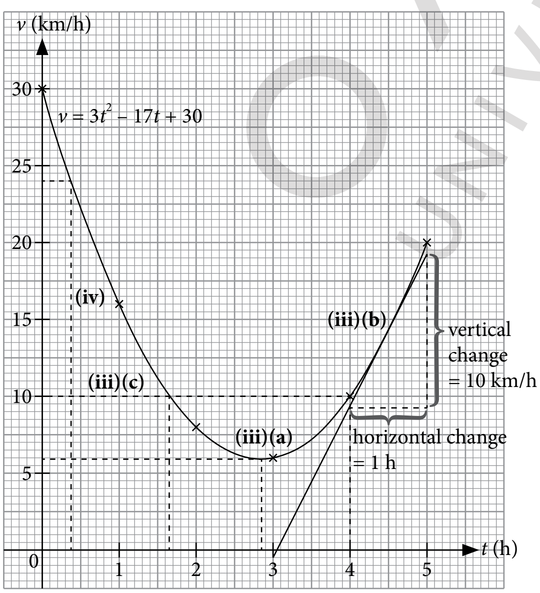
$\therefore$ $\boldsymbol{h = 8}$

When $t = 4$,

$v = 3(4)^2 - 17(4) + 30$
$= 10$

$\therefore$ $\boldsymbol{k = 10}$

(ii)



(iii) **(a)** From the graph, the time at which the speed is a minimum is **2.85 hours**.

(b) A tangent is drawn to the curve at $t = 4.5$.
From the graph,

$$\text{Gradient} = \frac{10\text{ km/h}}{1\text{ h}} = 10\text{ km/h}^2$$

This value represents the acceleration of the object at $t = 4.5$.

(c) From the graph, the time interval that the speed does not exceed 10 km/h is $\boldsymbol{1.65 \leqslant t \leqslant 4}$.

(iv) From the graph, the value of t at which both objects have the same speed is **0.4**.

16. **(i)** From the graph, the oven was switched off at 1300 hours and left to cool to 25 °C.
$\therefore$ the room temperature was **25 °C**.

(ii) From the graph,

$$\text{Gradient} = \frac{(220 - 100)\text{ °C}}{30\text{ min}} = 4\text{ °C/minute}$$

$\therefore$ the rate at which the oven first heated up was **4 °C/minute**.

(iii) From the graph, the second batch of pastries was baked at 150 °C.
$\therefore$ $\boldsymbol{k = 150}$

(iv) From the graph, the second batch of pastries was only placed in the oven at 1430 hours and the baking process stopped at 1630 hours.
$\therefore$ baking duration = **2 hours**

Chapter 7 Volume, Surface Area and Symmetry of Pyramids, Cones and Spheres

TEACHING NOTES

Suggested Approach

In Book 2, students have learnt to find the volume and surface area of cubes, cuboids, prisms and cylinders. Here, they will learn to determine the volume and surface area of pyramids, cones, spheres, as well as composite solids. By the end of this chapter, not only should students be familiar with the various formulae in calculating the volume and surface area of such solids, but they should also be able to solve related problems in real-world contexts.

For some problems, students are expected to recall and apply Pythagoras' Theorem. Thus, teachers may wish to revise the basics of Pythagoras' Theorem with students before they attempt such problems. As a side note, when the value of π is not stated in the question, students are expected to use the value on the calculator.

Section 7.1: Volume, surface area and symmetry of pyramids

As an introduction, teachers can show students some real-life examples of pyramids and question students on the properties of pyramids (see Class Discussion: What are pyramids?)

Teachers should go through the parts of a pyramid. After which, students should observe and recognise the various types of pyramids. Teachers may wish to point out that the pyramids in the scope of study are right pyramids, where the apex is vertically above the centre of the base and the base is a regular polygon.

Students should also be given the opportunity to compare the volumes of a prism and a pyramid whose bases are identical, so as to lead them to derive and appreciate the formula for the volume of a pyramid (see Investigation: Volume of pyramid). They should also find out more about the symmetry in pyramids with the use of a plane to cut each pyramid (see Investigation: Symmetry in pyramids).

Some students might have difficulties finding the surface areas of different pyramids. As such, teachers are recommended to guide students to find the surface areas with the use of nets.

Section 7.2: Volume, surface area and symmetry of cones

Similar to pyramids, teachers can start off with an activity to introduce cones (see Class Discussion: What are cones?).

To improve and enhance understanding, students should learn and explain the features of a cone and state the differences between a cone and a pyramid (see Investigation: Comparison between a cone and a pyramid).

Proceeding on, students should realise that the volume and total surface area of a cone is analogous to the volume and total surface area of a pyramid. The curved surface area of a cone is one unique calculation that must be noted.

Students might have difficulties understanding why the curved surface of a cone opens to form a sector (see Investigation: Curved surface area of cone). Teachers might wish to use a paper cut-out of a sector to form a cone to help students observe the relationship between the curved surface of a cone and a sector.

As an extension from the previous section, students should also explore the symmetry in cones using a plane (see Investigation: Symmetry in cones).

Section 7.3: Volume and surface area of spheres

Besides learning about the volume and surface area of a sphere, students should also be exposed to the volume and surface area of a hemisphere. Students should be given the opportunity to explore how the volume and surface area of a sphere can be obtained (see Investigation: Volume of sphere and Investigation: Surface area of sphere), as well as think about how the surface area of a hemisphere can be found (see Thinking Time on page 238). This will minimise the formulae students need to remember.

Section 7.4: Volume and surface area of composite solids

In this section, students are required to find the volume and the surface area of various solids made up of solids which they have learnt so far. Besides the ones covered in this chapter, other solids from Book 2, such as cubes, cuboids, prisms and cylinders are also included. As such, teachers may wish to do a revision of all solids before visiting this section.

One common mistake that students might make in calculating the total surface area is to include overlapping faces. Thus, teachers may guide students to make the calculations in an organised manner, such as listing and calculating the area of each face, one at a time.

Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Practise Now 8).

7.1 Volume, surface area and symmetry of pyramids

Class Discussion (What are pyramids?)

1. The pyramids are made up of one base and triangular faces joined to the sides of the base. The triangles are joined at a single point on the other end, opposite the base.

2. **(a)** The slanted faces of the Great Pyramid are isosceles triangles and the slanted faces of the tetrahedral dice are equilateral triangles.

(b) The base of the Great Pyramid is a rectangle (or a square) and the base of the tetrahedral dice is a triangle.

3. The cross sections of a pyramid are **not uniform** throughout the pyramid. They get smaller from the base upwards.

Thinking Time (Page 212)

1. The slant edge is the hypotenuse of a right-angled triangle, whose other sides are the height of the pyramid and half of the diagonal of the base.

The slant height is the hypotenuse of another right-angled triangle, whose other sides are the height of the pyramid and half of the side of the base.

The slant faces of regular pyramids are congruent, isosceles triangles.

2. **Yes**, they are pyramids. They have the features of a pyramid, i.e. slanted triangular faces that are joined to a polygonal base meet at an apex opposite the base.

Journal Writing (Page 212)

Prisms have two polygonal bases that are congruent and parallel to each other while pyramids have only one polygonal base with an apex vertically above it. The sides of a prism are rectangles while the sides of a pyramid are triangles joined at the apex. The cross section of a prism is uniform while the cross section of a pyramid is not.

Three real-life examples of pyramids are **tents, the roofs of houses and metronomes**.

Investigation (Volume of pyramid)

The prism is completely filled after pouring the sand from the pyramid three times.

Volume of pyramid = $\mathbf{\frac{1}{3}}$ × volume of corresponding prism

Practise Now 1

1. Volume of pyramid $= \frac{1}{3} \times 36 \times 7$

$= \mathbf{84\ cm^3}$

2. Volume of the Great Pyramid $= \frac{1}{3} \times 229^2 \times 146$

$= \mathbf{2\ 550\ 000\ m^3}$ (to 3 s.f.)

Practise Now 2

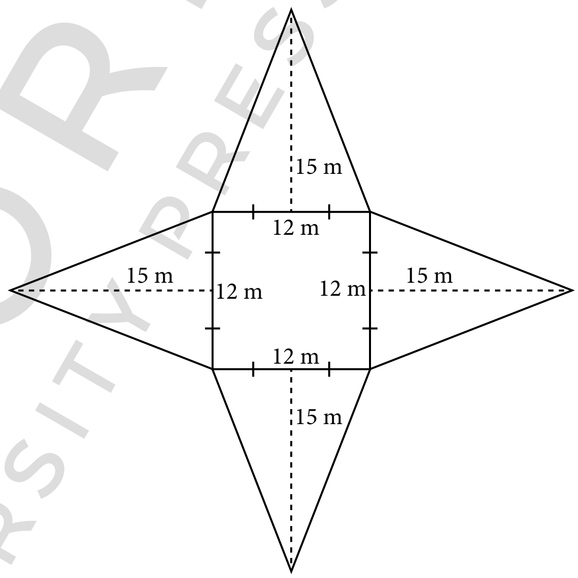
Volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height

$75 = \frac{1}{3} \times 5^2 \times \text{height}$

$75 = \frac{25}{3} \times \text{height}$

$\therefore$ height = **9 m**

Practise Now 3



Area of each triangular face $= \frac{1}{2} \times 12 \times 15$

$= 90\ m^2$

Area of square base $= 12^2$

$= 144\ m^2$

$\therefore$ total surface area $= 4 \times 90 + 144$

$= \mathbf{504\ m^2}$

Practise Now 4

(i) Area of each triangular face

$= \frac{161 - 7^2}{4}$

$= \frac{112}{4}$

$= 28\ cm^2$

Area of $\triangle VQR = \frac{1}{2} \times VB \times 7$

$= 28$

$VB = 8$

$\therefore$ the slant height of the pyramid is **8 cm**.

(ii) Let the point where the vertical from V meets the base be A such that VA is perpendicular to the base.

$AB = \frac{1}{2} \times 7$

$= 3.5$ cm

In $\triangle VAB$, $\angle A = 90°$.

Using Pythagoras' Theorem,

$VB^2 = VA^2 + AB^2$

$8^2 = VA^2 + 3.5^2$

$VA^2 = 8^2 - 3.5^2$

$= 51.75$

$VA = \sqrt{51.75}$ (since $VA > 0$)

$\therefore$ volume of pyramid $= \frac{1}{3} \times 7^2 \times \sqrt{51.75}$

$=$ **117 cm³** (to 3 s.f.)

Investigation (Symmetry in pyramids)

1. **(a)** **4**

(b) **4**

2. **(a)** **6**

(b) **5**

3.

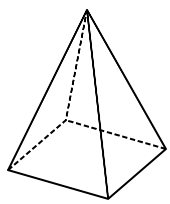
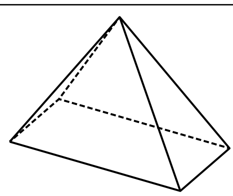
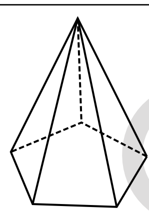
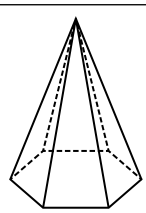
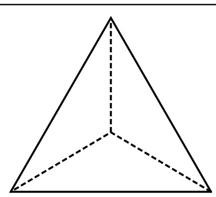
Name	Figure	Number of planes of symmetry	Order of rotational symmetry
Square pyramid		**4**	**4**
Rectangular pyramid		**2**	**2**
Regular pentagonal pyramid		**5**	**5**
Regular hexagonal pyramid		**6**	**6**
Regular tetrahedron		**6**	**12**

Table 7.2

Exercise 7A

1. Volume of pyramid $= \frac{1}{3} \times 15 \times 4$

$=$ **20 cm³**

2. Volume of pyramid $= \frac{1}{3} \times 23 \times 6$

$=$ **46 cm³**

3. Base area of pyramid $= \frac{1}{2} \times 7 \times 4$

$= 14$ m²

Volume of pyramid $= \frac{1}{3} \times 14 \times 5$

$=$ $\mathbf{23\frac{1}{3}}$ **m³** or **23.3 m³** (to 3 s.f.)

4. Base area of pyramid $= \frac{1}{2} \times 5 \times 8$

$= 20$ cm²

Volume of pyramid $= \frac{1}{3} \times 20 \times \text{height}$

$50 = \frac{20}{3} \times \text{height}$

$\therefore$ height $=$ **7.5 cm**

5. Volume of pyramid $= \frac{1}{3} \times \text{base area} \times 12$

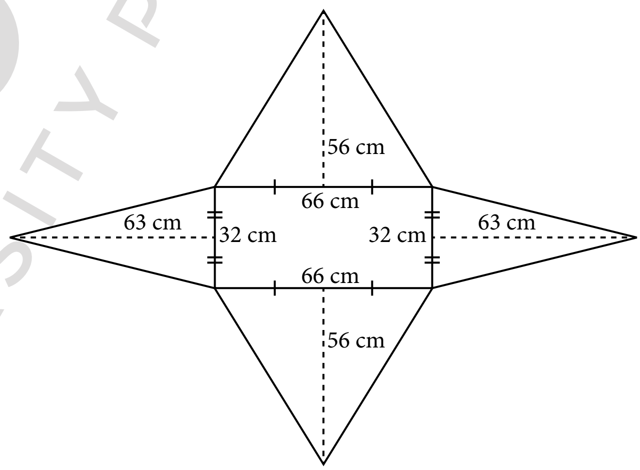
$100 = 4 \times \text{base area}$

$\therefore$ base area $= 25$ m²

Hence, length of square base $= \sqrt{25}$

$=$ **5 m**

6.



Total surface area $= 2 \times \left(\frac{1}{2} \times 66 \times 56\right) + 2 \times \left(\frac{1}{2} \times 32 \times 63\right) + (66 \times 32)$

$= 3696 + 2016 + 2112$

$=$ **7824 cm²**

7. **(a)** **4**

(b) **6**

8. **(a)** **(i)** **6**

(ii) **4**

(b) **(i)** **4**

(ii) **1**

9. Volume of one paperweight $= \frac{1}{3} \times 6^2 \times 7$

$= 84$ cm³

Mass of one paperweight = density × volume

$= 3.1 \times 84$

$= 260.4$ g

$\therefore$ mass of four paperweights $= 4 \times 260.4$

$=$ **1041.6 g**

10. (i) Let the point where the vertical from V meets the rectangular base be P such that VP is perpendicular to the base.

$PQ = \frac{1}{2} \times 15$

$= 7.5$ cm

In $\triangle VPQ$, $\angle P = 90°$.

Using Pythagoras' Theorem,

$VQ^2 = PV^2 + PQ^2$

$16^2 = PV^2 + 7.5^2$

$PV^2 = 16^2 - 7.5^2$

$= 199.75$

$PV = \sqrt{199.75}$ (since $PV > 0$)

$= 14.1$ (to 3 s.f.)

$\therefore$ the height of the pyramid is **14.1 cm**.

(ii) Volume of pyramid $= \frac{1}{3} \times (15 \times 9) \times \sqrt{199.75}$

$=$ **636 cm³** (to 3 s.f.)

11. Volume of pyramid $= \dfrac{\text{mass}}{\text{density}}$

$\frac{1}{3} \times 30 \times \text{height} = \frac{500}{6}$

$= \frac{250}{3}$

$\therefore$ height $=$ **$8\frac{1}{3}$ cm** or **8.33 cm** (to 3 s.f.)

12. Volume of pyramid $= \frac{1}{3} \times \text{base area} \times \text{height} = 100$

base area × height = 300

Let the height of the pyramid be 5 cm.

Then base area $= \frac{300}{5}$

$= 60$ cm²

$\therefore$ a possible set of dimensions is **height = 5 cm** and a rectangular base with **length = 10 cm** and **breadth = 6 cm**.

13. (i) Volume of pyramid $= \frac{1}{3} \times (10 \times 8) \times \text{height} = 180$

$\frac{80}{3} \times \text{height} = 180$

$\therefore$ height $=$ **6.75 cm**

(ii) Let the slant height from V to PQ be l_1 cm and the slant height from V to QR be l_2 cm.

Using Pythagoras' Theorem,

$l_1 = \sqrt{6.75^2 + \left(\frac{8}{2}\right)^2}$ (since $l_1 > 0$)

$= 7.8462$ cm (to 5 s.f.)

$l_2 = \sqrt{6.75^2 + \left(\frac{10}{2}\right)^2}$ (since $l_2 > 0$)

$= 8.4001$ cm (to 5 s.f.)

Total surface area

$= 2 \times \left(\frac{1}{2} \times 7.8462 \times 10\right) + 2 \times \left(\frac{1}{2} \times 8.4001 \times 8\right) + (10 \times 8)$

$=$ **226 cm²** (to 3 s.f.)

14. (i) Volume of pyramid $= \frac{1}{3} \times (16 \times 14) \times \text{height} = 700$

$\frac{224}{3} \times \text{height} = 700$

$\therefore$ height $=$ **9.375 m**

(ii) Let the slant height from the top of the pyramid to the side of 16 m be l_1 m and that to the side of 14 m be l_2 m.

Using Pythagoras' Theorem,

$l_1 = \sqrt{9.375^2 + \left(\frac{14}{2}\right)^2}$ (since $l_1 > 0$)

$= 11.700$ m (to 5 s.f.)

$l_2 = \sqrt{9.375^2 + \left(\frac{16}{2}\right)^2}$ (since $l_2 > 0$)

$= 12.324$ m (to 5 s.f.)

Total surface area

$= 2 \times \left(\frac{1}{2} \times 11.700 \times 16\right) + 2 \times \left(\frac{1}{2} \times 12.324 \times 14\right) + (16 \times 14)$

$=$ **584 m²** (to 3 s.f.)

15. Total surface area = total area of all faces

4 × area of triangular face + base area = 85

4 × area of triangular face + $(4.8)^2$ = 85

4 × area of triangular face = 61.96

area of triangular face = 15.49

Since one side of each triangular face is 4.8 m,

then $\frac{1}{2} \times \text{slant height} \times 4.8 = 15.49$.

$\therefore$ slant height $= 6.4542$ m (to 5 s.f.)

Using Pythagoras' Theorem,

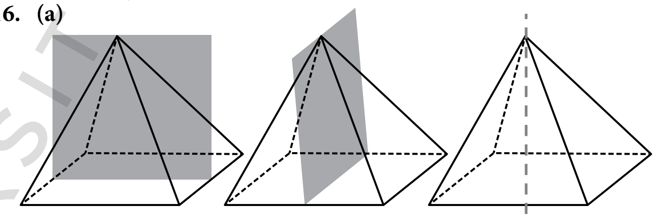
Height of pyramid $= \sqrt{6.4542^2 - \left(\frac{4.8}{2}\right)^2}$

$= 5.9914$ m (to 5 s.f.)

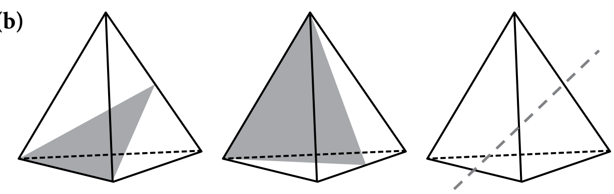
$\therefore$ volume of pyramid $= \frac{1}{3} \times (4.8)^2 \times 5.9914$

$=$ **46.0 m³** (to 3 s.f.)

16. (a)



(b)



17. Volume of pyramid $= \frac{1}{3} \times (15 \times 10) \times 20$

$= 1000$ cm³

Volume of tank $= 30^3$

$= 27\,000$ cm³

Volume of water in tank $= 27\,000 - 1000$

$= 26\,000$ cm³

Depth of remaining water in tank

$= \dfrac{26\,000}{30 \times 30}$

$=$ **$28\frac{8}{9}$ cm** or **28.9 cm** (to 3 s.f.)

18. Let WX be a, XY be b and the height of the pyramid be h.
Using Pythagoras' Theorem,

$$VA = \sqrt{h^2 + \left(\frac{b}{2}\right)^2}$$
$$= \sqrt{h^2 + \frac{b^2}{4}}$$
$$VB = \sqrt{h^2 + \left(\frac{a}{2}\right)^2}$$
$$= \sqrt{h^2 + \frac{a^2}{4}}$$

Since $a > b$, $VB > VA$.
$\therefore$ the slant height ***VA* is shorter than the slant height *VB***.

19. **(i)** Using Pythagoras' Theorem,

$$\text{Slant height} = \sqrt{8^2 - \left(\frac{8}{2}\right)^2}$$
$$= \sqrt{48}$$
$$= \mathbf{6.93\ cm} \text{ (to 3 s.f.)}$$

(ii) Using Pythagoras' Theorem,

$$\text{Height of tetrahedron} = \sqrt{8^2 - \left(\frac{2}{3} \times \sqrt{48}\right)^2}$$
$$= 6.5320 \text{ cm (to 5 s.f.)}$$
$$\therefore \text{volume of tetrahedron} = \frac{1}{3} \times \left(\frac{1}{2} \times 8 \times \sqrt{48}\right) \times 6.5320$$
$$= \mathbf{60.3\ cm^3} \text{ (to 3 s.f.)}$$

20. **(i)** Since the base of both the pyramid and the cuboid is a square which has 4 lines of symmetry, the structure has **4** planes of symmetry.

(ii) Since the base of both the pyramid and the cuboid is a square with 1 axis of rotational symmetry, the structure has **1** axis of rotational symmetry.

(iii) Since the base of both the pyramid and the cuboid is a square with an order of rotational symmetry 4, the order of rotational symmetry for the structure is **4**.

7.2 Volume, surface area and symmetry of cones

Class Discussion (What are cones?)

1. The cones have a circular base with a curved surface and an apex opposite the base.
2. **(a)** A cone has one circular base while a cylinder has two circular bases. A cone has an apex opposite its base while a cylinder does not have an apex. The cross section of a cone is non-uniform while the cross section of a cylinder is uniform.
   (b) The base of a cone is a circle while the base of a pyramid is a polygon. The side of a cone is a curved surface while the sides of a pyramid are triangles. The cross section of a cone is a circle while the cross section of a pyramid is a polygon.
   (c) A cone has one base while a triangular prism has two bases. A cone has an apex opposite its bases while a triangular prism does not have an apex. The cross section of a cone is non-uniform and a circle while the cross section of a triangular prism is uniform and a triangle.

Thinking Time (Page 226)

1. **No**, it is not a right circular cone. The apex is vertically above the edge of the base and not vertically above the centre of the circular base.
2. **No**, a pyramid is not a cone. The base of a pyramid is a polygon, while the base of a cone is a closed curve.
3. Three real-life examples of cones are **traffic cones, megaphones and volcanoes**.

Investigation (Comparison between a cone and a pyramid)

1. The polygon will start to look like a **circle**.
2. The pyramid will start to look like a **cone**.

Thinking Time (Page 228)

Volume of cone = $\frac{1}{3}\pi r^2 h$
Volume of cylinder = $\pi r^2 h$
Since the cone and the cylinder have the same base and the same height, i.e. r and h are equal, then **volume of cone = $\frac{1}{3}$ × volume of corresponding cylinder**.

Practise Now 5

1. $$\text{Volume of cone} = \frac{1}{3} \times \pi \times 8^2 \times 17$$
$$= \mathbf{1140\ cm^3} \text{ (to 3 s.f.)}$$
2. $$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$84\pi = \frac{1}{3} \times \pi \times 6^2 \times h$$
$$h = 7$$
$\therefore$ the height of the cone is **7 m**.

Investigation (Curved surface area of cone)

If the number of sectors is increased indefinitely, then the shape in Fig. 7.21**(b)** will look like a **rectangle** $PQRS$.
Since $PQ + RS$ = circumference of the base circle in Fig. 7.20**(a)**, then the length of the rectangle, $PQ = \boldsymbol{\pi r}$.
Since PS = slant height of the cone in Fig. 7.20**(a)**, then the breadth of the rectangle, $PS = \boldsymbol{l}$.

$$\therefore \text{curved surface area of cone} = \text{area of rectangle}$$
$$= \boldsymbol{PQ \times PS}$$
$$= \boldsymbol{\pi r l}$$

Thinking Time (Page 230)

Total surface area of a solid cone
= curved surface area of cone + base area of cone
= $\boldsymbol{\pi r l + \pi r^2}$

Practise Now 6

1. $$\text{Total surface area of cone} = \pi \times 2 \times 5 + \pi \times 2^2$$
$$= 10\pi + 4\pi$$
$$= 14\pi$$
$$= \mathbf{44.0\ cm^2} \text{ (to 3 s.f.)}$$
2. $$\text{Total surface area of cone} = \pi r l + \pi r^2$$
$$350 = 3.142 \times 8 \times l + 3.142 \times 8^2$$
$$25.136 l = 350 - 201.088$$
$$= 148.912$$
$$l = 5.92 \text{ (to 3 s.f.)}$$
$\therefore$ the slant height of the cone is **5.92 m**.

Practise Now 7

1. Let the slant height of the cone be l m.
Using Pythagoras' Theorem,
$l = \sqrt{8^2 + 15^2}$
$= 17$
$\therefore$ curved surface area of cone $= \pi \times 8 \times 17$
$= 136\pi$
$=$ **427 m²** (to 3 s.f.)

2. Let the height of the cone be h cm.
Using Pythagoras' Theorem,
$h = \sqrt{12^2 - 7^2}$
$= \sqrt{95}$
$\therefore$ volume of cone $= \frac{1}{3} \times \pi \times 7^2 \times \sqrt{95}$
$=$ **500 cm³** (to 3 s.f.)

Investigation (Symmetry in cones)

(a) **Infinite**
(b) **Infinite**

Exercise 7B

1. (a) Volume of cone $= \frac{1}{3} \times \pi \times 6^2 \times 14$
$= 168\pi$
$=$ **528 cm³** (to 3 s.f.)

(b) Volume of cone $= \frac{1}{3} \times 154 \times 5$
$= \frac{770}{3}$
$=$ **$256\frac{2}{3}$ cm³** or **257 cm³** (to 3 s.f.)

(c) Volume of cone $= \frac{1}{3} \times \pi \times \left(\frac{7}{2}\right)^2 \times 14$
$= \frac{343}{6}\pi$
$=$ **180 cm³** (to 3 s.f.)

(d) Let the radius of the base be r mm.
$2\pi r = 132$
$r = \frac{66}{\pi}$
Volume of cone $= \frac{1}{3} \times \pi \times \left(\frac{66}{\pi}\right)^2 \times 28$
$= \frac{40\ 656}{\pi}$
$=$ **12 900 mm³** (to 3 s.f.)

2. Volume of cone $= \frac{1}{3} \times$ base area $\times$ height
$160 = \frac{1}{3} \times 20 \times \text{height}$
$\therefore$ height $=$ **24 m**

3. Volume of cone $= \frac{1}{3} \times$ base area $\times$ height
$320\pi = \frac{1}{3} \times \pi r^2 \times \text{height}$
$r^2 \times \text{height} = 960$
Let $\boldsymbol{r = 5}$.
Corresponding height $= \frac{960}{5^2}$
$=$ **38.4 cm**

4. Volume of cone $= \frac{1}{3}\pi r^2 h$
$132 = \frac{1}{3}\left(\frac{22}{7}\right) r^2 (14)$
$r^2 = 9$
$r = 3$ (since $r > 0$)
$\therefore$ the radius of the circular base is **3 cm**.

5. (a) Total surface area of cone $= \pi \times 4 \times 7 + \pi \times 4^2$
$= 28\pi + 16\pi$
$= 44\pi$
$=$ **138 cm²** (to 3 s.f.)

(b) Total surface area of cone $= \pi \times \left(\frac{28}{2}\right) \times 30 + \pi \times \left(\frac{28}{2}\right)^2$
$= 420\pi + 196\pi$
$= 616\pi$
$=$ **1940 mm²** (to 3 s.f.)

(c) Let the radius of the base be r cm.
$2\pi r = 132$
$r = \frac{66}{\pi}$
Total surface area of cone $= \pi \times \left(\frac{66}{\pi}\right) \times 25 + \pi \times \left(\frac{66}{\pi}\right)^2$
$= 1650 + \frac{4356}{\pi}$
$=$ **3040 cm²** (to 3 s.f.)

6. Curved surface area $= \pi rl$
$84\pi = \pi \times 6 \times l$
$l = 14$
$\therefore$ the slant height of the cone is **14 mm**.

7. Total surface area of cone $= \pi rl + \pi r^2$
$1000 = 3.142 \times r \times l + 3.142 \times r^2$
$r \times l + r^2 = \frac{1000}{3.142}$
Let $r = 15$.
$15 \times l + 15^2 = \frac{1000}{3.142}$
$l = 6.22$ (to 3 s.f.)
$\therefore$ a possible set of dimensions is **radius = 15 cm** and **slant height = 6.22 cm**.

8. Curved surface area $= \pi rl$
$251 = \pi \times r \times 5$
$r = 16.0$ (to 3 s.f.)
$\therefore$ the radius of the circular base is **16.0 m**.

9. **Infinite**

10. (i) **Infinite**
(ii) **1**

11. Volume of conical funnel $= \frac{1}{3} \times \pi \times \left(\frac{23.2}{2}\right)^2 \times 42$
$= 1883.84\pi \text{ cm}^3$
Let the height of the cylindrical tin be h cm.
$\pi\left(\frac{16.2}{2}\right)^2 h = 1883.84\pi$
$65.61h = 1883.84$
$h = 28.7$ (to 3 s.f.)
$\therefore$ the least possible height of the tin is **29 cm**.

12. Volume of conical block of silver $= \frac{1}{3}\times\pi\times12^2\times16$

$= 768\pi \text{ cm}^3$

Volume of one coin $= \pi\times\left(\frac{3}{4}\right)^2\times\frac{1}{6}$

$= \frac{3}{32}\pi \text{ cm}^3$

Number of coins that can be made $= \dfrac{768\pi}{\frac{3}{32}\pi}$

$= \mathbf{8192}$

13. Circumference of base of cone $= 2 \times \pi \times 10$

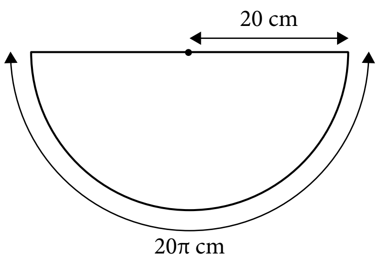
$= 20\pi$ cm

Radius of the sector of the circle forming the net of the cone
= slant height
= 20 cm.

Circumference of circle with radius 20 cm $= 2 \times \pi \times 20$

$= 40\pi$ cm

Since $\frac{20\pi}{40\pi} = \frac{1}{2}$, the net of the cone is a semicircle.



14. **(i)** Circumference of base of cone $= \frac{1}{2}\times\pi\times10$

$= 5\pi$ cm

Diameter of base of cone $= \frac{5\pi}{\pi}$

$= \mathbf{5}$ **cm**

(ii) Slant height of cone = radius of net = 5 cm

Curved surface area of cone $= \pi\times\left(\frac{5}{2}\right)\times5$

$= \mathbf{39.3}$ **cm²** (to 3 s.f.)

15. Using Pythagoras' Theorem,

Slant height of cone $= \sqrt{5^2+12^2}$

$= 13$ cm

Curved surface area of cone $= \pi \times 5 \times 13$

$= \mathbf{204}$ **cm²** (to 3 s.f.)

16. Using Pythagoras' Theorem,

Height of cone $= \sqrt{20^2-8^2}$

$= \sqrt{336}$ cm

Volume of cone $= \frac{1}{3}\times\pi\times8^2\times\sqrt{336}$

$= \mathbf{1230}$ **cm³** (to 3 s.f.)

17. **(i)** Using Pythagoras' Theorem,

Radius of base $= \sqrt{21^2-17^2}$

$= \sqrt{152}$ mm

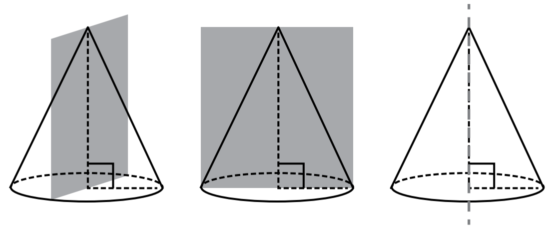
Volume of cone $= \frac{1}{3}\times\pi\times\left(\sqrt{152}\right)^2\times17$

$= \mathbf{2710}$ **mm³** (to 3 s.f.)

(ii) Total surface area $= \pi\times\sqrt{152}\times21+\pi\times\left(\sqrt{152}\right)^2$

$= \mathbf{1290}$ **mm²** (to 3 s.f.)

18.



19. **(i)** Radius of each cone $= \frac{14}{2}$

$= 7$ cm

Total surface area of solid $= 2\pi rl$

$= 2 \times \pi \times 7 \times 15$

$= \mathbf{660}$ **cm²** (to 3 s.f.)

(ii) Using Pythagoras' Theorem,

Height of each cone $= \sqrt{15^2-7^2}$

$= \sqrt{176}$ cm

Volume of solid $= 2\times\frac{1}{3}\pi r^2h$

$= \frac{2}{3}\times\pi\times7^2\times\sqrt{176}$

$= \mathbf{1360}$ **cm³** (to 3 s.f.)

20. Total surface area $= \pi rl + \pi r^2$

$1240 = \pi \times 13.5 \times l + \pi \times 13.5^2$

$l = \frac{1240-182.25\pi}{13.5\pi}$

$= 15.737$ m (to 5 s.f.)

Using Pythagoras' Theorem,

Height of cone $= \sqrt{15.737^2-13.5^2}$

$= 8.0872$ m (to 5 s.f.)

Volume of cone $= \frac{1}{3}\times\pi\times13.5^2\times8.0872$

$= \mathbf{1540}$ **m³**

7.3 Volume and surface area of spheres

Thinking Time (Page 234)

1. A hemisphere is **half of a sphere**. Some real-life examples of hemispheres are **bowls, stadium domes and call bells**.
2. **No**, a rugby ball and a chicken's egg are not spheres. A sphere has a constant radius throughout, but a rugby ball and a chicken's egg do not.

Class Discussion (Is the King's crown made of pure gold?)

Volume of 11.6 kg of pure gold $= \frac{11.6\times1000}{19.3}$

$= 601 \text{ cm}^3$ (to 3 s.f.)

Since the volume of water displaced $= 714 \text{ cm}^3 \neq 601 \text{ cm}^3$, the crown was **not made of pure gold**.

Investigation (Volume of sphere)

Volume of cylinder = $\pi r^2 h$

$= \pi \times r^2 \times 2r$

$= \mathbf{2\pi r^3}$

Volume of sphere = $\frac{2}{3}$ × volume of cylinder

$= \frac{2}{3} \times \mathbf{2\pi r^3}$

$= \mathbf{\frac{4}{3}\pi r^3}$

Practise Now 8

1. Volume of one ball bearing $= \frac{4}{3}\times\pi\times\left(\frac{0.4}{2}\right)^3$

$= \frac{4}{375}\pi \text{ cm}^3$

Mass of 5000 ball bearings $= 5000\times11.3\times\frac{4}{375}\pi$

$=$ **1890 g** (to 3 s.f.)

2. Volume of basketball $= \frac{4}{3}\pi r^3$

$5600 = \frac{4}{3}\pi r^3$

$r^3 = \frac{4200}{\pi}$

$r =$ **11.0 cm** (to 3 s.f.)

Introductory Problem Revisited

Based on the dimensions of the sphere, $h = 2r$.

Volume of cylinder = $\pi r^2 h$

$= \pi r^2(2r)$

$= 2\pi r^3$

Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2(2r)$

$= \frac{2}{3}\pi r^3$

Volume of sphere $= \frac{4}{3}\pi r^3$

Volume of pyramid $= \frac{1}{3}r^2 h$

$= \frac{1}{3}r^2(2r)$

$= \frac{2}{3}r^3$

$\frac{2}{3}r^3 < \frac{2}{3}\pi r^3 < \frac{4}{3}\pi r^3 < 2\pi r^3$

Arranging solids in ascending order of their volumes:

Pyramid, cone, sphere, cylinder

Investigation (Surface area of sphere)

Curved surface area of hemisphere = curved surface area of cylinder

$= 2\pi rh$

$= 2\pi \times r \times r$

$= \mathbf{2\pi r^2}$

$\therefore$ surface area of sphere = 2 × curved surface area of hemisphere

$= 2 \times \mathbf{2\pi r^2}$

$= \mathbf{4\pi r^2}$

4. **4** circles are covered completely with the orange skin.
5. Surface area of the orange = $\mathbf{4\pi r^2}$

Thinking Time (Page 238)

Total surface area of a solid hemisphere

= curved surface area of hemisphere + base area of hemisphere

$= \frac{1}{2}\times 4\pi r^2 + \pi r^2$

$= 2\pi r^2 + \pi r^2$

$= \mathbf{3\pi r^2}$

Practise Now 9

1. Surface area of sphere = $4 \times \pi \times 0.6^2$

$=$ **4.52 m²** (to 3 s.f.)

2. Surface area of sphere $= 4\times\pi\times\left(\frac{25}{2}\right)^2$

$=$ **1960 cm²** (to 3 s.f.)

Practise Now 10

Curved surface area of hemisphere $= \frac{1}{2}\times 4\pi r^2$

$200 = 2\pi r^2$

$r^2 = \frac{200}{2\pi}$

$r = \sqrt{\frac{200}{2\pi}}$ (since $r > 0$)

$=$ **5.64 cm** (to 3 s.f.)

Exercise 7C

1. (a) Volume of sphere $= \frac{4}{3}\times\pi\times 8^3$

$=$ **2140 cm³** (to 3 s.f.)

(b) Volume of sphere $= \frac{4}{3}\times\pi\times 14^3$

$=$ **11 500 mm³** (to 3 s.f.)

(c) Volume of sphere $= \frac{4}{3}\times\pi\times 4^3$

$=$ **268 m³** (to 3 s.f.)

2. (a) Volume of sphere $= \frac{4}{3}\pi r^3$

$1416 = \frac{4}{3}\pi r^3$

$r^3 = \frac{1416}{\frac{4}{3}\pi}$

$r = \sqrt[3]{\frac{1416}{\frac{4}{3}\pi}}$

$=$ **6.97 cm** (to 3 s.f.)

(b) Volume of sphere $= \frac{4}{3}\pi r^3$

$12\,345 = \frac{4}{3}\pi r^3$

$r^3 = \frac{12\,345}{\frac{4}{3}\pi}$

$r = \sqrt[3]{\frac{12\,345}{\frac{4}{3}\pi}}$

$=$ **14.3 mm** (to 3 s.f.)

(c) Volume of sphere $= \frac{4}{3}\pi r^3$

$$780 = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{780}{\frac{4}{3}\pi}$$

$$r = \sqrt[3]{\frac{780}{\frac{4}{3}\pi}}$$

$= \textbf{5.71 m}$ (to 3 s.f.)

(d) Volume of sphere $= \frac{4}{3}\pi r^3$

$$972\pi = \frac{4}{3}\pi r^3$$

$$r^3 = 729$$

$$r = \sqrt[3]{729}$$

$= \textbf{9 cm}$

(e) Volume of sphere $= \frac{4}{3}\pi r^3$

$$498\pi = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{747}{2}$$

$$r = \sqrt[3]{\frac{747}{2}}$$

$= \textbf{7.20 mm}$ (to 3 s.f.)

(f) Volume of sphere $= \frac{4}{3}\pi r^3$

$$15\frac{3}{16}\pi = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{729}{64}$$

$$r = \sqrt[3]{\frac{729}{64}}$$

$= \textbf{2.25 m}$

3. **(a)** Surface area of sphere $= 4 \times \pi \times 12^2$

$= \textbf{1810 cm}^{\mathbf{2}}$ (to 3 s.f.)

(b) Surface area of sphere $= 4 \times \pi \times 9^2$

$= \textbf{1020 mm}^{\mathbf{2}}$ (to 3 s.f.)

(c) Surface area of sphere $= 4 \times \pi \times \left(\frac{6}{2}\right)^2$

$= \textbf{113 m}^{\mathbf{2}}$ (to 3 s.f.)

4. Total surface area of solid hemisphere $= \pi r^2 + \frac{1}{2} \times 4\pi r^2$

$= 3\pi r^2$

$= 3 \times 7 \times 3.142^2$

$= \textbf{462 cm}^{\mathbf{2}}$ (to 3 s.f.)

5. **(a)** Surface area of sphere $= 4\pi r^2$

$$210 = 4\pi r^2$$

$$r^2 = \frac{210}{4\pi}$$

$$r = \sqrt{\frac{210}{4\pi}} \quad (\text{since } r > 0)$$

$= \textbf{4.09 cm}$ (to 3 s.f.)

(b) Surface area of sphere $= 4\pi r^2$

$$7230 = 4\pi r^2$$

$$r^2 = \frac{7230}{4\pi}$$

$$r = \sqrt{\frac{7230}{4\pi}} \quad (\text{since } r > 0)$$

$= \textbf{24.0 mm}$ (to 3 s.f.)

(c) Surface area of sphere $= 4\pi r^2$

$$3163 = 4\pi r^2$$

$$r^2 = \frac{3163}{4\pi}$$

$$r = \sqrt{\frac{3163}{4\pi}} \quad (\text{since } r > 0)$$

$= \textbf{15.9 m}$ (to 3 s.f.)

(d) Surface area of sphere $= 4\pi r^2$

$$64\pi = 4\pi r^2$$

$$r^2 = 16$$

$$r = \sqrt{16} \quad (\text{since } r > 0)$$

$= \textbf{4 cm}$

(e) Surface area of sphere $= 4\pi r^2$

$$911\pi = 4\pi r^2$$

$$r^2 = \frac{911}{4}$$

$$r = \sqrt{\frac{911}{4}} \quad (\text{since } r > 0)$$

$= \textbf{15.1 mm}$ (to 3 s.f.)

(f) Surface area of sphere $= 4\pi r^2$

$$49\pi = 4\pi r^2$$

$$r^2 = \frac{49}{4}$$

$$r = \sqrt{\frac{49}{4}} \quad (\text{since } r > 0)$$

$= \textbf{3.5 m}$

6. Curved surface area of hemisphere $= \frac{1}{2} \times 4\pi r^2$

$$364.5\pi = 2\pi r^2$$

$$r^2 = \frac{729}{4}$$

$$r = \sqrt{\frac{729}{4}} \quad (\text{since } r > 0)$$

$= \textbf{13.5 cm}$

7. Volume of one ball bearing $= \frac{4}{3} \times \pi \times \left(\frac{0.7}{2}\right)^3$

$= \frac{343}{6000}\pi \text{ cm}^3$

Mass of one ball bearing $= \frac{343}{6000}\pi \times 7.85$

$= 1.4098$ g (to 5 s.f.)

Number of ball bearings that can be made $= \frac{1000}{1.4098}$

$= \textbf{709}$ (to the nearest whole number)

8. Volume of aluminium used $= \frac{4}{3} \times \pi \times 30^3 - \frac{4}{3} \times \pi \times 20^3$

$= 79\ 587 \text{ cm}^3$ (to 5 s.f.)

Mass of sphere $= 79\ 587 \times 2.7$

$= 214\ 884.9$ g

$= \textbf{215 kg}$ (to 3 s.f.)

9. Volume of sphere formed $= 54 \times \frac{1}{2} \times \frac{4}{3} \times \pi \times \left(\frac{2}{2}\right)^3$

$= 36\pi \text{ cm}^3$

$$\frac{4}{3}\pi r^3 = 36\pi$$

$$r^3 = 27$$

$$r = 3$$

$\therefore$ the radius of the sphere is **3 cm**.

10. Volume of acid $= \frac{1}{2}\times\frac{4}{3}\times\pi\times\left(\frac{26.4}{2}\right)^3$

$= 1533.312\pi \text{ cm}^3$

Let the depth of acid in the beaker be h cm.

$\pi\left(\frac{16}{2}\right)^2 h = 1533.312\pi$

$64h = 1533.312$

$h = 24.0$ (to 3 s.f.)

$\therefore$ the depth of the acid in the beaker is **24.0 cm**.

11. Volume of water in tin $= \pi\times\left(\frac{18}{2}\right)^2\times 13.2$

$= 1069.2\pi \text{ cm}^3$

Volume of ball bearing $= \frac{4}{3}\times\pi\times\left(\frac{9.3}{2}\right)^3$

$= 134.0595\pi \text{ cm}^3$

New height of water in tin $= \dfrac{1069.2\pi + 134.0595\pi}{\pi\left(\frac{18}{2}\right)^2}$

$=$ **14.86 cm** (to 2 d.p.)

12. Let the radius of the sphere be r m.

$\frac{4}{3}\times\pi\times r^3 = 850$

$r^3 = \frac{1275}{2\pi}$

$r = \sqrt[3]{\frac{1275}{2\pi}}$

$= 5.8764$ (to 5 s.f.)

Surface area of sphere $= 4 \times \pi \times 5.8764^2$

$=$ **434 m²** (to 3 s.f.)

13. Let the radius of the basketball be r cm.

$4 \times \pi \times r^2 = 1810$

$r^2 = \frac{905}{2\pi}$

$r = \sqrt{\frac{905}{2\pi}}$ (since $r > 0$)

$= 12.001$ (to 5 s.f.)

Volume of basketball $= \frac{4}{3}\times\pi\times 12.001^3$

$=$ **7240 cm³** (to 3 s.f.)

14. **(i)** Radius of sphere = radius of base of can = 3.4 cm

Depth of water in can when the sphere was placed inside

$= 2 \times 3.4$

$= 6.8$ cm

Surface area of can in contact with water

$= \pi \times 3.4^2 + 2 \times \pi \times 3.4 \times 6.8$

$= 57.8\pi$

$=$ **182 cm²** (to 3 s.f.)

(ii) Volume of water in can $= \pi\times 3.4^2\times 6.8 - \frac{4}{3}\times\pi\times 3.4^3$

$= \frac{9826}{375}\pi \text{ cm}^3$

Depth of water in can before the sphere was placed inside

$= \dfrac{\frac{9826}{375}\pi}{\pi\times 3.4^2}$

$= \mathbf{2\frac{4}{15}}$ **cm** or **2.27 cm** (to 3 s.f.)

15. For each hemisphere,

Area painted red = curved surface area

$= \frac{1}{2}\times 4\pi r^2$

$= 2\pi r^2$

Area painted yellow = base area

$= \pi r^2$

$\therefore$ amount of red paint : amount of yellow paint

$= 2\pi r^2 : \pi r^2$

$=$ **2 : 1**

7.4 Volume and surface area of composite solids

Practise Now 11

Height of cone $= \frac{3}{4}\times 3r$

$= \frac{9}{4}r$ cm

Volume of cone $= \frac{1}{3}\times\pi\times r^2\times\frac{9}{4}r$

$10 \times 1000 = \frac{3}{4}\pi r^3$

$r^3 = \frac{40\,000}{3\pi}$

$r = \sqrt[3]{\frac{40\,000}{3\pi}}$

$= 16.191$ (to 5 s.f.)

Volume of cylinder $= \pi \times 16.191^2 \times 3(16.191)$

$= 40\,003 \text{ cm}^3$ (to 5 s.f.)

$= 40.0\ l$ (to 3 s.f.)

$\therefore$ amount of water needed to fill the container completely

$= 40 + 10$

$=$ **50 *l***

Practise Now 12

(a) **(i)** Radius of hemisphere $= \frac{30}{2}$

$= 15$ cm

Height of cone $= 50 - 15$

$= 35$ cm

Volume of solid

= volume of cone + volume of hemisphere

$= \frac{1}{3}\times\pi\times 15^2\times 35 + \frac{1}{2}\times\frac{4}{3}\times\pi\times 15^3$

$= 2625\pi + 2250\pi$

$= 4875\pi$

$=$ **15 300 cm³** (to 3 s.f.)

(ii) Using Pythagoras' Theorem,

Slant height of cone $= \sqrt{35^2 + 15^2}$

$= \sqrt{1450}$

Total surface area of solid

= curved surface area of cone + curved surface area of hemisphere

$= \pi\times 15\times\sqrt{1450} + \frac{1}{2}\times 4\times\pi\times 15^2$

$=$ **3210 cm²** (to 3 s.f.)

(b) (i) Let the height of the cylinder be h cm.

$\pi \times 12.5^2 \times h = 4875\pi$

$h = 31.2$

$\therefore$ the height of the cylinder is **31.2 cm**.

(ii) Surface area of cylinder

$= 2 \times \pi \times 12.5 \times 31.2 + 2 \times \pi \times 12.5^2$

$= 780\pi + 312.5\pi$

$=$ **1092.5π cm²**

Exercise 7D

1. Radius of base $= \frac{12}{2}$

$= 6$ m

Total surface area of rocket

= curved surface area of cone + curved surface area of cylinder + base area

$= \pi \times 6 \times 15 + 2 \times \pi \times 6 \times 42 + \pi \times 6^2$

= **1980 m²** (to 3 s.f.)

2. Volume of remaining solid

= volume of cylinder − volume of cone

$= \pi \times 6^2 \times 15 - \frac{1}{3} \times \pi \times 3^2 \times 15$

$= 540\pi - 45\pi$

$= 495\pi$

= **1560 cm³** (to 3 s.f.)

3. (i) Volume of solid

= volume of cylinder + volume of hemisphere

$= \pi \times 7^2 \times 10 + \frac{1}{2} \times \frac{4}{3} \times \pi \times 7^3$

$= 490\pi + \frac{686}{3}\pi$

$= \frac{2156}{3}\pi$

= **2260 cm³** (to 3 s.f.)

(ii) Total surface area of solid

= curved surface area of hemisphere + curved surface area of cylinder + base area

$= \frac{1}{2} \times 4 \times \pi \times 7^2 + 2 \times \pi \times 7 \times 10 + \pi \times 7^2$

$= 98\pi + 140\pi + 49\pi$

$= 287\pi$

= **902 cm²** (to 3 s.f.)

4. (i) Volume of solid

= volume of hemisphere + volume of cone

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times 21^3 + \frac{1}{3} \times \pi \times 21^2 \times 28$

$= 6174\pi + 4116\pi$

$= 10\,290\pi$

= **32 300 cm³** (to 3 s.f.)

(ii) Total surface area of solid

= curved surface area of hemisphere + curved surface area of cone

$= \frac{1}{2} \times 4 \times \pi \times 21^2 + \pi \times 21 \times 35$

$= 882\pi + 735\pi$

$= 1617\pi$

= **5080 cm²** (to 3 s.f.)

5. Height of cone $= \frac{3}{5} \times 4r$

$= \frac{12}{5}r$

Volume of cone $= \frac{1}{3}\pi r^2 h$

$7 \times 1000 = \frac{1}{3} \times \pi \times r^2 \times \frac{12}{5}r$

$\frac{12}{15}\pi r^3 = 7000$

$r^3 = \frac{8750}{\pi}$

$r = \sqrt[3]{\frac{8750}{\pi}}$

$= 14.070$ (to 5 s.f.)

Volume of cylinder $= \pi \times 14.070^2 \times 4(14.070)$

$= 35\,002$ cm³ (to 5 s.f.)

= **35.0 *l*** (to 3 s.f.)

$\therefore$ amount of water needed to fill container completely

$= 35 + 7$

= **42 *l***

6. (i) Radius of cylinder = radius of base of cone

$= \frac{8}{2}$

$= 4$ m

Total surface area of solid

= curved surface area of 2 cones + curved surface area of cylinder

$= 2 \times \pi \times 4 \times 6 + 2 \times \pi \times 4 \times 8$

$= 48\pi + 64\pi$

$= 112\pi$

= **352 m²** (to 3 s.f.)

(ii) Using Pythagoras' Theorem,

Height of cone $= \sqrt{6^2 - 4^2}$

$= \sqrt{20}$

Volume of solid

= volume of 2 cones + volume of cylinder

$= 2 \times \frac{1}{3} \times \pi \times 4^2 \times \sqrt{20} + \pi \times 4^2 \times 8$

= **552 m³** (to 3 s.f.)

7. Radius of base $= \frac{4.7}{2}$

$= 2.35$ m

Height of cylinder $= 16.5 - 2.35$

$= 14.15$ m

Capacity of tank

= volume of hemisphere + volume of cylinder

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times 2.35^3 + \pi \times 2.35^2 \times 14.15$

= **273 m³** (to 3 s.f.)

8. Volume of metal ball $= \frac{4}{3} \times \pi \times 3^3$

$= 36\pi$ cm³

Let the height of the cone be h cm.

$\frac{1}{3} \times \pi \times 4^2 \times h = 36\pi$

$\frac{16}{3}h = 36$

$h = 6.75$

$\therefore$ the height of the cone is **6.75 cm**.

9. **(i)** Volume of hemisphere $= \frac{1}{2}\times\frac{4}{3}\times\pi\times35^3$

$= \frac{85\ 750}{3}\pi \text{ cm}^3$

Volume of cone $= 1\frac{1}{5}\times\frac{85\ 750}{3}\pi$

$= 34\ 300\pi \text{ cm}^3$

Let the height of the cone be h cm.

$\frac{1}{3}\times\pi\times35^2\times h = 34\ 300\pi$

$\frac{1225}{3}h = 34\ 300$

$h = 84$

$\therefore$ the height of the cone is **84 cm**.

(ii) Using Pythagoras' Theorem,

Slant height of cone $= \sqrt{35^2+84^2}$

$= 91$ cm

Total surface area of solid

= curved surface area of cone + curved surface area of hemisphere

$= \pi\times35\times91+\frac{1}{2}\times4\times\pi\times35^2$

$= 3185\pi + 2450\pi$

$= \mathbf{5635\pi\ cm^2}$

10. **(i)** Volume of solid

= volume of pyramid + volume of cuboid

$= \frac{1}{3}\times30\times30\times28+30\times30\times40$

$= 8400 + 36\ 000$

$= \mathbf{44\ 400\ cm^3}$

(ii) Using Pythagoras' Theorem,

Slant height of pyramid $= \sqrt{28^2+\left(\frac{30}{2}\right)^2}$

$= \sqrt{1009}$ cm

Total surface area of solid

$= 4\times\frac{1}{2}\times30\times\sqrt{1009}+2\times30\times40+2\times30\times40+30\times30$

$= \mathbf{7610\ cm^2}$ (to 3 s.f.)

Chapter 8 Averages of Statistical Data

TEACHING NOTES

Suggested Approach

In primary school, students have learnt that the average of a set of data is the sum of all the data divided by the number of data. Teachers can further explain that in statistics, there are other types of 'averages'. The average that students are familiar with is also known as the mean. In this chapter, besides mean, students will also learn about the properties of median and mode.

By the end of the chapter, students should be able to calculate mean, median and mode from a set of raw data, as well as data from various statistical diagrams. Students should also be able to decide on the preferred measure of average in various real-world contexts.

Section 8.1: Mean

Teachers can guide students through the worked examples to show how the mean is calculated. Students should be reminded to be careful not to miss out any values or use any wrong values in the calculation. Teachers are also recommended to go through with students on how mean helps to distribute items equally, as well as possible misconceptions (see Class Discussion: Understanding the concept of mean).

Since calculating the mean from a frequency table as well as estimating the mean of a set of grouped data are new to students, teachers should anticipate more practice and guidance for students in this section.

Section 8.2: Median

The definition and purpose of a median should be well-explained to students. The example on page 258 of the textbook is a good example why the median is preferred over the mean. Students may need to be reminded that the purpose of a numerical average is to give the best representation or summary of any set of data.

The steps in finding the median, such as to determine whether the number of data values is even or odd, arranging the data in order, are important and must be emphasised to students. The activities are also meant to test and reinforce students' understanding (see Class Discussion: Understanding the concept of median).

Section 8.3: Mode

The mode is arguably the easiest numerical average that students will need to learn, as it involves identifying the most frequent data without any calculations involved. As such, students should also be able to quickly learn to find the mode from various statistical diagrams.

Students should also be exposed to cases where a data set has more than one mode (See Worked Example 11), or has no mode at all (see Exercise 9B Question 3(c)).

Section 8.4: Measures of central tendency

Questions involving all three numerical averages will be covered in this section. Students may need to recall the solving of algebraic equations.

In this section, students should also be exposed to cases which require them to compare the mean, median and mode, as well as to decide on the most suitable numerical average depending on the set of data provided (see Class Discussion: Comparing different measures of central tendency). As a consolidation and a reinforcement of students' understanding in this chapter, teachers may also wish to get students to create data sets based on certain criteria involving the 3 averages (see Journal Writing on page 271).

Introductory Problem

The solutions to this problem can be found in Introductory Problem Revisited (after Exercise 8A).

8.1 Mean

Practise Now 1

Mean height of students $= \frac{1.54+1.67+1.49+1.54+1.76+1.69+1.58}{7}$

$= \mathbf{1.61\text{ m}}$

Practise Now 2

$44 + 47 + y + 58 + 55 = 52 \times 5$

$204 + y = 260$

$y = \mathbf{56}$

Practise Now 3

1. **(i)** Since mean $= \frac{\text{sum of the 7 numbers}}{7}$,

then sum of the 7 numbers $= 7 \times \text{mean}$

$= 7 \times 11$

$= \mathbf{77}$

(ii) Let the remaining two numbers be y and y.

$3 + 17 + 20 + 4 + 15 + y + y = 77$

$59 + 2y = 77$

$2y = 18$

$y = \mathbf{9}$

$\therefore$ the remaining two numbers are **9** and **9**.

2. Since mean $= \frac{\text{sum of the heights of 20 boys and 14 girls}}{34}$

then sum of the heights of 20 boys and 14 girls $= 34 \times \text{mean}$

$= 34 \times 161$

$= 5474$ cm

Since mean $= \frac{\text{sum of the heights of 14 girls}}{14}$

then sum of the heights of 14 girls $= 14 \times \text{mean}$

$= 14 \times 151$

$= 2114$ cm

Sum of the heights of 20 boys $= 5474 - 2114$

$= 3360$ cm

Mean height of the 20 boys $= \frac{3360}{20}$

$= \mathbf{168\text{ cm}}$

3. $\frac{16+w+17+9+x+2+y+7+z}{9} = 11$

$16 + w + 17 + 9 + x + 2 + y + 7 + z = 99$

$51 + w + x + y + z = 99$

$w + x + y + z = 48$

Mean of w, x, y and $z = \frac{48}{4}$

$= \mathbf{12}$

Practise Now 4

Total number of toys sold in March $= 31 \times 8$

$= 248$

Total number of toys sold in April $= 11 \times 30$

$= 330$

$\therefore$ mean number of toys sold per day in the two months

$= \frac{248+330}{31+30}$

$= \frac{578}{61}$

$= \mathbf{9.48}$ (to 3 s.f.)

Class Discussion (Understanding the concept of mean)

1. **No**, it is not possible. The most number of sweets that a student brought was 7, hence the mean number of sweets, x, must be less than 7 and x cannot be 8.
2. **No**, the mean number of sweets may or may not be equal to someone starting with that quantity originally.
3. **Yes**. The excess sweets from the students who brought more are distributed to those who brought less, without any sweets being added or removed. Hence, this number of sweets remains equal.
4. **0.25 ball**. This might not make sense as the ball cannot be divided, but we can view this as four students sharing one ball.
5. **Yes**. It means that there are 324 people in 100 households.
6. The "average household size" is obtained by dividing the number of persons in households by the number of households. Hence, it need not be an integer.

Practise Now 5

(i) Total number of visitors $= 12 + 32 + 54 + 68 + 18 + 16$

$= \mathbf{200}$

(ii) Total amount of money spent by the visitors

$= 12 \times \$40 + 32 \times \$60 + 54 \times \$80 + 68 \times \$100 + 18 \times \$160 + 16 \times \$200$

$= \mathbf{\$19\,600}$

(iii) Mean amount of money spent by the visitors $= \frac{\$19\,600}{200}$

$= \mathbf{\$98}$

Practise Now 6

Mean time taken by the group of students

$= \frac{\Sigma fx}{\Sigma f}$

$= \frac{5\times20+3\times30+10\times40+1\times50+1\times60}{5+3+10+1+1}$

$= \frac{700}{20}$

$= \mathbf{35\text{ minutes}}$

Practise Now 7

1.

Age (x years old)	Frequency (f)	Mid-value (x)	fx
$20 < x \leqslant 30$	12	25	300
$30 < x \leqslant 40$	10	35	350
$40 < x \leqslant 50$	20	45	900
$50 < x \leqslant 60$	15	55	825
$60 < x \leqslant 70$	18	65	1170
	$\Sigma f = 75$		$\Sigma fx = 3545$

Estimated mean age of the employees $= \frac{3545}{75}$

$= \mathbf{47.3\ years\ old}$ (to 3 s.f.)

2.

Mid-value (x)	Frequency (f)	fx
27.5	2	55
32.5	4	130
37.5	7	262.5
42.5	10	425
47.5	8	380
52.5	6	315
57.5	3	172.5
	$\Sigma f = 40$	$\Sigma fx = 1740$

Estimated mean length of the leaves $= \frac{1740}{40}$

$= \mathbf{43.5\ mm}$

Exercise 8A

1. Mean number of passengers

$= \frac{\text{sum of number of passengers}}{\text{number of coaches}}$

$= \frac{29+42+45+39+36+41+38+37+43+35+32+40}{12}$

$= \frac{457}{12}$

$= \mathbf{38.1}$ (to 3 s.f.)

2. Mean price of books

$= \frac{\text{sum of prices of books}}{\text{number of books}}$

$= \frac{19.90+24.45+34.65+26.50+44.05+38.95+56.40+48.75+29.30+35.65}{10}$

$= \frac{358.6}{10}$

$= \mathbf{\$35.86}$

3. $\frac{7+15+12+5+h+13}{6} = 10$

$52 + h = 60$

$h = \mathbf{8}$

4. Since mean of masses of five boys $= \frac{\text{sum of the masses of five boys}}{5}$

then sum of the masses of five boys $= 5 \times \text{mean}$

$= 5 \times 62$

$= 310$ kg

Since mean of masses of four boys $= \frac{\text{sum of the masses of four boys}}{4}$

then sum of the masses of four boys $= 4 \times \text{mean}$

$= 4 \times 64$

$= 256$ kg

Mass of boy excluded $= 310 - 256$

$= \mathbf{54\ kg}$

5. **(i)** Since mean of eight numbers $= \frac{\text{sum of the eight numbers}}{8}$,

then sum of the eight numbers $= 8 \times \text{mean}$

$= 8 \times 12$

$= \mathbf{96}$

(ii) $6 + 8 + 5 + 10 + 28 + k + k + k = 96$

$57 + 3k = 96$

$3k = 39$

$k = \mathbf{13}$

6. Total number of people living in Building X $= 60 \times 3$

$= 180$

Total number of people living in Building Y $= 80 \times 4$

$= 320$

$\therefore$ mean number of people living in each unit $= \frac{180+320}{60+80}$

$= \frac{500}{140}$

$= \mathbf{3\frac{4}{7}}$ or $\mathbf{3.57}$ (to 3 s.f.)

7. Total mass of students in Class A $= 50.5 \times 28$

$= 1414$ kg

Total mass of students in Class B $= 52.6 \times 35$

$= 1841$ kg

$\therefore$ mean mass of the students in the two classes

$= \frac{1414+1841}{28+35}$

$= \frac{3255}{63}$

$= \mathbf{51\frac{2}{3}\ kg}$ or $\mathbf{51.7\ kg}$ (to 3 s.f.)

8. **(i)** Total number of matches played $= 6 + 8 + 5 + 6 + 2 + 2 + 1$

$= \mathbf{30}$

(ii) Total number of goals scored

$= 6 \times 0 + 8 \times 1 + 5 \times 2 + 6 \times 3 + 2 \times 4 + 2 \times 5 + 1 \times 6$

$= \mathbf{60}$

(iii) Mean number of goals scored per match $= \frac{60}{30}$

$= \mathbf{2}$

9. Mean number of days of absence

$= \frac{\Sigma fx}{\Sigma f}$

$= \frac{23\times0+4\times1+5\times2+2\times3+2\times4+1\times5+2\times6+1\times9}{23+4+5+2+2+1+2+1}$

$= \frac{54}{40}$

$= \mathbf{1.35}$

10. Mean $= \dfrac{\Sigma fx}{\Sigma f}$

$= \dfrac{3 \times 3 + 4 \times 5 + 5 \times 6 + 6 \times 4 + 7 \times 2}{3 + 5 + 6 + 4 + 2}$

$= \dfrac{97}{20}$

$= \mathbf{4.85}$

11. **(i)** Since mean of 10 numbers $= \dfrac{\text{sum of 10 numbers}}{10}$,

then sum of 10 numbers $= 10 \times \text{mean}$

$= 10 \times 14$

$= 140$

Since mean of 3 numbers $= \dfrac{\text{sum of 3 numbers}}{3}$,

then sum of 3 numbers $= 3 \times \text{mean}$

$= 3 \times 4$

$= 12$

Sum of the remaining seven numbers $= 140 - 12$

$= \mathbf{128}$

(ii) $15 + 18 + 21 + 5 + m + 34 + 14 = 128$

$107 + m = 128$

$m = \mathbf{21}$

12. **(i)** Since mean height $= \dfrac{\text{sum of heights of three plants}}{3}$,

then sum of heights of three plants $= 3 \times \text{mean}$

$= 3 \times 30$

$= 90$ cm

Height of Plant B $= \dfrac{3}{2+3+5} \times 90$

$= \dfrac{3}{10} \times 90$

$= \mathbf{27\ cm}$

(ii) Since mean height $= \dfrac{\text{sum of heights of four plants}}{4}$

then sum of heights of four plants $= 4 \times \text{mean}$

$= 4 \times 33$

$= 132$ cm

Height of Plant D $= 132 - 90$

$= \mathbf{42\ cm}$

13. Since mean monthly wage of twelve workers

$= \dfrac{\text{sum of monthly wages of twelve workers}}{12}$,

then sum of monthly wages of twelve workers $= 12 \times \text{mean}$

$= 12 \times 1000$

$= \$12\,000$

Since mean monthly wage of five inexperienced workers

$= \dfrac{\text{sum of monthly wages of five inexperienced workers}}{5}$,

then sum of monthly wages of five inexperienced workers

$= 5 \times \text{mean}$

$= 5 \times 846$

$= \$4230$

Sum of monthly wages of seven experienced workers

$= 12\,000 - 4230$

$= \$7770$

Mean monthly wage of seven experienced workers $= \dfrac{7770}{7}$

$= \mathbf{\$1110}$

14. **(a)** Mean mark of students for English

$= \dfrac{\Sigma fx}{\Sigma f}$

$= \dfrac{0\times0+1\times1+6\times2+14\times3+4\times4+8\times5+2\times6+4\times7+0\times8+1\times9+0\times10}{40}$

$= \dfrac{160}{40}$

$= \mathbf{4}$

Mean mark of students for Mathematics

$= \dfrac{\Sigma fx}{\Sigma f}$

$= \dfrac{0\times0+4\times1+1\times2+6\times3+5\times4+10\times5+3\times6+5\times7+3\times8+1\times9+2\times10}{40}$

$= \dfrac{200}{40}$

$= \mathbf{5}$

(b) **(i)** Passing mark for English $= 50\% \times 10$

$= \dfrac{50}{100} \times 10$

$= 5$

Number of students who passed English $= 8 + 2 + 4 + 1$

$= 15$

Percentage of students who passed English $= \dfrac{15}{40} \times 100\%$

$= \mathbf{37\frac{1}{2}\%}$ or **37.5%**

(ii) Passing mark for Mathematics $= 50\% \times 10$

$= \dfrac{50}{100} \times 100$

$= 5$

Number of students who did not pass Mathematics

$= 4 + 1 + 6 + 5$

$= 16$

Percentage of students who did not pass Mathematics

$= \dfrac{16}{40} \times 100\%$

$= \mathbf{40\%}$

15. **(i)** Mean amount of time that the 30 students slept for

$= \dfrac{\Sigma fx}{\Sigma f}$

$= \dfrac{3\times2+4\times3+5\times7+6\times4+7\times3+8\times5+9\times4+10\times2}{30}$

$= \dfrac{194}{30}$

$= \mathbf{6\frac{7}{15}\ h}$ or **6.47 h** (to 3 s.f.)

(ii) Total amount of time the other 20 students slept for

$= 7.5 \times 20$

$= 150$ h

Total amount of time the 50 students slept for $= 194 + 150$

$= 344$ h

$\therefore$ mean amount of time that the 50 students slept for

$= \dfrac{344}{50}$

$= \mathbf{6.88\ h}$

16. (i)

Height (x cm)	Frequency (f)	Mid-value (x)	fx
$0 < x \leqslant 10$	4	5	20
$10 < x \leqslant 20$	6	15	90
$20 < x \leqslant 30$	14	25	350
$30 < x \leqslant 40$	6	35	210
$40 < x \leqslant 50$	10	45	450
	$\Sigma f = 40$		$\Sigma fx = 1120$

Estimate for the mean height of the plants $= \frac{1120}{40}$

$= \mathbf{28\ cm}$

(ii) Number of plants not taller than 40 cm $= 4 + 6 + 14 + 6$

$= 30$

P(plant not taller than 40 cm) $= \frac{30}{40}$

$= \mathbf{\frac{3}{4}}$

17. (i)

Time taken (t minutes)	Frequency (f)	Mid-value (x)	fx
$116 \leqslant t < 118$	1	117	117
$118 \leqslant t < 120$	6	119	714
$120 \leqslant t < 122$	23	121	2783
$122 \leqslant t < 124$	28	123	3444
$124 \leqslant t < 126$	27	125	3375
$126 \leqslant t < 128$	9	127	1143
$128 \leqslant t < 130$	5	129	645
$130 \leqslant t < 132$	1	131	131
	$\Sigma f = 100$		$\Sigma fx = 12\,352$

Estimate for the mean travelling time of the lorries

$= \frac{12\,352}{100}$

$= \mathbf{123.52\ minutes}$

(ii) Number of lorries which took less than 124 minutes

$= 1 + 6 + 23 + 28$

$= 58$

Fraction of lorries which took less than 124 minutes $= \frac{58}{100}$

$= \mathbf{\frac{29}{50}}$

18. (i)

Speed (x km/h)	Frequency (f)	Mid-value (x)	fx
$30 < x \leqslant 40$	16	35	560
$40 < x \leqslant 50$	25	45	1125
$50 < x \leqslant 60$	35	55	1925
$60 < x \leqslant 70$	14	65	910
$70 < x \leqslant 80$	10	75	750
	$\Sigma f = 100$		$\Sigma fx = 5270$

Estimate for the mean speed of the vehicles $= \frac{5270}{100}$

$= \mathbf{52.7\ km/h}$

(ii) Required ratio $= 16 : (14 + 10)$

$= 16 : 24$

$= \mathbf{2 : 3}$

19. (i)

Mean distance (d million km)	Frequency
$21.0 \leqslant d < 21.5$	**7**
$21.5 \leqslant d < 22.0$	**0**
$22.0 \leqslant d < 22.5$	**1**
$22.5 \leqslant d < 23.0$	**1**
$23.0 \leqslant d < 23.5$	**8**
$23.5 \leqslant d < 24.0$	**3**

(ii)

Mean distance (d million km)	Frequency (f)	Mid-value (x)	fx
$21.0 \leqslant d < 21.5$	7	21.25	148.75
$21.5 \leqslant d < 22.0$	0	21.75	0
$22.0 \leqslant d < 22.5$	1	22.25	22.25
$22.5 \leqslant d < 23.0$	1	22.75	22.75
$23.0 \leqslant d < 23.5$	8	23.25	186
$23.5 \leqslant d < 24.0$	3	23.75	71.25
	$\Sigma f = 20$		$\Sigma fx = 451$

Estimate for the mean of the mean distances of the moons from Jupiter

$= \frac{451}{20}$

$= \mathbf{22.55\ million\ km}$

20. Since mean of x, y and $z = \frac{\text{sum of } x, y \text{ and } z}{3}$,

then sum of x, y and $z = 3 \times \text{mean}$

$= 3 \times 6$

$= 18$

Since mean of x, y, z, a and $b = \frac{\text{sum of } x, y, z, a \text{ and } b}{5}$,

then sum of x, y, z, a and $b = 5 \times \text{mean}$

$= 5 \times 8$

$= 40$

Sum of a and $b = 40 - 18$

$= 22$

Mean of a and $b = \frac{22}{2}$

$= \mathbf{11}$

21. (i) Mean $= \frac{\text{sum of the lifespans of 30 light bulbs}}{30}$

$$= \frac{\begin{array}{l}167+171+179+167+171+165+175+179\\+169+168+171+177+169+171+177+173\\+165+175+167+174+177+172+164+175\\+179+179+174+174+168+171\end{array}}{30}$$

$= \frac{5163}{30}$

$= \mathbf{172.1\ hours}$

(ii)

Lifespan (x hours)	Frequency (f)
$164 \leqslant x < 167$	3
$167 \leqslant x < 170$	7
$170 \leqslant x < 173$	6
$173 \leqslant x < 176$	7
$176 \leqslant x < 179$	3
$179 \leqslant x < 182$	4

(iii)

Lifespan (x hours)	Frequency (f)	Mid-value (x)	fx
$164 \leqslant x < 167$	3	165.5	496.5
$167 \leqslant x < 170$	7	168.5	1179.5
$170 \leqslant x < 173$	6	171.5	1029
$173 \leqslant x < 176$	7	174.5	1221.5
$176 \leqslant x < 179$	3	177.5	532.5
$179 \leqslant x < 182$	4	180.5	722
	$\Sigma f = 30$		$\Sigma fx = 5181$

Estimate for the mean lifespan of the lightbulbs

$= \frac{5181}{30}$

$=$ **172.7 hours**

(iv) The two values are different. The value in part **(iii)** is an estimate of the actual value in part **(i)**.

8.2 Median

Introductory Problem Revisited

1. **2400** is the middle value.
2. **Yes**, 2400 will still be the middle value.
3. **Yes**, the median salary presents a fitting picture as most of the employees in the company earn a salary that is close to \$2400.
4. The median salary is not affected by the extreme value as the median is determined by position, unlike the mean which involves adding all the values during its computation.
5. There are **two** extreme values, which are **800** and **10 500**.
6. Mean monthly salary $= \$\frac{800+5500+5900+6300+10\,500}{5}$

   $= \$\frac{29\,000}{5}$

   $=$ **\$5800**
7. The median salary is **\$5900**.
8. **Yes**, the mean and median monthly salaries are close to each other.
9. The mean monthly salary is not adversely affected in this case because the two extreme values on either end of the data set (i.e. \$800 and \$10 500) more or less balance out.
10. **Yes.** The mean of \$5800 lies in the middle of the monthly salaries of the three salaries that are not the extreme values, unlike the mean of the company above. The median of \$5900 also describes general average monthly salary of the employees and it is more or less in the middle of the other two monthly salaries (i.e. \$800 and \$10 500).

Practise Now 8

(a) Total number of data values = 7

Position of median $= \frac{7+1}{2}$

$= 4$

Rearranging the data in ascending order:

3, 9, 11, 15, 16, 18, 20

$\therefore$ median = data in the 4th position

= **15**

(b) Total number of data values = 5

Position of median $= \frac{5+1}{2}$

$= 3$

Rearranging the data in ascending order:

11.2, 15.6, 17.3, 18.2, 30.2

$\therefore$ median = data in the 3rd position

= **17.3**

Practise Now 9

1. (a) Total number of data values = 6

Position of median $= \frac{6+1}{2}$

$= 3.5$

Rearranging the data in ascending order:

12, 15, 15, 20, 25, 32

$\therefore$ median = mean of 3rd and 4th values

$= \frac{15+20}{2}$

= **17.5**

(b) Total number of data values = 8

Position of median $= \frac{8+1}{2}$

$= 4.5$

Rearranging the data in ascending order:

6.7, 6.8, 7.3, 8, 8.8, 8.9, 8.9, 10

$\therefore$ median = mean of 4th and 5th values

$= \frac{8+8.8}{2}$

= **8.4**

2. Rearranging the data in ascending order,

OPEN We can have x, 3, 5, 6, 7, 9 or 3, 5, 6, 7, 9, x or x is between any two numbers.

Total number of data values = 6

Position of median $= \frac{6+1}{2}$

$= 3.5$

The median is the mean of the 3rd and 4th values.

Since $\frac{6+7}{2} = 6.5$, 6 and 7 are the 3rd and 4th values respectively.

Hence, x can be **any number equal to or more than 7** and a possible value of x is 7.

Class Discussion (Understanding the concept of median)

1. **No.** For example, consider this data set: 2, 6, 8, 11. There are two middle values: 6 and 8. But the median is 7, which is not even a data value in the data set.
2. **No.** For example, consider this data set: 2, 7, 10, 14, 16. The median is 10, but only 40% of the data values are greater than 10, and 40% are less than 10. Another example is this data set: 10, 10, 10, 14. The median is still 10, but now only 25% of the data values are greater than 10, and 0% of the data values are less than 10.

Practise Now 10

1. Total number of data values = 28

   Position of median $= \frac{28+1}{2}$

   $= 14.5$

   $\therefore$ median = mean of 14th and 15th values

   $= \frac{7+7}{2}$

   $=$ **7 minutes**
2. Total number of data values = 55

   Position of median $= \frac{55+1}{2}$

   $= 28$

   $\therefore$ the class interval which contains the median price is $\mathbf{5 < x \leqslant 10}$.
3. Total number of data values = 16

   Position of median $= \frac{16+1}{2}$

   $= 8.5$

   $\therefore$ median = mean of 8th and 9th values

   $= \frac{3+3}{2}$

   $=$ **3**

8.3 Mode

Class Discussion (Mode as an average)

1. **(a)** Mean $= \frac{8+8+10+8+10+12+10+8+8+12}{10}$

   $=$ **9.4**

   (b) Rearranging the data in ascending order:

   8, 8, 8, 8, 8, 10, 10, 10, 12, 12

   Median position $= \frac{10+1}{2}$

   $= 5.5$

   Median = mean of 5th and 6th values

   $= \frac{8+10}{2}$

   $=$ **9**

   (c) Mode = **8**
2. The mean of 9.4 and the median of 9 are not very meaningful because there are no such sizes for the blouses. Hence, the **mode** presents the most fitting picture of the most popular size of blouses.

Practise Now 11

(i) Modal lengths = **60 cm and 110 cm**

(ii) New modal length = **60 cm**

Class Discussion (Understanding the concept of mode)

1. **No**, 7 was the largest data value.
2. **Yes**. The mode is the data value with the highest frequency and thus must correspond to the number of sweets that some students brought.

Practise Now 12

(a) Mode = **PKR 3000**

(b) Modal class: $\mathbf{45 < x \leqslant 60}$

(c) Mode = **1**

(d) Mode = **Caramel and hazelnut**

8.4 Measures of central tendency

Class Discussion (Comparing different measures of central tendency)

1. **(i)** The mean is preferred when there are no extreme values in the data set.

   (ii) The median is preferred when there are extreme values in the data set.

   (iii) The mode is preferred when the most common value in the data set is of interest.
2. Cinema X:

   Median. The mean of 76.2 is way below the daily attendance of 4 of the 5 days (i.e. 87, 88, 92 and 98), but the median of 88 is not affected by the extreme value of 16.

   Cinema Y:

   Mean. The median of 62 only describes the daily attendance of 3 days well (i.e. 60, 61 and 62), but it does not take into account the daily attendance of the other two days (i.e. 98 and 98). The mode of 98 is at the extreme end of the data set when arranged in ascending order, which does not account for the daily attendance of the other 3 days well. We are not interested in the daily attendance that occurs the most often because it does not tell us about the typical daily attendance.

   Cinema Z:

   Mean or median. The two extreme values on either end of the data set when arranged in ascending order (i.e. 54 and 98) more or less balance out in this case. The mean of 75.6 is not adversely affected by them and lies in the middle of the daily attendance of the other 3 days (i.e. 75, 75 and 76). The median of 75 also describes the daily attendance of 3 days well (i.e. 75, 75 and 76) and it is more or less in the middle of the daily attendance of the other days (i.e. 54 and 98), unlike the median of Cinema X. Although the mode of 75 is equal to the median of 75, we are not interested in the daily attendance that occurs the most often because it does not tell us about the typical daily attendance.

Practise Now 13

(a) $\frac{2\times0+x\times1+3\times2+4\times3+1\times4}{2+x+3+4+1} = 1.8$

$$\begin{aligned} \frac{x+22}{x+10} &= 1.8 \\ x + 22 &= 1.8(x + 10) \\ x + 22 &= 1.8x + 18 \\ 0.8x &= 4 \\ x &= \mathbf{5} \end{aligned}$$

(b) Greatest possible value of x = **3**

(c) **Method 1:**

We write the data as follows:

$$0, 0, \underbrace{1, ..., 1}_{x}, 2, 2, 2, 3, 3, 3, 3, 4$$

The greatest value of x occurs when the median is here.

$\therefore 2 + x = 2 + 4 + 1$

$2 + x = 7$

$x = 5$

Hence, greatest value of $x = 5$

The smallest value of x occurs when the median is here.

$\therefore 2 + x + 2 = 4 + 1$

$4 + x = 5$

$x = 1$

Hence, smallest value of $x = 1$

$\therefore$ the possible values of x are **1**, **2**, **3**, **4** and **5**.

Method 2:

$$\text{Position of median} = \frac{(2+x+3+4+1)+1}{2} = \frac{x+11}{2}$$

For greatest value of x, position of median $= 2 + x + 1 = x + 3$

$$\begin{aligned} \therefore \frac{x+11}{2} &= x + 3 \\ x + 11 &= 2x + 6 \\ x &= 11 - 6 \\ &= 5 \end{aligned}$$

For smallest value of x, position of median $= 2 + x + 3 = x + 5$

$$\begin{aligned} \therefore \frac{x+11}{2} &= x + 5 \\ x + 11 &= 2x + 10 \\ x &= 11 - 10 \\ &= 1 \end{aligned}$$

$\therefore$ the possible values of x are **1, 2, 3, 4** and **5**.

Practise Now 14

Since the smallest number has an infinite number of factors, then the smallest number is 0. Arranging the numbers in ascending order:

0, ___, ___, ___, ___, ___

Given that there are six numbers, the median is the mean of the third and fourth number.

Since the median is 1.5, then the sum of the third and the fourth number is 3.

Considering that the smallest number is 0, then the third number can be 0 or 1 and the fourth number will be 3 or 2 respectively. The third and the fourth numbers cannot be 0 and 3 respectively as that will mean that the set of numbers does not contain 1. Hence, the third and the fourth numbers are 1 and 2 respectively:

0, ___, 1, 2, ___, ___

Since the mode is 1, this number must occur at least twice:

0, 1, 1, 2, ___, ___

$$\begin{aligned} \text{Sum of the six numbers} &= \text{mean} \times 6 \\ &= 2 \times 6 \\ &= 12 \end{aligned}$$

So the sum of the last two numbers is $12 - 0 - 1 - 1 - 2 = 8$.

Since half of the numbers are prime and there is only one prime number in the data set so far, the last two numbers must be prime and their sum is 8, i.e. 3 and 5.

$\therefore$ the six numbers are **0, 1, 1, 2, 3** and **5**.

Practise Now 15

Correct modal mass = incorrect modal mass – 8
= 60 – 8
= **52 kg**

Correct median mass = incorrect median mass – 8
= 62 – 8
= **54 kg**

Method 1:

Correct mean mass = incorrect mean mass – 8
= 65.3 – 8
= **57.3 kg**

Method 2:

Total incorrect mass of 30 students = incorrect mean mass × 30
= 65.3 × 30
= 1959 kg

Total correct mass of 30 students = 1959 – 30 × 8
= 1719 kg

Correct mean mass = 1719 ÷ 30
= **57.3 kg**

Journal Writing (Page 271)

A wide variety of data sets can satisfy the conditions.

Examples of possible data sets are:

1, 1, 2, 5, 10, 16, 21 (mode = 1, median = 5, mean = 8)

8, 8, 8, 9, 9, 14, 28 (mode = 8, median = 9, mean = 12)

10, 10, 10, 15, 20, 25, 30 (mode = 10, median = 15, mean = 17.1 (to 3 s.f.))

Exercise 8B

1. **(a)** Total number of data values = 7

$$\text{Position of median} = \frac{7+1}{2} = 4$$

Rearranging the data in ascending order:

1, 3, 5, 5, 5, 6, 6

$\therefore$ median = **5**

(b) Total number of data values = 7

$$\text{Position of median} = \frac{7+1}{2} = 4$$

Rearranging the data in ascending order:

1.1, 1.2, 1.6, 2.8, 3.2, 4.1, 4.1

$\therefore$ median = **2.8**

(c) Total number of data values = 6

Position of median = $\frac{6+1}{2}$

$= 3.5$

Rearranging the data in ascending order:

25, 28, 29, 30, 33, 37

$\therefore$ median = $\frac{29+30}{2}$

= **29.5**

(d) Total number of data values = 8

Position of median = $\frac{8+1}{2}$

$= 4.5$

Rearranging the data in ascending order:

4.7, 5.5, 8.4, 12, 13.5, 22.6, 31.3, 39.6

$\therefore$ median = $\frac{12+13.5}{2}$

= **12.75**

2. (a) Total number of data values = 15

Position of median = $\frac{15+1}{2}$

$= 8$

$\therefore$ median = **400**

(b) Total number of data values = 44

Position of median = $\frac{44+1}{2}$

$= 22.5$

$\therefore$ the class interval which contains the median is $\mathbf{15 < x \leqslant 20}$.

(c) Total number of data values = 21

Position of median = $\frac{21+1}{2}$

$= 11$

$\therefore$ median = **70**

(d) Total number of data values = 40

Position of median = $\frac{40+1}{2}$

$= 20.5$

$\therefore$ median = $\frac{5+6}{2}$

= **5.5**

3. (a) Mode = **3**

(b) Modes = **7.7 and 9.3**

(c) **No mode**

4. (i) Modal temperature = **27 °C**

(ii) New modal temperatures = **22 °C and 27 °C**

5. (a) **No mode**

(b) Mode = **30**

(c) Mode = **White bread**

6. (a) Modes = **Blue and purple**

(b) Modal class: $\mathbf{0 \leqslant x < 10}$

(c) Mode = **$32**

(d) Mode = **Japan**

7. Let the eighth number be x.

Total number of data values = 8

Position of median = $\frac{8+1}{2}$

$= 4.5$

Rearranging the numbers in ascending order,
we can have x, 1, 2, 3, 4, 9, 12, 13 or 1, 2, 3, 4, 9, 12, 13, x or x is between any 2 numbers.

The median is the mean of the 4th position and 5th position.

Since $\frac{3+4}{2} = 3.5 \neq 4.5$ and $\frac{4+9}{2} = 6.5 \neq 4.5$, x must be in the 5th position.

If x is in the 5th position, $\frac{4+x}{2} = 4.5$

$4 + x = 9$

$x = 5$

The eighth number is **5**.

8. (i) Mean score of Albert = $\frac{3+2+5+7+3+2+2+4+17}{9}$

$= \frac{45}{9}$

= **5**

Mean score of Bernard = $\frac{4+4+6+8+3+3+2+6+6}{9}$

$= \frac{42}{9}$

$= \mathbf{4\frac{2}{3}}$

(ii) **Bernard** scored better on most of the holes.
Since the mean score of Albert is greater than that of Bernard, the mean scores do **not** indicate which player scored better on most of the holes.

(iii) Total number of data values = 9

Position of median = $\frac{9+1}{2}$

$= 5$

Median score of Albert = **3**

Median score of Bernard = **4**

(iv) Modal score of Albert = **2**

Modal score of Bernard = **6**

(v) The **median** gives the best comparison of the abilities of Albert and Bernard because it is not affected by extreme scores, unlike the mean. In addition, the median gives an overview of the abilities, but the mode does not, especially in this case where the scores are spread out.

9. (i) The number of students who did less than or equal to 5 pull-ups, or more than or equal to 10 pull-ups are grouped together.

(ii) Total number of data values for each class = 21

Position of median = $\frac{21+1}{2}$

$= 11$

$\therefore$ median number of pull-ups for 2A = **7**

$\therefore$ median number of pull-ups for 2B = **7**

(iii) Modal number of pull-ups for 2A = **6**

Modal number of pull-ups for 2B = **8**

(iv) The **mode** gives a better comparison. The mode shows the most common number of pull-ups done by the students in each class, which gives a better comparison of the number of pull-ups that most students in the class did. On the other hand, the median does not show any difference between the two classes.

10. **(a)** **(i)** $x + 2 + y + 6 + 14 = 40$

$$x + y + 22 = 40$$
$$x + y = 40 - 22$$
$$= 18 \text{ (shown)}$$

(ii) $2x + 2 \times 4 + 6y + 14 \times 10 + 6 \times 8 = 6.4 \times 40$

$$2x + 6y + 196 = 256$$
$$2x + 6y = 256 - 196$$
$$= 60$$
$$x + 3y = 30 \text{ (shown)}$$

(iii) $x + y = 18 \quad \text{— (1)}$

$x + 3y = 30 \quad \text{— (2)}$

(2) – (1): $2y = 30 - 18$

$= 12$

$y = 6$

Substitute $y = 6$ into (1): $x + 6 = 18$

$x = 18 - 6$

$= 12$

$\therefore$ $\boldsymbol{x = 12, y = 6}$

(b) **(i)** Position of median $= \dfrac{40+1}{2}$

$= 20.5$

$\therefore$ median $= \dfrac{6+8}{2}$

$= \mathbf{7}$

(ii) Mode = **10**

11. **(a)** $0 \times 5 + 1 \times 2 + 2 \times 1 + 3x = 1 + 2 + 5) \times 2 + x)$

$$4 + 3x = 16 + 2x$$
$$3x - 2x = 16 - 4$$
$$x = \mathbf{12}$$

(b) **Method 1:**

$$0, 0, 0, 0, 0, 1, 1, 2, \underbrace{3, \ldots, 3}_{x}$$

The smallest value of x occurs when the median is here.

$\therefore$ $5 = 1 + 1 + x$

$x = 3$

Hence, smallest value of $x = 3$.

The greatest value of x occurs when the median is here.

$\therefore$ $5 + 1 = 1 + x$

$x = 5$

Hence, greatest value of $x = 5$.

$\therefore$ x can be either **3, 4 or 5**.

Method 2:

Position of median $= \dfrac{(5+2+1+x)+1}{2}$

$= \dfrac{x+9}{2}$

For greatest value of x, position of median $= 5 + 2$

$= 7$

$\therefore$ $\dfrac{x+9}{2} = 7$

$x + 9 = 14$

$x = 5$

For smallest value of x, position of median $= 5 + 1$

$= 6$

$\therefore$ $\dfrac{x+9}{2} = 6$

$x + 9 = 12$

$x = 3$

$\therefore$ x can be either **3, 4 or 5**.

12. Given that there are seven numbers, the median is the fourth number.

Since the median is 8, then the fourth number of the data set that has been arranged in ascending order is 8:

□, □, □, 8, □, □, □

Since the modes are 2 and 10, these numbers must occur at least twice. The largest number cannot be 10 as it is a perfect square.

We can have:

2, 2, □, 8, 10, 10, □ or □, 2, 2, 8, 10, 10, □

Sum of the seven numbers = mean × 7

$= 7 \times 7$

$= 49$

The sum of the remaining two numbers is

$49 - 2 - 2 - 8 - 10 - 10 = 17$.

Since the largest number is a perfect square, which is more than 10 but less than 17, the largest number is 16. The remaining number is $17 - 16 = 1$.

$\therefore$ the seven numbers are **1, 2, 2, 8, 10, 10** and **16**.

13. Since half of the numbers are the same and the mode is 9, then three of the numbers are 9.

Given that there are six numbers, the median is the mean of the third and fourth number of the data set that has been arranged in ascending order.

Since the median is 9, then there are two possible arrangements of the three 9s:

□, □, 9, 9, 9, □ or □, 9, 9, 9, □, □

Note that the first number, being the smallest, must be the single digit negative number.

Sum of the six numbers = mean × 6

$= 7 \times 6$

$= 42$

Sum of the remaining 3 numbers $= 42 - 3 \times 9$

$= 15$

Since two of the positive numbers are prime, then two of the remaining three unknown numbers must be prime numbers since 9 is not a prime number.

Prime numbers that are greater than 9 but smaller than 15 = {11, 13}

Prime numbers that are smaller than 9 = {2, 3, 5, 7}

Suppose that there is 1 number larger than 9 in the data set and the number is 13.

Then the other prime number can be 3, 5 or 7 and the single digit negative number will be −1, −3 or −5 respectively:

$\therefore$ a possible set of the six numbers is **−3, 5, 9, 9, 9, 13**.

14. Correct modal time = incorrect modal time + 3

$= 13 + 3$

$=$ **16 min**

Correct median time = incorrect median time + 3

$= 13 + 3$

$=$ **16 min**

Method 1:

Correct mean time = incorrect mean time + 3

$= 15 + 3$

$=$ **18 min**

Method 2:

Total incorrect time of 12 students = incorrect mean time × 12

$= 15 \times 12$

$= 180$ min

Total correct time of 12 students = $180 + 3 \times 12$

$= 216$ min

Correct mean time = $216 \div 12$

= **18 min**

15. New modal amount = previous modal amount – \$10

= \$50 – \$10

= **\$40**

New median amount = previous median amount – \$10

= \$40 – \$10

= **\$30**

Method 1:

New mean amount = previous mean amount – \$10

= \$35 – \$10

= **\$25**

Method 2:

Total previous amount = previous mean amount × 6

= \$35 × 6

= \$210

Total new amount = \$210 – \$10 × 6

= \$150

New mean mass = \$150 ÷ 6

= **\$25**

16. (i) $0 \times 5 + 1 \times 13 + 2 \times 15 + 3x + 5 + 1 \times 4y + 2 \times 6 = 50 \times 2.18$

$13 + 30 + 3x + 4 + 5y + 12 = 109$

$59 + 3x + 5y = 109$

$3x + 5y = 50$ — (1)

$5 + 13 + 15 + x + 1 + y + 2 = 50$

$36 + x + y = 50$

$x + y = 14$ — (2)

(2) × 3: $3x + 3y = 42$ — (3)

(1) – (3): $2y = 8$

$y = 4$

Substitute $y = 4$ into (2): $x + 4 = 14$

$x = 10$

$\therefore$ $\mathbf{x = 10, y = 4}$

(ii) (a) Position of median = $\frac{50+1}{2}$

$= 25.5$

Median = $\frac{2+2}{2}$

= **2**

(b) Mode = **2**

(iii) Number of years with at most p major hurricanes

$= \frac{36}{100} \times 50$

$= 18$

Since $5 + 13 = 18$, $p =$ **1**.

17. (a) $0 \times 4 + 1 \times 6 + 2 \times 3 + 3x + 5 \times 2 = 2.2 \times (4 + 6 + 3 + x + 3 + 2)$

$6 + 6 + 3x + 12 + 10 = 2.2 \times (18 + x)$

$34 + 3x = 39.6 + 2.2x$

$0.8x = 5.6$

$x = 7$

(b) Method 1:

$$0, 0, 0, 0, \underbrace{1, \ldots, 1}_{6}, 2, 2, 2, \underbrace{3, \ldots, 3}_{x}, 4, 4, 4, 5, 5$$

The greatest value of x occurs when the median is here.

$\therefore 4 + 6 + 2 = x + 3 + 2$

$x = 7$

Hence, the greatest value of x is **7**.

Method 2:

Position of median = $\frac{(4+6+3+x+3+2)+1}{2}$

$= \frac{x+19}{2}$

For greatest value of x, position of median = $4 + 6 + 3$

$= 13$

$\therefore \frac{x+19}{2} = 13$

$x + 19 = 26$

$x = 7$

Hence, the greatest value of x is **7**.

(c) If the modal number of social networking accounts the students own is 3, then $x > 6$. $\therefore$ the smallest possible value of x is **7**.

18. (i) Total number of students

$= x + 1 + x - 2 + x + 2 + x + x - 2 + x - 4 + x - 3$

$= 7x - 8$

$$\text{Mean} = \frac{(x+1)\times 0+(x-2)\times 1+(x+2)\times 2+x\times 3+(x-2)\times 4+(x-4)\times 5+(x-3)\times 6}{7x-8}$$

$$= \frac{x-2+2x+4+3x+4x-8+5x-20+6x-18}{7x-8}$$

$$= \frac{21x-44}{7x-8}$$

Mode = 2

$\therefore \frac{21x-44}{7x-8} = 2$

$21x - 44 = 2(7x - 8)$

$21x - 44 = 14x - 16$

$7x = 28$

$x =$ **4**

(ii)

Number of books	0	1	2	3	4	5	6
Number of students	5	2	6	4	2	0	1

Total number of data values = $7(4) - 8$

$= 20$

Position of median = $\frac{20+1}{2}$

$= 10.5$

$\therefore$ median = mean of the data in the 10th and the 11th positions

$= \frac{2+2}{2}$

= **2**

Notes

Notes